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Research Article

Reservoirs in Permutations

Aubrey Blecher, Arnold Knopfmacher*

The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of Witwatersrand, South Africa

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Abstract

The capacity of a permutation of n is the amount of water its bargraph would retain under the usual laws of physics when the water is allowed to escape left or right. A reservoir is a single body of water after all water has been allowed to escape under the same physical laws. For each n, we obtain formulae for the total number of reservoirs and their total width across all permutations of n. For each $1 \leq d \leq n-2$, we also compute the number of reservoirs of depth d.

Keywords: permutations; generating functions; reservoirs; key comparisons in quicksort.

2020 Mathematics Subject Classification: 05A05, 05A15.

1. Introduction

A permutation $a_1, a_2, \cdots, a_n$ of n is an ordering from left to right of all the letters from the alphabet $[n] = \{1, 2, \cdots, n\}$. We represent each such permutation as a bargraph with n columns in which column i is of height a_i, and we define the capacity of a permutation as the amount of water the bargraph representation would retain if it was considered as a full container with the water allowed to escape left and right, subject to the usual rules of physics.

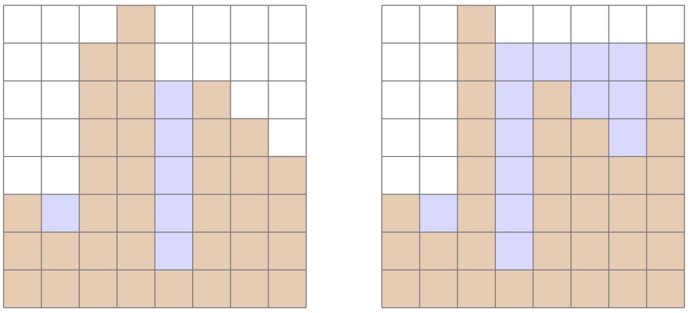

Figure 1.1: The water capacity of two permutations of 8 (brown colour). Each has two reservoirs (with capacity in blue colour).

We further define the capacity of any given part in the permutation as the amount of water retained above that part after the water has been allowed to escape left and right. This simple definition is the subject of some speculative discussion in Remark 3.2 found in Section 3. In the current paper, we consider the separate reservoirs in permutation rather than the general notion of the full capacity. Roughly speaking, a reservoir is a single connected body of water. Formally, a reservoir in a permutation is a contiguous subword $a_{s+1}, a_{s+2}, \cdots, a_{s+i}$ (where $i \geq 3$) of the original permutation (here understood as a word) where capacity(a_{s+1}) = capacity$(a_{s+i}) = 0$ and for $s+2 \leq r \leq s+i-1$, capacity$(a_r) > 0$. This notion of a reservoir was originally defined in an equivalent way in [10] and is illustrated in Figure 1.1. The notion is general enough to include reservoirs in word structures such as compositions, Dyck and Motzkin paths, permutations, set partitions (in the form of restricted growth functions) and Catalan words. Some of these cases are dealt with in further articles, see [5, 6].

The approach which we use in this paper is as follows: we let $R(n)$ be the total number of reservoirs across all permutations of n. We then derive a formula relating $R(n)$ to $R(n-1)$.

*Corresponding author (arnold.knopfmacher@wits.ac.za).

2. Formula Relating $R(n)$ to $R(n-1)$

Here is a construction we will use in this section. Given a particular permutation of $n-1$, we add one to each part of the permutation ("raising") and then insert a 1 ("inserting 1"). In terms of the bargraph representations, this is equivalent to raising the original bargraph by 1 and then constructing n unique bargraphs by inserting a single square between any pair of columns or at the beginning or end. This process of raising and then inserting ones creates the entire set of bargraphs of permutations of n from those of $n-1$ in a unique way. So we obtain $n!$ bargraphs from the original $(n-1)!$ that we start with. We also note that the raising step of the construction preserves all the reservoirs that were in the original permutations of $n-1$.

New Reservoirs

We first consider the case where the placement of the one is in position m which adds a new reservoir. This implies that $2 \le m \le n-1$ (i.e., m is not the first or last position). We also let the part in position $m-1$ be p and that in position $m+1$ be q. Without loss of generality we assume $p < q$. The case $p > q$ is handled by reversing the permutation, which turns it into the original case and simply doubling the answer obtained for $p < q$. The sub-permutation $p, 1, q$ in the permutations of n will be a new reservoir iff each part in positions 1 to $m-2$ is smaller than p. This implies that $p \ge m$. Those in positions $m+2$ to n are any sub-permutation of the remaining parts. Assembling all these restrictions we obtain the number of new reservoirs produced by the construction of permutations of n from those of $n-1$ as

$$2\sum_{m=2}^{n-1}\sum_{p=m}^{n-1}\sum_{q=p+1}^{n}(m-2)!\binom{p-2}{m-2}(n-(m+1))! \tag{1}$$

The One is Placed in a Position Which is Inside a Reservoir from a Permutation of $n-1$

The raising part of the construction preserves all the reservoirs that were in the original permutations of $n-1$. The insertion of the one in any of the n available positions produces n copies of the original reservoirs So the number of (copies of) existing reservoirs is

$$nR(n-1) \tag{2}$$

Now let $R(n)$ be the total number of reservoirs in permutations of n. Clearly $R(1) = 0$. Using (1) and (2), for $n \ge 2$, we have

$$R(n) = nR(n-1) + 2\sum_{m=2}^{n-1}\sum_{p=m}^{n-1}\sum_{q=p+1}^{n}(m-2)!\binom{p-2}{m-2}(n-(m+1))!. \tag{3}$$

The sequence obtained from Equation (3) begins

$$\{0, 0, 2, 16, 116, 888, 7416, 67968, 682272, 7467840\}. \tag{4}$$

Remark 2.1. *According to A052582 in OEIS [11],* $2(n-1)(n-1)!$ *is the total number of pairs* a_i, a_{i+1} *in all permutations of* n *such that* a_i, a_{i+1} *are consecutive increasing or decreasing integers. To prove this latter, let* a_i, a_{i+1} *be any pair of consecutive integers in any permutation of* n*. We have* $1 \le i \le n-1$*. The other parts in the permutation can be counted in* $(n-2)!$ *ways, and the number of consecutive pairs in* $2(n-1)^2$ *ways, giving a total number of such permutations as* $2(n-1)^2(n-2)! = 2(n-1)(n-1)!$ *.*

By using this in Equation (3), we obtain the following proposition

Proposition 2.1. *The total number of reservoirs* $R(n)$ *in permutations of* n *satisfies the recurrence given by*

$$R(n) = nR(n-1) + 2(n-2)(n-2)!. \tag{5}$$

Proof. We need to prove that

$$\sum_{m=2}^{n-1}\sum_{p=m}^{n-1}\sum_{q=p+1}^{n}(m-2)!\binom{p-2}{m-2}(n-(m+1))! = (n-2)(n-2)!$$

Here, we have

$$\sum_{m=2}^{n-1}\sum_{p=m}^{n-1}\sum_{q=p+1}^{n}(m-2)!\binom{p-2}{m-2}(n-(m+1))!$$
$$=\sum_{m=2}^{n-1}\sum_{p=m}^{n-1}(m-2)!(n-p)\binom{p-2}{m-2}(n-(m+1))!$$
$$=\sum_{p=2}^{n-1}\sum_{m=2}^{p}(m-2)!(n-p)\binom{p-2}{m-2}(n-(m+1))! = \sum_{p=2}^{n-1}(n-2)! = (n-2)(n-2)!. \tag{6}$$

□

Solving the recurrence in Proposition 2.1, we obtain the following result.

Theorem 2.1. *The number of reservoirs $R(n)$ in permutations of n is given by*

$$R(n) = 2(n-1)!(nH_{n-1} - 2n + 2) \tag{7}$$

where H_n is the nth harmonic number.

According to A288964 in [11], $R(n+1)$ is the number of key comparisons to sort all $n!$ permutations of n by quicksort. We show this as follows: The average number of key comparisons in quicksort (see for example [7], p.6) satisfies the recurrence,

$$nc(n) - (n+1)c(n-1) = 2(n-2). \tag{8}$$

Therefore, the total number of key comparisons $C(n)$ is given by

$$C(n) = n!c(n) = (n+1)C(n-1) + 2(n-1)(n-1)! \tag{9}$$

where $C(1) = 0$. This is the same recurrence satisfied by $R(n)$ in Proposition 2.1 where n is replaced by $n+1$. Therefore $R(n+1) = C(n)$.

Remark 2.2. *We pose the following open question to the interested researcher. Show a direct relationship between the number of reservoirs in permutations of $n+1$ and the total number of key comparisons to sort all permutations of n using quicksort.*

3. Width of Reservoirs

The construction we are using inserts a one in position m of a raised permutation of $n-1$ where $1 \le m \le n$. Other than the first position, $m = 1$, and last position, $m = n$, either these are inside an old reservoir or begin a new reservoir. Both these cases add 1 to the total width of the reservoirs. The total number of permutations of n in which m marks an inside position is given as follows:

$$n!\frac{n-2}{n} = (n-1)!(n-2). \tag{10}$$

When m is the first or last position, there are no new reservoirs. Let $W(n)$ be the total width of reservoirs in permutations of n. Clearly $W(1) = 0$. The above argument implies

$$W(n) = \underbrace{2W(n-1)}_{m\in\{1,n\}} + \underbrace{(n-2)W(n-1) + n!\frac{n-2}{n}}_{m\in\{2,3,\cdots,n-1\}} = nW(n-1) + n!\frac{n-2}{n}. \tag{11}$$

By solving recurrence (11), we obtain the following theorem.

Theorem 3.1. *The total width $W(n)$ of the reservoirs over all permutations of n is*

$$W(n) = n!(1 + n - 2H_n) \tag{12}$$

where H_n is the nth harmonic number.

The sequence obtained from Equation (12) begins

$$\{0, 0, 2, 20, 172, 1512, 14184, 143712, 1575648, 18659520\}. \tag{13}$$

According to the OEIS, [11], sequence A321853, $a(n)$ is the sum of the fill times of all 1-dimensional fountains given by the permutations of n. Here the 1-dimensional fountain given by a permutation of n is a $1 \times n$ grid of squares with a source on the left and a sink on the right, where the permutation gives the height of each square of the fountain. Starting from the source, the water fills up each square of the fountain at the rate of one unit volume per unit time. The water immediately flows to all adjacent regions of lower height. The water flows out immediately upon reaching the sink. The expected amount of time to fill a fountain given by a random permutation in n is $a(n)/n!$.

This sequence A321853 can be interpreted in terms of water capacity as follows. To the left of the bargraph of a permutation of n, insert a part of size $n+1$ and compute the water capacity of this modified permutation of $n+1$. The total water capacity over all such modified permutations of $n+1$ is $a(n)$. Furthermore $W(n+1) = 2a(n)$. This is proved simply by comparing (12) with the formula for $a(n)$ given in [11].

Remark 3.1. *Once again we pose the following open question. Show directly that the total width of the reservoirs in all permutations of* $n+1$ *is equal to twice the water capacity of the modified permutations of* n*. For example, when* $n = 3$*, the total width of all permutations of* $n+1 = 4$ *is 16, whereas the total water capacity of the modified permutations of 3 is 8.*

Remark 3.2. *This remark is speculative in nature and if you wish to skip it, feel free to do so because you will not miss out on any of the mathematical arguments. The sequence* $a(n)$ *is given the name time as opposed to capacity (a spatial concept) of a single column as per its definition in the introduction. There, it was defined as the amount of water retained in a single column after the water has been allowed to escape left and right from the reservoir in which the single column was located. We might have added: "under the usual rules of gravity" because water always flows downward except where it is forced to flow horizontally in the case where there is no space for downward movement. The latter occurs if the downward direction is already full of water underlain by a physical barrier. The amount of time to fill up a given fountain is the same as the capacity of its modified permutation as described immediately above Remark 3.1. Let us try to make explicit what is contrasted here. The time count explicitly adds together the time required to fill all single (horizontal) rows of contiguous or adjacent cells. It is explicitly a horizontal count of the capacity of any given reservoir as opposed to the vertical (column by column) count in any of the previous papers [1–4, 8, 9]. In this discussion, time is distinguished from space as a different way of accounting for the same quantity, especially those quantities which are fundamentally dependent on gravity (which is the case here) where gravity is understood as a phenomenon that distorts space. In this discussion gravity biases the flow of water downwards except where there is no downward opportunity for the flow.*

4. Reservoirs of Depth 1

Let us consider permutations of n. A reservoir in such a permutation is of depth 1 iff it has exactly one part of depth 1 (see introduction). For an illustration of depth 1 reservoirs, reading from left to right, see the first and third reservoirs in Figure 1.1. If the part which has depth one is 1 itself, we call this a new reservoir. Otherwise, we call it an old reservoir. Both the depth 1 reservoirs above are old reservoirs. We shall separately count the number of new and old reservoirs in permutations of n.

Number of New Reservoirs of Depth 1

From the definition of a new reservoir, the 1 must occur in an internal position m. Because the depth is one, we must have a 2 occurring in an adjacent position next to the 1. Without loss of generality we will assume that these occur in the order 21 and account for the case 12 using symmetry. If $n \geq 4$ and if the 1 is not in position two (i.e., $m \neq 2$), then the first 4 positions will be $p21q$ where both $p > 2$ and $q > 2$. This implies that $p21q$ is not a reservoir of depth 1 and therefore all such reservoirs begin 21 with 1 in the second position, or by symmetry end 12 with 1 in the second last position. For $n \geq 3$ the number b_n of new reservoirs in permutations of n is given by

$$b_n = 2(n-2)!. \tag{14}$$

(The case $n = 3$ above also satisfies this formula as can be verified by examining the only two such cases with reservoirs).

Number of Old Reservoirs of Depth 1

By definition, such an old reservoir must be in the ordered form $p, p-1, q$ where $p \geq 3$ and $q > p$. The reservoir in reverse order is again handled by symmetry. Again, in order for this to be a reservoir of depth 1, the i parts preceding the p must all be less than $p-1$. Counting the number of possibilities where these i parts $< p-1$ precede p, we obtain

$$\sum_{i=0}^{p-2} i! \binom{p-2}{i} (n-3-i)! = \frac{(n-2)!}{n-p}.$$

Therefore the number, c_n, of old reservoirs in permutations of n is given by

$$c_n = 2 \sum_{p=3}^{n-1} \sum_{q=p+1}^{n} \frac{(n-2)!}{n-p} = 2(n-3)(n-2)!, \tag{15}$$

where the coefficient 2 again accounts for the symmetric case where the reservoir is in reverse order. By adding together Equations (14) and (15), we have proved

Proposition 4.1. *The number of reservoirs of depth 1 in permutations of n is given by*

$$2(n-2)(n-2)!. \tag{16}$$

5. Reservoirs of Depth d

The depth of a reservoir is defined to be d iff the reservoir has exactly one part of depth d (see introduction) and all other parts are of depth $< d$. For example, from left to right the reservoirs in Figure 1.1 are of depth 1,5,1 and 6. Let us consider permutations of n and reservoirs of depth d and width w. Such a reservoir has the general form $p, p_1, \cdots, p_w, q$ where $q > p$, $p_r = p - d$ for some $r \in [w]$ and for $s \in [w] - \{r\}$ $p - d < p_s < p$. (As in the previous section, the case in reverse order will be handled using symmetry). The bounds for w,d and p are $1 \leq d \leq p-1$, $1 \leq w \leq d$ and $2 \leq p \leq n-1$. Can we decide how many reservoirs there are for fixed d and fixed w? For $i \in [w]$, we have that $p_1, p_2, \ldots, p_w$ is a permutation with parts chosen from the set $\{p - k : k \in [w]\}$ and as already stated, one of these is $p_r = p - d$. The number of such reservoirs is therefore given by

$$\binom{d-1}{w-1} w!. \tag{17}$$

Also as before, in order for $p, p_1, \cdots, p_w, q$ to be a reservoir of depth d, the i parts preceding the p must all be less than p. Counting the number of possibilities where these i parts $< p$ precede p, we obtain the total number of reservoirs of depth d, summed over all possible widths w, as

$$\begin{aligned} 2 \sum_{w=1}^{d} \binom{d-1}{w-1} w! \sum_{p=d+1}^{n-1} \sum_{q=p+1}^{n} \sum_{i=0}^{p-1-w} i! \binom{p-1-w}{i} (n-i-2-w)! &= 2 \sum_{w=1}^{d} \binom{d-1}{w-1} w! \sum_{p=d+1}^{n-1} (n-w-1)! \\ &= \frac{2(n-d-1)n!}{(n-d)(n-d+1)}, \end{aligned}$$

where the coefficient 2 is obtained from the reverse order symmetry. So, we have proved the following proposition.

Proposition 5.1. *The number of reservoirs of depth d in permutations of n is given by*

$$\frac{2(n-d-1)n!}{(n-d)(n-d+1)}, \tag{18}$$

where $1 \leq d \leq n-2$.

Example 5.1. *The permutations of 4 which have reservoirs of depth 2 are:*

$$\{3,1,2,4\}, \{3,2,1,4\}, \{4,1,2,3\}, \{4,2,1,3\},$$
$$\{2,3,1,4\}, \{3,1,4,2\}, \{2,4,1,3\}, \{4,1,3,2\},$$

with a count of 8 reservoirs, as predicted by (18).

References

[1] A. Blecher, C. Brennan, A. Knopfmacher, The water capacity of integer compositions, *Online J. Anal. Comb.* **13** (2018) #06.
[2] A. Blecher, C. Brennan, A. Knopfmacher, Capacity of words, *J. Comb. Math. Comb. Comput.* **107** (2018) 245–258.
[3] A. Blecher, C. Brennan, A. Knopfmacher, M. Shattuck, Capacity of permutations, *Ann. Math. Inform.* **50** (2019) 39–56.
[4] A. Blecher, C. Brennan, A. Knopfmacher, Water capacity of Dyck paths, *Adv. Appl. Math.* **112** (2020) #101945.
[5] A. Blecher, A. Knopfmacher, Reservoirs in words over the alphabet $[k]$, Submitted.
[6] A. Blecher, A. Knopfmacher, Reservoirs in compositions, Submitted.
[7] M. Kauers, P. Paule, *The Concrete Tetrahedron*, Springer, New York, 2011.
[8] T. Mansour, M. Shattuck, Counting water cells in bargraphs of compositions and set partitions, *Appl. Anal. Discrete Math.* **12** (2018) 413–438.
[9] T. Mansour, M. Shattuck, Counting water cells in pattern restricted compositions, *Turkish J. Anal. Number Theory* **7(4)** (2019) 98–112.
[10] T. Mansour, M. Shattuck, Enumeration of Catalan and smooth words according to capacity, *Integers* **25** (2025) #A5.
[11] The On-Line Encyclopedia of Integer Sequences, http://oeis.org/.