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Research Article

Short Proofs of Three Combinatorial Results in the Johnson Scheme

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Abstract

In this paper, we give short proofs of three theorems concerning extremal problems in the Johnson scheme, or, in other terminology, on (n, k, L)-systems. The main result is a proof of the Aljohani–Bamberg–Cameron conjecture which claims that if $n > n_0(k)$ and there are an (n, k, L)-system and an $(n, k, \{0, \dots, k-1\} \setminus L)$-system whose sizes have product $\binom{n}{k}$, then they are a t-intersecting family and a Steiner system $S(t, k, n)$ for some t.

Keywords: L-systems; Johnson scheme; Erdős–Sós problem; Steiner system.

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1. Introduction

For non-negative integers n, a, b, put $[n] = \{1, \dots, n\}$ and, more generally, $[a, b] = \{a, a+1, \dots, b\}$. Given a set X and an integer $k \geqslant 0$, denote by $\binom{X}{k}$ the collection of all k-element subsets (k-sets) of X. A *family* is simply a collection of sets. For $L \subset [0, k-1]$ we say that $\mathcal{F} \subset \binom{[n]}{k}$ is an (n, k, L)-*system* if $|F_1 \cap F_2| \in L$ for any distinct $F_1, F_2 \in \mathcal{F}$. The problem of determining the maximum size of an (n, k, L)-system is a classical problem in extremal set theory, introduced and studied by Deza, Erdős and Frankl.

A 2-star is the family of all k-sets containing a fixed 2-element set. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called t-intersecting if $|F_1 \cap F_2| \geq t$ for all distinct $F_1, F_2 \in \mathcal{F}$.

Let $G = (V, E)$ be a simple graph, denote by $\omega(G)$ its clique number, which is the size of a largest clique in G. By $\alpha(G)$ we denote the size of a largest coclique (independent set), equivalently, the size of a largest clique in the complement of G. A graph is called vertex-transitive if, given any two vertices v_1 and v_2 of G, there is an automorphism f such that $f(v_1) = v_2$. Denote by e_U the number of edges induced by a subset $U \subset V(G)$.

Sometimes it is convenient to represent problems on (n, k, L)-systems in terms of the generalized Johnson graph $J(n, k, L)$. The vertex set of this graph is $\binom{[n]}{k}$, and two vertices are joined by an edge if the size of the intersection of the corresponding sets belongs to L. In these terms, the Deza–Erdős–Frankl problem is to determine the clique number $\omega(J(n, k, L))$.

These graphs (for specific L) are also studied as constant-weight codes. From the point of view of algebraic combinatorics, all graphs $J(n, k, L)$, for fixed n and k, belong to the same Johnson association scheme and therefore have a common eigenbasis.

The graph $J(n, k, L)$ is vertex-transitive, and hence the clique-coclique bound applies to it (see, for instance, [9]):

$$\alpha(J(n, k, L)) \cdot \omega(J(n, k, L)) \leq |V(J(n, k, L))| = \binom{n}{k}. \tag{1}$$

In Section 2, we show that, when n is sufficiently large in terms of k, the clique-coclique bound (1) can also be derived from the celebrated theorem of Deza, Erdős and Frankl.

Aljohani, Bamberg and Cameron [2] studied when equality can occur in (1). In their terminology this means that the Johnson scheme $J(n, k)$ is non-separating. This question is also motivated by the theory of synchronizing automata and transformation semigroups: the conjecture determines, for n sufficiently large in terms of k, whether the natural action of S_n on the k-subsets of $[n]$ is separating.

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There are several natural examples.

Example 1.1 (Aljohani–Bamberg–Cameron, [2]). *Suppose that there exists a projective plane of order q, and put $n = q^2+q+1$ and $k = q+1$. The lines of the projective plane form a clique in $J(n,k,\{1\})$, since any two lines intersect in exactly one point. On the other hand, the family of all k-subsets containing a fixed pair of points is a coclique in this graph. The sizes of these two families multiply to $\binom{n}{k}$.*

A more general source of examples comes from designs. Recall that a Steiner system $S(t,k,n)$ is a family $\mathcal{D} \subset \binom{[n]}{k}$ such that every t-element subset of $[n]$ is contained in exactly one member of $\mathcal{D}$. A projective plane of order q, used in Example 1.1, is an example of a Steiner system $S(2,q+1,q^2+q+1)$. In particular, any two distinct sets in $\mathcal{D}$ have intersection of size at most $t-1$. Thus, if $L = \{0,1,\dots,t-1\}$, then the members of $\mathcal{D} = S(t,k,n)$ form a clique in $J(n,k,L)$. The family of all k-sets containing a fixed t-element set is a coclique in $J(n,k,L)$, and we have

$$|\mathcal{D}|\binom{n-t}{k-t} = \frac{\binom{n}{t}}{\binom{k}{t}}\binom{n-t}{k-t} = \binom{n}{k}.$$

The complementary case $L = \{t,t+1,\dots,k-1\}$ gives the same construction with the roles of the clique and the coclique interchanged. By the result of Keevash on the existence of designs [10], such Steiner systems exist for all sufficiently large n satisfying the necessary divisibility conditions, namely

$$\binom{k-i}{t-i} \,\Big|\, \binom{n-i}{t-i} \quad \text{for all } i = 0,1,\dots,t-1.$$

See also [8, 11, 12].

Aljohani, Bamberg and Cameron [2] formulated the following conjecture. We prove it in Section 2.

Theorem 1.1. *Let L be a non-empty proper subset of $\{0,\dots,k-1\}$. If $\alpha(J(n,k,L))\cdot\omega(J(n,k,L)) = \binom{n}{k}$ and n is sufficiently large in terms of k, then there exists t such that either L or $\{0,\dots,k-1\}\setminus L$ is equal to $\{0,1,\dots,t-1\}$.*

In [2], this conjecture was proved for $k \le 4$. We also note that this conjecture appears as Problem 30.6 in the problem list from the 30th British Combinatorial Conference [3], 2024.

Now let us make a more detailed analysis of Example 1.1. Projective planes of order q exist for a prime power q; a major conjecture is that this does not hold for other values of q. Example 1.1 and inequality (1) give

$$\alpha(J(q^2+q+1,q+1,\{1\})) \le \binom{q^2+q-1}{q-1}.$$

This inequality was proven for every q by Cherkashin [4] and then Linz [17] strengthened this result by proving that

$$\alpha(J(n,k,\{1\})) \le \binom{n-2}{k-2}$$

for $3k-3 \le n \le k^2-k+1$. Aljohani, Bamberg and Cameron also conjectured that every coclique of size $\binom{q^2+q-1}{q-1}$ in $J(q^2+q+1,q+1,\{1\})$ is a 2-star provided that $q > 2$; Cherkashin stated the same question for $J(k^2-k+1,k,\{1\})$ and arbitrary $k > 3$ (for $k = 3$ there is a simple counterexample). Here we note that the Delsarte linear programming method (applied by Linz) together with the celebrated Ahlswede–Khachatrian Complete Intersection Theorem give the classification of all maximum cocliques in a graph $J(n,k,\{1\})$ for $3k-3 \le n < k^2-k+1$.

Theorem 1.2. *Let I be a coclique in the graph $J(n,k,\{1\})$ of size $\binom{n-2}{k-2}$, $k \ge 3$. If $3k-3 < n < k^2-k+1$ then I is a 2-star. If $n = 3k-3$ then I is either a 2-star or the union of all k-sets having at least 3 elements among fixed 4 elements.*

Note that the most interesting case $n = k^2-k+1$ remains open. Finally, let us note that for the case of a large enough k these results are covered by very general results of Kupavskii and Zakharov [16] and Keller and Lifshitz [13]; these results also cover the uniqueness issue.

After one determines the size of a maximum coclique and classifies all maximum cocliques in G, it is natural to consider stability and supersaturation problems. The results of Keller and Lifshitz were extended to the stability setting by Ellis, Keller and Lifshitz [7]. The third result concerns the supersaturation problem, which is to determine the minimal number of edges in a vertex subset of a given size. Many general tight results on this problem in the graphs $J(n,k,\{t\})$ were obtained in [15].

On the other hand, a complete asymptotic solution of this question is very complicated even in the case $J(n,3,\{1\})$, see [6]. This problem has a threshold when the size of the vertex subset is cn^2 for a constant c; in other regimes the supersaturation problem for the graph sequence $J(n,3,\{1\})$ is asymptotically solved.

Theorem 1.3. *Let $c > 2/9$ be a constant and U be a vertex subset of size $\lfloor cn^2 \rfloor$ in $J(n, 3, \{1\})$. Then U contains at least*

$$\left(\frac{9c^2}{2} - c + o(1)\right) n^3$$

edges.

If $2c$ is an integer, then Theorem 9 from [6] shows that Theorem 1.3 is asymptotically tight. Also, the proof of Theorem 1.3 gives tight lower bound in the case $n^2 = o(|U|)$, solved in [18].

Theorem 1.3 can be rephrased as follows. Let H be a 3-uniform hypergraph with n vertices and cn^2 edges for some constant $c > 2/9$. Then H has at least $\left(\frac{9c^2}{2} - c + o(1)\right) n^3$ pairs of edges with unit intersection.

2. Proof of Theorem 1.1

First, we state the condition of the celebrated theorem of Deza, Erdős and Frankl, which gives an upper bound on the size of an (n, k, L)-system for $n > n_0(k)$. We then show how Theorem 1.1 follows from it.

We will only need two parts of the theorem, so for convenience we state only those. For the proof and further information on (n, k, L)-systems, see [14].

Theorem 2.1 (Deza–Erdős–Frankl, [5])**.** *Fix a positive integer $k > 0$ and a set*

$$L = \{\ell_1 < \ldots < \ell_r\} \subset [0, k-1].$$

Assume that $n \geqslant 2^k k^3$, and that $\mathcal{F}$ is an (n, k, L)-system.
(i) If $|\mathcal{F}| \geqslant k^2 2^{r-1} n^{r-1}$ then

$$(\ell_2 - \ell_1) \mid (\ell_3 - \ell_2) \mid \ldots \mid (\ell_r - \ell_{r-1}) \mid (k - \ell_r).$$

(ii) We have

$$|\mathcal{F}| \leqslant \prod_{i=1}^{r} \frac{n - \ell_i}{k - \ell_i}.$$

We now turn to our problem. Write $L = \{\ell_1 < \ldots < \ell_r\} \subset [0, k-1]$ and $S = [0, k-1] \setminus L = \{s_1 < \ldots < s_{k-r}\}$. Let $\mathcal{F} \subset \binom{[n]}{k}$ and $\mathcal{G} \subset \binom{[n]}{k}$ be families corresponding to a maximum clique and a maximum coclique in the graph $J(n, k, L)$, respectively. Clearly, $\mathcal{F}$ is an (n, k, L)-system, while $\mathcal{G}$ is an (n, k, S)-system. By the assumption of Theorem 1.1, we have $|\mathcal{F}| \cdot |\mathcal{G}| = \binom{n}{k}$.

Assume that $n \geqslant 2^k k^3$. By Theorem 2.1, we have

$$|\mathcal{F}| \leqslant \prod_{i=1}^{r} \frac{n - \ell_i}{k - \ell_i} \qquad \text{and} \qquad |\mathcal{G}| \leqslant \prod_{i=1}^{k-r} \frac{n - s_i}{k - s_i}.$$

For the equality $|\mathcal{F}| \cdot |\mathcal{G}| = \binom{n}{k}$ to hold, both upper bounds must be tight, since

$$\prod_{i=1}^{r} \frac{n - \ell_i}{k - \ell_i} \cdot \prod_{i=1}^{k-r} \frac{n - s_i}{k - s_i} = \prod_{j=0}^{k-1} \frac{n - j}{k - j} = \binom{n}{k}.$$

We may also assume that $|\mathcal{F}| \geqslant k^2 2^{r-1} n^{r-1}$ and $|\mathcal{G}| \geqslant k^2 2^{k-r-1} n^{k-r-1}$. Indeed, otherwise, using the upper bounds from part (ii) of Theorem 2.1, we would have $|\mathcal{F}| \cdot |\mathcal{G}| \leqslant c(k) n^{k-1}$ for some constant $c(k)$, which is impossible for sufficiently large n, since $|\mathcal{F}| \cdot |\mathcal{G}| = \binom{n}{k}$.

We know that $k-1$ belongs to either L or S. First assume that $k - 1 \in L$. Then $\ell_r = k - 1$. By part (i) of Theorem 2.1, we have

$$(\ell_2 - \ell_1) \mid (\ell_3 - \ell_2) \mid \ldots \mid (\ell_r - \ell_{r-1}) \mid (k - \ell_r).$$

Since $k - \ell_r = 1$, all the differences $\ell_2 - \ell_1, \ldots, \ell_r - \ell_{r-1}$ are equal to 1. Hence

$$L = \{k - r, k - r + 1, \ldots, k - 1\}.$$

If, on the other hand, $k-1 \in S$, then the same argument applied to the (n, k, S)-system $\mathcal{G}$ shows that $S = \{r, r+1, \ldots, k-1\}$. Consequently, $L = \{0, 1, \ldots, r-1\}$, again as required.

Remark 2.1. *Inspecting the proof, it is easy to see that the condition $n \geq k^{k+o(k)}$ is sufficient for our proof of Theorem 1.1. We make no attempt to optimize this bound.*

3. Proof of Theorem 1.2

Define the Frankl family $\mathcal{F}_{n,k,t,r}$ by the following set of vertices in $J(n, k, \{t\})$

$$\mathcal{F}_{n,k,t,r} := \left\{ S \in \binom{[n]}{k} : |S \cap [t+2r]| \geq t + r \right\}$$

for some $n > k > t \geq 1$ and $r \geq 0$.

Theorem 3.1 (Ahlswede–Khachatrian, [1])**.** *A maximum t-intersecting family is always a Frankl set (up to a permutation of $[n]$). Namely, $\mathcal{F}_{n,k,t,r}$ is the largest exactly in the range*

$$(k-t+1)\left(2 + \frac{t-1}{r+1}\right) \leq n \leq (k-t+1)\left(2 + \frac{t-1}{r}\right).$$

(By convention $\frac{t-1}{r} = +\infty$ for $r = 0$.)

Now let us return to the graph $J(n, k, \{1\})$. Linz in [17, Theorem 2.3] shows that the matrix $M = M(n, k)$ explicitly constructed by Wilson in [19] satisfies the following properties.

(i) M is a symmetric matrix of size $\binom{n}{k}$. There is a natural bijection π between the rows/columns and the vertices of $J(n, k, \{1\})$ in which M_{ij} depends only on $|\pi(i) \cap \pi(j)|$.†

(ii) For $3k - 3 \leq n$ the largest eigenvalue of M is $\binom{n-2}{k-2}$.

(iii) For $n \leq k^2 - k + 1$ the entries $M_{i,j}$ with $|\pi(i) \cap \pi(j)| \neq 1$ are at least 1.

The proof of item (iii) in [17] shows that $M_{i,j} = 1$ for $|\pi(i) \cap \pi(j)| \geq 2$ and $M_{i,j}$ with $|\pi(i) \cap \pi(j)| = 0$ is at least 1 if and only if n is at most $k^2 - k + 1$. The same proof actually gives that $M_{i,j}$ with $|\pi(i) \cap \pi(j)| = 0$ is at strictly greater than 1 if and only if n is strictly smaller than $k^2 - k + 1$.

Now let χ_I be a characteristic vector of a coclique I in the graph $J(n, k, \{1\})$ (according to the bijection π). Then in the range $3k - 3 \leq n \leq k^2 - k + 1$ we have

$$\binom{n-2}{k-2}|I| = \lambda_{max}(M) \cdot |I| \geq \langle M\chi_I, \chi_I \rangle \geq |I|^2,$$

which is the Linz argument for $|I| \leq \binom{n-2}{k-2}$.

Now suppose we have $3k - 3 \leq n < k^2 - k + 1$ and $|I| = \binom{n-2}{k-2}$. Then

$$\langle M\chi_I, \chi_I \rangle = |I|^2,$$

and the mentioned more precise version of item (iii) implies that a couple of sets from I never have an empty intersection. Thus I is a 2-intersecting family and Theorem 3.1 finishes the proof.

4. Proof of Theorem 1.3

Consider the adjacency matrix A of the graph $J(n, 3, \{1\})$. It has spectrum consisting of only four eigenvalues

$$\lambda_0 = \frac{3(n-3)(n-4)}{2}, \quad \lambda_1 = \frac{(n-4)(n-9)}{2}, \quad \lambda_2 = -2n + 11, \quad \lambda_3 = 3.$$

(The calculation is standard and described for instance in [4].) From now on, assume that $n \geq 7$, then the smallest eigenvalue is λ_2.

Let U be a vertex subset of $J(n, 3, \{1\})$, and let χ_U be the corresponding characteristic vector. Let E_i be the eigenspace corresponding to λ_i, and consider the decomposition $\chi_U = \sum_{i=0}^{3} \alpha_i u_i$, where $u_i \in E_i$ are unit vectors. Then

$$2e_U = \langle A\chi_U, \chi_U \rangle = \sum_{i=0}^{3} \alpha_i^2 \lambda_i \geq \alpha_0^2 \lambda_0 + \sum_{i=1}^{3} \alpha_i^2 \lambda_2.$$

†This means that M belongs to the Bose–Mesner algebra of the Johnson scheme.

Note that $\sum_{i=0}^{3} \alpha_i^2 = |U|$ which is the squared length of χ_U. Also, since the graph is regular and connected, the eigenspace for λ_0 is one-dimensional and contains the all-ones vector. Thus $\alpha_0 = |U|/\sqrt{\binom{n}{3}}$. Putting everything together

$$2e_U \geq \frac{3(n-3)(n-4)}{2} \frac{|U|^2}{\binom{n}{3}} - (2n-11)\left(|U| - \frac{|U|^2}{\binom{n}{3}}\right) = \frac{(3+o(1))n^2}{2} \frac{|U|^2}{\binom{n}{3}} - (2+o(1))n \cdot |U| = (9c^2 - 2c + o(1))n^3.$$

Remark 4.1. *The bound in Theorem 1.3 can be obtained by using other matrices than just the adjacency matrix. On the other hand, it should be a linear programming (Delsarte–Schrijver) bound, and since it is asymptotically tight for infinitely many values of c, the result should be asymptotically the same.*

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