

Research Article

A Complete Solution to the Conjecture on the Inverse Collatz–Sinogowitz Irregularity Problem

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Abstract

The Collatz–Sinogowitz irregularity index of a graph G is defined as $CS(G) = \lambda(G) - \bar{d}(G)$, where $\lambda(G)$ is the spectral radius and $\bar{d}(G)$ is the average degree of G . Alazemi, Anđelić and Dimitrov [*Open Math.* **21** (2023) #20220572] conjectured that, for any positive integer i larger than a given constant and integer $c \geq 2$, (i) there exists a connected c -cyclic graph of order n with $CS(G) = i$, provided that n divides $2(c-1)$; and (ii) there are infinitely many connected c -cyclic graphs whose Collatz–Sinogowitz irregularity indices lie in the interval $(i, i+1)$. They proved that the conjecture holds for $c = 1$. In this paper, we give a complete solution to the conjecture. We show that statement (i) does not hold and statement (ii) is true. Our method relies on a graph operation that preserves both the spectral radius and the average degree. Additionally, we improve a known upper bound for the Collatz–Sinogowitz irregularity index. Aouchiche et al. [*European J. Oper. Res.* **191** (2008) 661–676] established the upper bound $CS(G) \leq \frac{n}{4} - 1 + \frac{1}{n}$. We refine this result by proving that, for any connected graph G with n vertices, $CS(G) \leq \frac{n}{4} - 1$, with only finitely many exceptions.

Keywords: irregularity measure; Collatz–Sinogowitz irregularity index; inverse irregularity problem.

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1. Introduction

In this paper, we only consider simple, undirected, and connected graphs. Let $G = (V(G), E(G))$ be the graph with vertex set $V(G)$ and edge set $E(G)$. Let K_n , S_n and C_n denote the complete graph, star and the cycle of order n , respectively. The complement of G is denoted by G^c . The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ (or $d(v)$ for simplicity). A graph is called regular if all its vertices have the same degree; otherwise, it is irregular. It is hard to measure the irregularity of a given graph, or compare the irregularity of two graphs. Several approaches have been proposed to quantify how irregular a graph is.

Let $\bar{d}(G)$ be the average degree of G , defined as $\bar{d}(G) = \frac{\sum_{v \in V(G)} d(v)}{n} = \frac{2m}{n}$, where $n = |V(G)|$ and $m = |E(G)|$. In view of the definition of irregularity, taking the vertex degrees into consideration, Bell [11] suggested making the variance $Var(G)$ as a direct measure

$$Var(G) = \frac{1}{n} \sum_{v \in V(G)} d(v)^2 - \bar{d}(G)^2.$$

Applying the imbalance of an edge (see [6]), Albertson [5] defined the irregularity of G as the sum of imbalances of all edges:

$$irr(G) = \sum_{uv \in E(G)} |d(u) - d(v)|.$$

A modification to the Albertson irregularity index is the so-called σ -irregularity [3, 18], defined as

$$\sigma(G) = \sum_{uv \in E(G)} (d(u) - d(v))^2.$$

Recently, Dimitrov and Stevanović [15] introduced the total σ -irregularity as

$$\sigma_t(G) = \frac{1}{2} \sum_{(u,v) \in V^2(G)} (d(u) - d(v))^2,$$

where $V^2(G)$ is the Cartesian product of $V(G)$ with itself.

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In the last decades, these irregularity measures have attracted a lot of attention from the researchers, of which one may refer to [1, 2, 10, 17, 18].

The adjacency matrix of a graph G is defined as $A(G) = (a_{ij})$, where $a_{ij} = 1$ if the vertex i is adjacent to j and $a_{ij} = 0$ otherwise. The eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ of $A(G)$ are the roots of its characteristic polynomial $\phi_G(\lambda) = \det(\lambda I - A(G))$. The set of eigenvalues together with their multiplicities of $A(G)$ is the spectrum of G , denoted by $\text{spec}(G)$. In particular, $\lambda_1(G)$ is called the spectral radius of G , which we write as $\lambda(G)$ for short.

It is proved that $\lambda(G) \geq \bar{d}(G)$, the equality holds if and only if G is a connected regular graph [13]. Differing from the above measures, Collatz and Sinogowitz [13] in 1957 proposed an index of graph G to measure the irregularity with the spectral radius and average degree:

$$CS(G) = \lambda(G) - \bar{d}(G).$$

This index is commonly referred to as Collatz–Sinogowitz irregularity index. Clearly, $CS(G) = 0$ if G is regular, and $CS(G) > 0$ otherwise.

If a connected graph G has exactly n vertices and m edges, then $c(G) = m - n + 1$ is called the cyclomatic number of G . A graph with cyclomatic number c is said to be c -cyclic graph. Specifically, a graph G with $c(G) = 1$ (respectively, $c(G) = 2$) is said to be unicyclic (respectively, bicyclic).

Recently, Alazemi, Anđelić and Dimitrov [4] studied the inverse Collatz–Sinogowitz irregularity problem of unicyclic graphs. They obtained the following result.

Theorem 1.1 (see [4]). *For every non-negative integer i ,*

- (i) *there exists a unicyclic graph whose Collatz–Sinogowitz irregularity index equals i ;*
- (ii) *there are infinitely many unicyclic graphs whose Collatz–Sinogowitz irregularity indices lie in the interval $(i, i + 1)$.*

Let G and H be two graphs with $u \in V(G)$ and $v \in V(H)$. The coalescence of G and H , denoted $(G, u) * (H, v)$, is obtained by identifying u and v from the disjoint union of G and H .

Alazemi, Anđelić and Dimitrov [4] proved that if $p = i^2 + 3i$, then $F_{1,p}$ (see Figure 1.1) satisfies Theorem 1.1(i). Moreover, the graph $(F_{1,p}, u) * (P_n, v)$ satisfies Theorem 1.1(ii), where u is a vertex of degree $p + 2$ in $F_{1,p}$ and v is an endpoint of the path P_n .

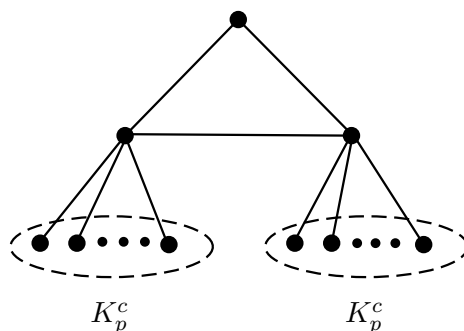


Figure 1.1: The graph $F_{1,p}$.

Note that $c(F_{1,p}) = 1$. They also proposed the following more general conjecture considering the cyclomatic number.

Conjecture 1.1 (see [4]). *For every positive integer i larger than a given constant and for every $c \geq 2$,*

- (i) *there is a connected c -cyclic graph of order n , whose Collatz–Sinogowitz irregularity index equals i , provided that n divides $2(c - 1)$;*
- (ii) *there are infinitely many connected c -cyclic graphs whose Collatz–Sinogowitz irregularity indices lie in the interval $(i, i + 1)$.*

In this paper, we give a complete solution to Conjecture 1.1.

The paper is organized as follows. In Section 2, we present some lemmas which are needed to show our results. In Section 3, Conjecture 1.1 is completely solved. In Section 4, we improve a known upper bound for the Collatz–Sinogowitz irregularity index.

2. Preliminaries

In this section, we introduce some notation and lemmas that will be used later.

Given a graph $G = (V(G), E(G))$ and a natural number k , a partition π of its vertex set $V(G) = X_1 \cup \dots \cup X_k$ into k parts is called equitable if there exists non-negative integers $b_{i,j}$ for $1 \leq i, j \leq k$, such that for any $i, j \in [k]$ and any vertex $u \in X_i$, the number of neighbors of u that are contained in X_j equals $b_{i,j}$. The $k \times k$ matrix $B = (b_{i,j})$ is called the quotient matrix of the partition π .

Lemma 2.1 (see [12]). *Let G be a connected graph and let π be an equitable partition of G with quotient matrix B . Then the spectral radius of G equals the spectral radius of B .*

Lemma 2.2 (see [14]). *Let H be a proper subgraph of a connected graph G . Then, $\lambda(H) < \lambda(G)$.*

Lemma 2.3 (see [13]). *Let G be a connected graph with the maximum degree Δ . Then, $\lambda(G) \leq \Delta$. Moreover, equality holds if and only if G is regular.*

Lemma 2.4 (see [19]). *Let T be a tree of order n , then $\lambda(T) \leq \lambda(S_n)$. Moreover, the equality holds if and only if $T = S_n$.*

Let $\mathcal{H}_n = \{S_n, S_{n-1}^1, K_n, K_{n-1}^1\}$, where S_{n-1}^1 (or K_{n-1}^1 , respectively) is the graph obtained by identifying a vertex of P_2 with a pendant vertex of star S_{n-1} (or a vertex of a complete graph K_{n-1} , respectively).

Lemma 2.5 (see [21]). *If G is a connected graph with $n \geq 10$ vertices and m edges, and $G \notin \mathcal{H}_n$, then $\lambda(G) < \sqrt{2m - n}$.*

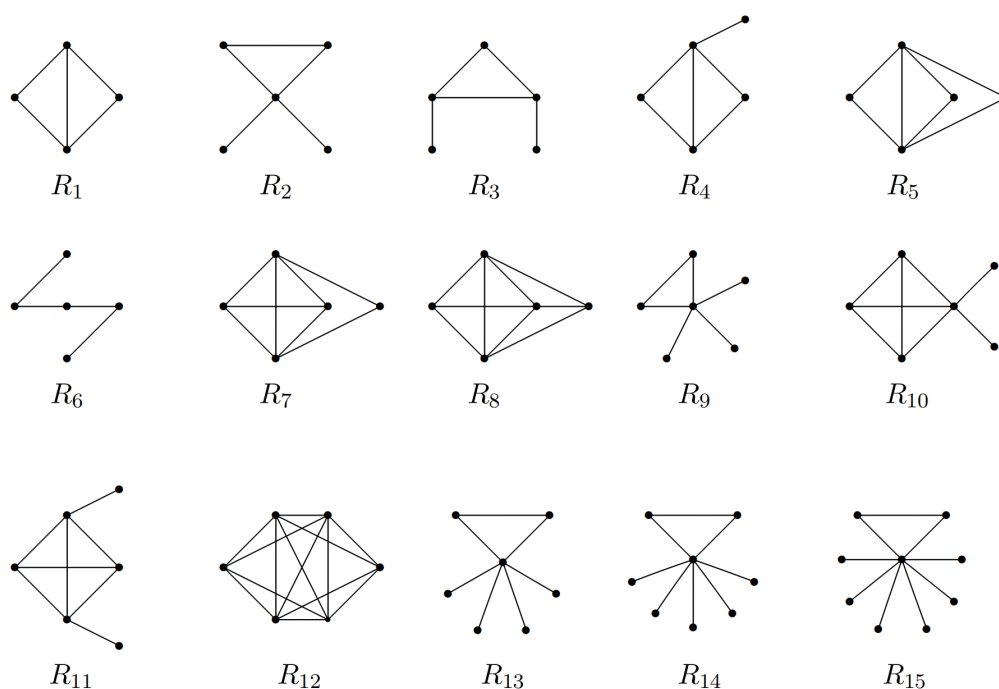


Figure 2.1: Small graphs with spectral radius at least $\sqrt{2m - n}$.

Lemma 2.6 (see [21]). *Let $n \leq 9$ and G be a graph with n vertices and m edges. If $\lambda(G) \geq \sqrt{2m - n}$, then $G \in \mathcal{H}_n$ or G is one of graphs R_1 – R_{15} shown in Figure 2.1.*

For later use, the approximate Collatz–Sinogowitz irregularity indices of $R_1, \dots, R_{15}$ are listed in Table 2.1.

Graph	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}	R_{14}	R_{15}
$CS(G)$	0.06	0.34	0.30	0.29	0.20	0.13	0.12	0.05	0.51	0.51	0.50	0.03	0.68	0.84	1

Table 2.1: Approximate Collatz–Sinogowitz irregularity indices of $R_1, \dots, R_{15}$.

Lemma 2.7 (see [4]). *The Collatz–Sinogowitz irregularity index of a graph is an integer if and only if both $\lambda(G)$ and $\bar{d}(G)$ are integers.*

Lemma 2.8 (see [22]). *Let $(G, u) * (H, v)$ be the coalescence of two connected graphs G and H , where $u \in G$ and $v \in H$. Then*

$$\lambda((G, u) * (H, v)) \leq \sqrt{\lambda(G)^2 + \lambda(H)^2},$$

with equality if and only if $(G, u) * (H, v)$ is a star.

Lemma 2.9. *Let G be a connected graph containing a non-cut edge uv . Suppose $G_1, G_2, \dots, G_k$ are k copies of G and the edges corresponding to uv are $u_1v_1, u_2v_2, \dots, u_kv_k$, respectively. The graph G' is obtained from $G_1, G_2, \dots, G_k$ by deleting edges $u_1v_1, u_2v_2, \dots, u_kv_k$ and adding edges $u_1v_2, u_2v_3, \dots, u_{k-1}v_k, u_kv_1$. Then G' is connected and $\text{spec}(G) \subset \text{spec}(G')$. Particularly, $\lambda(G) = \lambda(G')$ and $CS(G) = CS(G')$.*

Proof. Since each G_i is connected and the copies $G_1, G_2, \dots, G_k$ are connected cyclically by the added edges $u_1v_2, u_2v_3, \dots, u_{k-1}v_k, u_kv_1$, it follows that G' is connected. Let x be an eigenvector of G corresponding to an eigenvalue $\mu(G)$. Define

$$y = \overbrace{(x^T, x^T, \dots, x^T)^T}^k.$$

Then, for $i = 1, \dots, k$,

$$A(G')y_{u_i} = \sum_{u \in N(u_i) \setminus \{v_i\}} x_u + x_{v_{i+1}} = \sum_{u \in N(u_i)} x_u = A(G)x_{u_i} = \mu(G)x_{u_i} = \mu(G)y_{u_i}.$$

Similarly, we can prove that $A(G')y_{v_i} = \mu(G)y_{v_i}$, and the remaining vertices satisfy the eigen-equation as well. Therefore, $A(G')y = \mu(G)y$, which implies that $\mu(G)$ is also an eigenvalue of G' . When $\mu(G) = \lambda(G)$, according to the Perron-Frobenius theorem [16], x is a positive eigenvector, so y is also a positive eigenvector. Thus, $\lambda(G) = \lambda(G')$. Note that $\bar{d}(G) = \bar{d}(G')$. This gives $CS(G) = CS(G')$. $\square$

We note that a related construction of infinite families of graphs with the same spectral radius was recently presented by Andelić, da Fonseca and Stanić [8]. Furthermore, observe that in Lemma 2.9, if G is a c -cyclic graph and uv is not a cut edge, then G' is a $(kc - k + 1)$ -cyclic graph.

3. Proof of Conjecture 1.1

The following theorem shows that Conjecture 1.1(i) does not hold.

Theorem 3.1. *Let G be a c -cyclic ($c \geq 2$) graph with n vertices and m edges, and let $CS(G) = i$ be a positive integer. Then*

$$c > 1 + \frac{(i+3)^2}{4}.$$

Proof. Note that $m = n + c - 1$ and $\bar{d}(G) = 2 + 2(c-1)/n$. Without loss of generality, we consider the following two cases.

Case 1. $G \notin \mathcal{H}_n$.

For $n \geq 10$, by Lemma 2.5, we obtain

$$i = \lambda - \left(2 + \frac{2(c-1)}{n}\right) < \sqrt{n+2c-2} - 2 - \frac{2(c-1)}{n}. \quad (1)$$

After some computation, it can be shown that $\sqrt{n+2c-2} - 2 - 2(c-1)/n$ is monotonically increasing with respect to n . By Lemma 2.7, if i is an integer, then $n|2(c-1)$, and hence $n \leq 2(c-1)$. According to Eq. (1), $i < 2\sqrt{c-1} - 3$, which implies

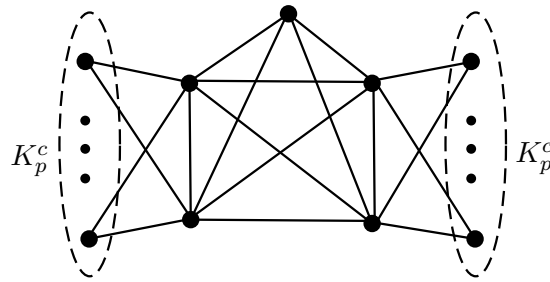
$$c > 1 + \frac{(i+3)^2}{4}.$$

For $n \leq 9$, if $\lambda(G) < \sqrt{2m-n}$, then the same argument as in the case $n \geq 10$ yields $c > 1 + \frac{(i+3)^2}{4}$. Otherwise, since $G \notin \mathcal{H}_n$, Lemma 2.6 implies that $G \in \{R_1, \dots, R_{15}\}$. However, the values listed in Table 2.1 show that none of these graphs with $c \geq 2$ has an integral Collatz–Sinogowitz irregularity index. This contradicts the assumption that $CS(G) = i$ is an integer.

Case 2. $G \in \mathcal{H}_n$.

Since $c \geq 2$, the graph G cannot belong to $\{S_n, S_{n-1}^1\}$. If $G = K_{n-1}^1$, then $\bar{d}(G) = n - 3 + \frac{4}{n}$. Since i is an integer, it follows from Lemma 2.7 that $n = 2$ or $n = 4$. In this case $c = 0$ or $c = 1$, which contradicts the assumption that $c \geq 2$. Therefore, $G \neq K_{n-1}^1$. If $G = K_n$, then $i = 0$, which is a contradiction. $\square$

Let $F_{2,p}$ be the graph obtained from K_5 by joining two disjoint 2-subsets of its vertices to two independent sets of order p , respectively (see Figure 3.1). For positive integers p, q and r , let $K_{p,q,r}$ denote the complete tripartite graph whose three partite sets have orders p, q and r , respectively.

Figure 3.1: The graph $F_{2,p}$.

Theorem 3.2. For every non-negative integer i ,

- (i) For $i \geq 1$, there exist graphs $F_{2,p}$ ($p = \frac{i(i+5)}{2}$) and $K_{q,q,r}$ ($q = 5i, r = 15i$) with Collatz–Sinogowitz irregularity indices equal i and $c(F_{2,p}) = i^2 + 5i + 6$, $c(K_{q,q,r}) = 175i^2 - 25i + 1$. For $i = 0$, the graph $F_{2,0}$ has Collatz–Sinogowitz irregularity index 0 and $c(F_{2,0}) = 6$. Furthermore, there are infinitely many non-unicyclic graphs whose Collatz–Sinogowitz irregularity indices equal i ;
- (ii) there are infinitely many non-unicyclic graphs whose Collatz–Sinogowitz irregularity indices lie in the interval $(i, i + 1)$.

Proof. (i) It is easy to see that $F_{2,p}$ has $2p + 5$ vertices and $4p + 10$ edges, and $c(F_{2,p}) = 2p + 6$.

If $p = 0$, then $CS(F_{2,p}) = 0$. Next, we assume $p \neq 0$. It enables the following partition $V(F_{2,p}) = V_1 \cup V_2 \cup V_3$, where $V_1 = \{v \in V(F_{2,p}) : d(v) = 2\}$, $V_2 = \{v \in V(F_{2,p}) : d(v) = p + 4\}$ and $V_3 = \{v \in V(F_{2,p}) : d(v) = 4\}$. The corresponding quotient matrix is

$$Q_1 = \begin{pmatrix} 0 & 2 & 0 \\ p & 3 & 1 \\ 0 & 4 & 0 \end{pmatrix} \quad (2)$$

Indeed,

$$\det(\lambda I - Q_1) = \det \begin{pmatrix} \lambda & -2 & 0 \\ -p & \lambda - 3 & -1 \\ 0 & -4 & \lambda \end{pmatrix} = \lambda^3 - 3\lambda^2 - (2p + 4)\lambda.$$

By Lemma 2.1, the spectral radius of $F_{2,p}$ is the largest root of $\phi_{Q_1}(\lambda) = \lambda^3 - 3\lambda^2 - (2p + 4)\lambda = 0$, i.e.,

$$\lambda(F_{2,p}) = \lambda(Q_1) = \frac{3 + \sqrt{8p + 25}}{2}.$$

Observe that $\lambda(F_{2,p})$ is an integer if and only if $\sqrt{8p + 25}$ is an odd integer. Let $p = \frac{i(i+5)}{2}$ for $i = 0, 1, 2, \dots$. Then $\sqrt{8p + 25} = 2i + 5$, i.e. $\lambda(F_{2,p}) = i + 4$. Since $\bar{d}(F_{2,p}) = 4$, we obtain

$$CS(F_{2,p}) = \lambda(F_{2,p}) - \bar{d}(F_{2,p}) = i.$$

Similarly, $K_{q,q,r}$ has $2q + r$ vertices and $q^2 + 2qr$ edges, and $c(K_{q,q,r}) = q^2 + 2qr - 2q - r + 1$.

If $r = q$, then $CS(K_{q,q,r}) = 0$. So, we assume $r \neq q$. Partition the vertices of $K_{q,q,r}$ into its three partite sets of sizes q, q , and r . The corresponding quotient matrix is

$$Q_2 = \begin{pmatrix} 0 & q & r \\ q & 0 & r \\ q & q & 0 \end{pmatrix}. \quad (3)$$

By Lemma 2.1, $\lambda(K_{q,q,r})$ is the largest root of $\phi_{Q_2}(\lambda) = \lambda^3 - (q^2 + 2qr)\lambda - 2q^2r = 0$, i.e.,

$$\lambda(K_{q,q,r}) = \lambda(Q_2) = \frac{q + \sqrt{q^2 + 8qr}}{2}.$$

It is obvious that

$$\bar{d}(G) = q + \frac{3qr}{2q + r}.$$

Set $r = 3q$. Then $\lambda(K_{q,q,r}) = 3q$ and $\bar{d}(G) = q + \frac{9q}{5}$. Note $CS(G)$ is integer if and only if $\frac{9q}{5}$ is integer. Taking $q = 5i$, we obtain $CS(G) = i$.

Choose a non-cut edge of $F_{2,p}$ or $K_{5i,5i,15i}$ and applying Lemma 2.9, we obtain infinitely many non-unicyclic graphs with Collatz–Sinogowitz irregularity index i .

(ii) Consider the graph $F_{2,p}$. Let $p \in \left(\frac{i(i+5)}{2}, \frac{(i+1)(i+6)}{2}\right)$. Then there are at least two integers in the interval $\left(\frac{i(i+5)}{2}, \frac{(i+1)(i+6)}{2}\right)$. Since $\lambda(F_{2,p}) = \frac{3+\sqrt{8p+25}}{2}$ and $\bar{d}(F_{2,p}) = 4$, we have

$$CS(F_{2,p}) = \frac{3+\sqrt{8p+25}}{2} - 4 \in (i, i+1).$$

Applying Lemma 2.9 to a non-cut edge of $F_{2,p}$, we get infinitely many non-unicyclic graphs whose Collatz–Sinogowitz irregularity indices lie in the interval $(i, i+1)$. $\square$

We first note that the graphs constructed in Theorem 3.2 have symmetric structures. Hence, the method developed in [7] can also be applied to determine their full spectra. Moreover, although both $c(F_{2,p})$ and $c(K_{q,q,r})$ are quadratic functions of i , the two families are structurally different and thus provide distinct realizations of the same prescribed Collatz–Sinogowitz irregularity index.

The Cartesian product of graphs G and H is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1 u_2 \in E(G)$ and $v_1 = v_2$ or $v_1 v_2 \in E(H)$ and $u_1 = u_2$. For $n \geq 3$, the Cartesian product $C_n \square K_2$ is called the n -prism.

Theorem 3.2 guarantees the existence of graphs with Collatz–Sinogowitz irregularity index i . By Theorem 3.1, for any positive integer i , a c -cyclic graph with $CS(G) = i$ must satisfy $c > 1 + \frac{(i+3)^2}{4}$. This leads to the following refinement of Conjecture 1.1(i): for any non-negative integer i and any integer $c > 1 + \frac{(i+3)^2}{4}$, does there exist a connected c -cyclic graph with index i ? The following theorem answers this question for the case $i = 0$.

Theorem 3.3. *For any $c \geq 3$, there exists a connected regular c -cyclic graph.*

Proof. For $c \geq 4$, the graph $C_{c-1} \square K_2$ has $2(c-1)$ vertices and $3(c-1)$ edges, and hence its cyclomatic number is c . Since $C_{c-1} \square K_2$ is 3-regular, we have $CS(C_{c-1} \square K_2) = 0$. For $c = 3$, the graph K_4 is a connected 3-cyclic regular graph. $\square$

If G and H are graphs with $V(G) = \{v_1, v_2, \dots, v_n\}$, then denoted by $G \circ H$, the *corona* of G and H is constructed from G by inserting n copies of H corresponding to each v_i ($i = 1, \dots, n$) and then adjoining v_i to all the vertices of i th copy of H .

Now we prove the following theorem, confirming Conjecture 1.1(ii). Write the unicyclic graph $C_g \circ K_p^c$ as $C_{g,p}$ for simplification.

Theorem 3.4. *For every non-negative integer i and a fixed integer $c \geq 2$, there exist infinitely many c -cyclic graphs whose Collatz–Sinogowitz irregularity indices lie in the interval $(i, i+1)$.*

Proof. We first consider the case $c = 2$, that is, we construct infinitely many bicyclic graphs G with $CS(G) \in (i, i+1)$.

Let H_1 be the graph shown in Figure 3.2, obtained by identifying a vertex u of $C_{g,p}$ with a vertex v of the cycle $C_{n-g(p+1)+1}$. For each integer $i \geq 1$, set $p = (i+1)^2$. On the one hand, using the method of quotient matrix and Lemma 2.1, we get $\lambda(C_{g,p}) = 1 + \sqrt{(i+1)^2 + 1}$ and $\lambda(C_{n-g(p+1)+1}) = 2$. By Lemmas 2.2 and 2.8, we have

$$\lambda(H_1) < \sqrt{\lambda^2(C_{g,p}) + \lambda^2(C_{n-g(p+1)+1})} = \sqrt{(1 + \sqrt{(i+1)^2 + 1})^2 + 4} < i + 3.$$

On the other hand, let $n \geq \max\{g(p+1) + 2, 4(i+2)\}$, then by Lemma 2.2,

$$\lambda(H_1) > \lambda(C_{g,p}) = \sqrt{i^2 + 2i + 2} + 1 > i + 2 + \frac{2}{n}.$$

Since H_1 is bicyclic, $\bar{d}(H_1) = \frac{2(n+1)}{n} = 2 + \frac{2}{n}$. Therefore,

$$CS(H_1) = \lambda(H_1) - \bar{d}(H_1) > i + 2 + \frac{2}{n} - \left(2 + \frac{2}{n}\right) = i.$$

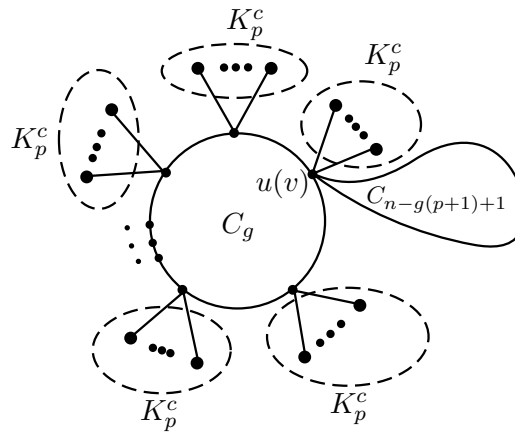
On the other hand,

$$CS(H_1) < i + 3 - \left(2 + \frac{2}{n}\right) = i + 1 - \frac{2}{n} < i + 1.$$

Hence, $CS(H_1) \in (i, i+1)$. Since n can be chosen arbitrarily large, this construction yields infinitely many bicyclic graphs satisfying the condition.

Next, we prove that there are infinitely many bicyclic graphs whose Collatz–Sinogowitz irregularity indices lie in the interval $(0, 1)$. We consider $H_2 = (C_g, u) * (C_{n-g+1}, v)$, where $n-g \geq 2$ and u (respectively, v) is some vertex of C_g (respectively, C_{n-g+1}). Then by Lemmas 2.2 and 2.8,

$$2 < \lambda((C_g, u) * (C_{n-g+1}, v)) \leq \sqrt{\lambda^2(C_g) + \lambda^2(C_{n-g+1})} = \sqrt{4 + 4} < 3.$$

**Figure 3.2:** The bicyclic graph H_1 .

Since H_2 is irregular, we have $\lambda(H_2) - \bar{d}(H_2) > 0$. Moreover, $\bar{d}(H_2) = 2 + \frac{2}{n} > 2$. Thus, $CS(H_2) \in (0, 1)$, and we get the desired result for the arbitrariness of order n .

We now consider the general case of $c \geq 3$. Set

$$G = \begin{cases} H_1, & i \geq 1; \\ H_2, & i = 0. \end{cases}$$

Let uv be an edge of G on a cycle, and $G_1, G_2, \dots, G_{c-1}$ be $c - 1$ copies of G and the edges corresponding to uv are $u_1v_1, u_2v_2, \dots, u_{c-1}v_{c-1}$, respectively. The graph G' is obtained from $G_1, G_2, \dots, G_{c-1}$ by deleting edges $u_1v_1, \dots, u_{c-1}v_{c-1}$ and adding edges $u_1v_2, \dots, u_{c-2}v_{c-1}, u_{c-1}v_1$. Then G' is a connected c -cyclic graph. By Lemma 2.9, $CS(G') = CS(G) \in (i, i + 1)$, where i is a non-negative integer. $\square$

4. An Upper Bound for the Collatz–Sinogowitz Index

A pineapple graph, denoted by $PA(n, p)$, is a graph on n vertices consisting of a complete graph K_p on p vertices and an independent set on $n - p$ vertices, where every vertex in the independent set is adjacent to exactly one common vertex of K_p .

In 2008, Aouchiche et al. [9] proposed the following conjecture.

Conjecture 4.1 (see [9]). *The connected graph on n ($n \geq 10$) vertices that maximizes the Collatz–Sinogowitz index is a pineapple graph $PA(n, \lceil \frac{n}{2} \rceil + 1)$.*

For sufficiently large n , the above conjecture was proved by Tait and Tobin [20]. For any positive integer n , Aouchiche et al. [9] established an upper bound for the Collatz–Sinogowitz irregularity index:

$$CS(G) \leq \frac{n}{4} - 1 + \frac{1}{n}.$$

Next we improve this upper bound.

Theorem 4.1. *If $G \notin \{R_1, R_2, R_3, R_4, R_9, R_{10}, S_4, S_5, S_6, S_3^1, K_3^1, K_4^1\}$ is a connected graph with n ($n \geq 4$) vertices, then*

$$CS(G) \leq \frac{n}{4} - 1.$$

The equality holds if and only if $G = K_4$.

Proof. Recall that $m = |E(G)|$. Without loss of generality, we consider the following two cases.

Case 1. $G \notin \mathcal{H}_n$.

For $n \geq 10$, by Lemma 2.5 and the AM-GM inequality, we have

$$CS(G) < \sqrt{2m - n} - \frac{2m}{n} = 2\sqrt{\frac{2m - n}{4}} - \frac{2m}{n} = 2\sqrt{\frac{n}{4} \cdot \frac{2m - n}{n}} - \frac{2m}{n} \leq \frac{n}{4} + \frac{2m - n}{n} - \frac{2m}{n} = \frac{n}{4} - 1.$$

For $4 \leq n \leq 9$, we consider two subcases.

Subcase 1.1. $\lambda < \sqrt{2m - n}$.

In this case, by similar reasoning as in the case $n \geq 10$, we have $CS(G) < \frac{n}{4} - 1$.

Subcase 1.2. $\lambda \geq \sqrt{2m - n}$.

By Lemma 2.6 and the values listed in Table 2.1, we have $CS(G) > \frac{n}{4} - 1$ for $G \in \{R_1, R_2, R_3, R_4, R_9, R_{10}\}$, whereas $CS(G) < \frac{n}{4} - 1$ for the remaining exceptional graphs. Since $G \notin \{R_1, R_2, R_3, R_4, R_9, R_{10}\}$, we conclude that $CS(G) < \frac{n}{4} - 1$.

Case 2. $G \in \mathcal{H}_n$.

If $G = S_n$, then

$$CS(S_n) = \sqrt{n-1} - \frac{2(n-1)}{n}.$$

Since $CS(S_n) = \sqrt{n-1} - 2 + \frac{2}{n}$, a comparison with $\frac{n}{4} - 1$ shows that $CS(S_n) < \frac{n}{4} - 1$ for $n \geq 7$, whereas $CS(S_n) > \frac{n}{4} - 1$ for $4 \leq n \leq 6$.

If $G = S_{n-1}^1$, then by Lemma 2.4,

$$CS(S_{n-1}^1) \leq \sqrt{n-1} - \frac{2(n-1)}{n}.$$

This upper bound coincides with that of $CS(S_n)$. Thus, $CS(S_{n-1}^1) < \frac{n}{4} - 1$ for $n \geq 7$. A direct computation shows that $CS(S_{n-1}^1) < \frac{n}{4} - 1$ for $n = 5, 6$, while $CS(S_{n-1}^1) > \frac{n}{4} - 1$ for $n = 4$.

If $G = K_{n-1}^1$, then by Lemma 2.3, we have

$$CS(K_{n-1}^1) \leq n-1 - (n-3 + \frac{4}{n}) = 2 - \frac{4}{n}.$$

For $n \geq 11$, $2 - \frac{4}{n} < \frac{n}{4} - 1$. For $4 \leq n \leq 10$, numerical evaluation gives $CS(K_{n-1}^1) < \frac{n}{4} - 1$ for $6 \leq n \leq 10$, whereas $CS(K_{n-1}^1) > \frac{n}{4} - 1$ for $4 \leq n \leq 5$.

If $G = K_n$, then $CS(G) = 0$. Thus, $CS(G) < \frac{n}{4} - 1$ for $n \geq 5$, $CS(G) = 0 = \frac{n}{4} - 1$ for $n = 4$. Hence, equality holds if and only if $G = K_4$. $\square$

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