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Research Article

Irregularity and Near Star Graphs

Allan Bickle*

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

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Abstract

The (Albertson) irregularity of a graph G is $irr(G) = \sum |d(u) - d(v)|$, where the sum is over all edges of G. In 2005, Hansen and Mélot proved an upper bound on the irregularity of a graph on n vertices and m edges that is attained for all feasible values, and characterized the extremal graphs. We give a much shorter proof of this result, and discuss some of its implications. Denoting the extremal graphs as near k-star graphs, we show that they sometimes occur as extremal graphs for related problems on sigma irregularity and first Zagreb index.

Keywords: irregularity; sigma irregularity; first Zagreb index; k-star.

2020 Mathematics Subject Classification: 05C07, 05C09, 05C75.

1. Introduction

A graph is **regular** if all of its vertices have the same degree. Otherwise, it is **irregular**. The term "irregularity" of a graph is used for several measures of how far from regular it is. Among various measures of graph irregularity, the earliest one was proposed by Albertson [2].

Definition 1.1. *The **imbalance** of an edge uv is $imb_G(uv) = |d_G(u) - d_G(v)|$. The **(Albertson) irregularity** of a graph G is*

$$irr(G) = \sum_{uv \in E(G)} |d_G(u) - d_G(v)| = \sum_{uv \in E(G)} imb_G(uv)$$

*The **sigma irregularity** of G is*

$$\sigma(G) = \sum_{uv \in E(G)} (d_G(u) - d_G(v))^2 = \sum_{uv \in E(G)} imb_G^2(uv)$$

There are many bounds on irregularity using various graph parameters. The problem of minimizing irregularity for fixed order n and size m is considered in [7]. Bounds on irregularity are surveyed in [11]. Hansen and Mélot [12] found a sharp upper bound on irregularity for fixed order n and size m. However, the proof of their result is about 4 pages, so a shorter proof is desirable.

They also characterized the extremal graphs as "fanned split graphs". However, we will use a different term for these graphs.

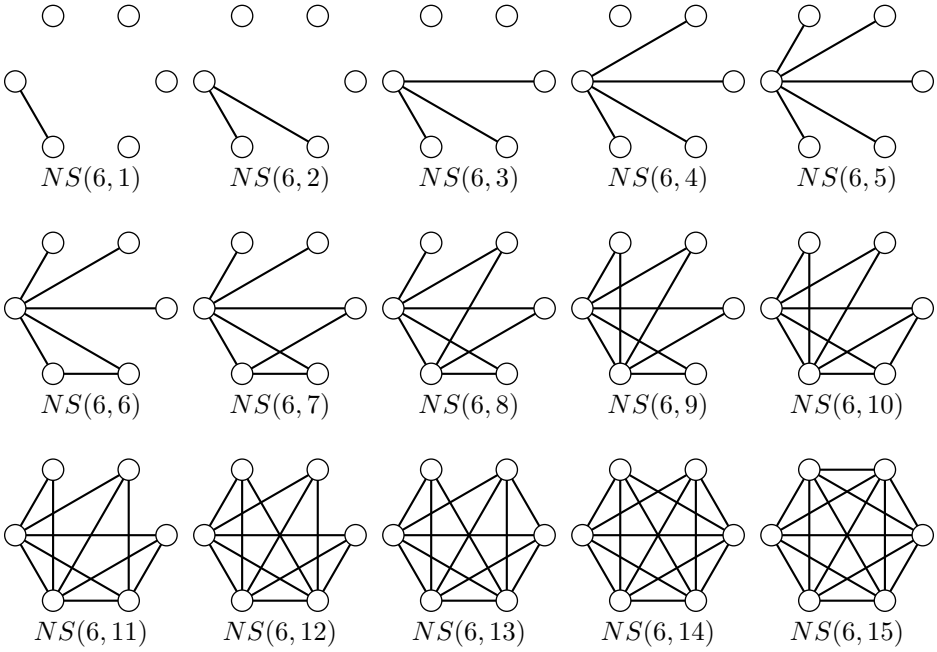
Definition 1.2. *Given integers n and m, $0 \le m < \binom{n}{2}$, let k be the unique integer so that*

$$kn - \binom{k+1}{2} \le m < (k+1)n - \binom{k+2}{2}.$$

*The **near** k**-star** $NS(n,m)$ is the unique graph so that $K_k + \overline{K}_{n-k} \subseteq NS(n,m) \subseteq K_{k+1} + \overline{K}_{n-k-1}$. Also, let $NS\left(n, \binom{n}{2}\right) = K_n$.*

*E-mail address: aebickle@purdue.edu

Let $c = m - \left(kn - \binom{k+1}{2}\right)$. Equivalently, $NS(n,m)$ can be formed by adding a star $K_{1,c}$ on the degree k vertices of the k**-star** $K_k + \overline{K}_{n-k}$. When $m = kn - \binom{k+1}{2}$ (for some k), $NS(n,m) = K_k + \overline{K}_{n-k}$. Some near star graphs for $n = 6$ are shown below.



A **dominating vertex** of a graph is a vertex adjacent to all other vertices. A **threshold graph** is a graph that can be formed from K_1 by the operations of adding an isolated vertex and adding a dominating vertex. It is not hard to see that any near k-star is a threshold graph. This implies that it is also a split graph [12] and is unigraphic (it is the only graph with its degree sequence).

Undefined notation and terminology will generally follow [4]. In particular, $n(G)$ and $m(G)$ are the number of vertices and edges of G, respectively. We will use n for the number of vertices of a graph when the context is clear. The **neighborhood** of a vertex v in G is denoted by $N_G(v)$ and its cardinality is the vertex **degree** $d_G(v)$. The join of graphs G and H is denoted $G + H$. Let K_n, $\overline{K}_n$, and $K_{1,n-1}$ represent a complete graph, empty graph, and star with n vertices.

2. Maximum Irregularity

In this section, we give a short proof of Hansen and Melot's theorem. We begin with a lemma that will be the base case for a proof by induction.

Lemma 2.1. *Let G be a graph with n vertices and m edges. Then $irr(G) \leq m(m-1)$. Equality holds if and only if $G = K_{1,m} \cup (n-m-1)K_1$.*

Proof. For each edge, the maximum possible imbalance is $\Delta(G) - 1 \leq m - 1$. Thus $irr(G) \leq m(m-1)$. Equality requires that each edge join a leaf to a degree m vertex. That is, G has a single nontrivial component, which is a star. □

Lemma 2.2. *Let G maximize irregularity among all graphs with fixed order n and size m. Let u have maximum degree in G. Then u is adjacent to all non-isolated vertices of G.*

Proof. Suppose to the contrary that there are vertices v and w so that G contains vw and not uw. Form graph G' by deleting vw and adding uw. Then only the degrees of u and v change. (We refer to the degrees in G in the following paragraph.)

The irregularity will increase by $d(u)$ due to edges incident with u. It will increase by $d(u) + 1 - d(w) - |d(v) - d(w)|$ due to deleting vw and add uw. It will decrease by at most $d(v) - 1$ due to edges incident with v. Thus the irregularity increases by at least $d(u) + d(u) + 1 - d(w) - |d(v) - d(w)| - (d(v) - 1) = 2d(u) - 2\max\{d(v), d(w)\} + 2 > 0$.

Thus u is adjacent to all non-isolated vertices of G. □

Lemma 2.3. *Let G be a graph containing vertices u, v, w with $d(u) \geq d(v) \geq 2$, $d(w) = 0$, u adjacent to all non-isolated vertices, and v has smallest degree among vertices with degree at least 2. Then deleting uv and adding uw will increase the irregularity.*

Proof. Form graph G' by deleting uv and adding uw. Then only the degrees of v and w change. Since u is adjacent to any leaves, v is not, so v has no neighbor with smaller degree. Thus the irregularity will increase by $d_G(v)-1$ due to edges incident with v. It will increase by $d_G(u)-1-(d_G(u)-d_G(v))=d_G(v)-1$ due to deleting uv and adding uw. Thus the irregularity increases by at least $d_G(v)-1+d_G(v)-1=2d_G(v)-2>0$. □

Theorem 2.1. [12] *Let G maximize irregularity among all graphs with fixed order n and size m. Then G is a near k-star.*

Proof. We use induction on m. Lemma 2.1 proves the result when $0\le m\le n-1$. Assume the result holds for graphs with fewer than m edges, and let G be a graph with n vertices and $m\ge n$ edges that maximizes irregularity. If there were an isolated vertex, we could use Lemma 2.3 to eliminate it. If the maximum degree vertex were not dominating, we could use Lemma 2.2 to increase its degree. Thus G has a dominating vertex u.

Let $H=G-u$. Now for $v\in H$, $d_H(v)=d_G(v)-1$. Thus the imbalance of each edge in H is the same as the corresponding edge in G. Thus $irr(G)=irr(H)+\sum_{v\in H}(n-1-(d_H(v)+1))=irr(H)+(n-1)(n-2)-2m(H)$. Now $(n-1)(n-2)-2m(H)$ is independent of the structure of H. Thus $irr(G)$ is maximized exactly when $irr(H)$ is maximized. Thus H is a near k-star. Thus G is also. □

A somewhat similar proof of Theorem 2.1 occurs in [13]. It is not hard to find the irregularity of near k-star graphs.

Corollary 2.1. [12] *Let k be the unique integer so that $kn-\binom{k+1}{2}\le m<(k+1)n-\binom{k+2}{2}$, and let $c=m-\left(kn-\binom{k+1}{2}\right)$. Then* for any c with $0\le c\le n-k-1$,

$$irr\left(NS\left(n,kn-\binom{k+1}{2}+c\right)\right)=k(n-k)(n-k-1)+c(c-2k-1).$$

Proof. Note that $irr\left(K_k+\overline{K}_{n-k}\right)=k(n-k)(n-1-k)$, since it has $k(n-k)$ edges with imbalance $n-1-k$, and all others have imbalance 0. If we add a star with c additional edges, we produce one vertex with degree $k+c$, and c vertices with degree $k+1$. The former reduces the imbalance of k edges by c, and the latter reduces the imbalance of ck edges by 1. We adjust the irregularity of $K_k+\overline{K}_{n-k}$ by $c(c-1)-ck-ck=c(c-2k-1)$. This gives the formula. □

Small values of $irr(NS(n,m))$ are given in the table below.

n\m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	0	0																			
3	0	0	2	0																	
4	0	0	2	6	4	4	0														
5	0	0	2	6	12	10	10	12	8	6	0										
6	0	0	2	6	12	20	18	18	20	24	20	18	18	12	8	0					
7	0	0	2	6	12	20	30	28	28	30	34	40	36	34	34	36	30	26	24	16	10
8	0	0	2	6	12	20	30	42	40	40	42	46	52	60	56	54	54	56	60	54	50

Consider all graphs with fixed order n. The maximum irregularity is about $k(n-k)(n-1-k)$, and the maximum of this expression occurs at

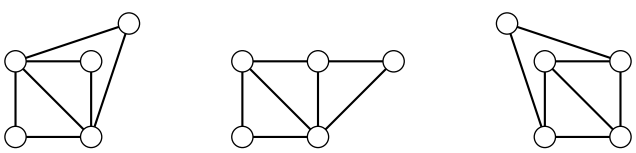
$$k=\frac{-1+2n-\sqrt{n^2-n+1}}{3}\approx\frac{n-1}{3}.$$

Plugging this in for k, we find the maximum possible irregularity is approximately $\frac{4}{27}n^3$. This was also observed in [2, 13].

Any tree on n vertices has $n-1$ edges. Theorem 2.1 implies that the maximum irregularity over all trees with n vertices occurs for the star $K_{1,n-1}$. We can generalize this observation.

Definition 2.1. *Let k be a positive integer. A graph is k-**degenerate** if its vertices can be successively deleted so that when deleted, they have degree at most k in the remaining graph. A graph is **maximal** k-**degenerate** if no edges can be added without violating this condition.*

See [5] for a survey of maximal k-degenerate graphs, including many results on degree sequences and degree sets. The size of a maximal k-degenerate graph with order $n\ge k$ is $k\cdot n-\binom{k+1}{2}$. The three maximal 2-degenerate graphs of order 5 are shown below [4].



By Theorem 2.1, a near k-star maximizes the irregularity for graphs of order n and size $m = kn - \frac{k(k+1)}{2}$. Since $K_k + \overline{K}_{n-k}$ is the only near k-star with this size, no other maximal k-degenerate graph can have the same or larger irregularity. This proves the following.

Corollary 2.2. [6] *Let G be a maximal k-degenerate graph with order $n \geq k$. Then $irr(G) \leq k(n-k)(n-1-k)$, and equality holds exactly for the k-stars $K_k + \overline{K}_{n-k}$.*

There is also a fairly short self-contained proof of this result in [6].

3. Maximum Sigma Irregularity

A similar approach might be used to find the graphs that maximize sigma irregularity for fixed order n and size m. The following lemma is proved by Ali et al. [3], where the proof is about 1 page. We present a shorter proof.

Lemma 3.1. [3] *Let G be a graph with n vertices and m edges. Then $\sigma(G) \leq m(m-1)^2$. Equality holds if and only if $G = K_{1,m} \cup (n-m-1)K_1$.*

Proof. For each edge, the maximum possible imbalance is $\Delta(G) - 1 \leq m - 1$. Thus $\sigma(G) \leq m(m-1)^2$. Equality requires that each edge join a leaf to a degree m vertex. That is, G has a single nontrivial component, which is a star. □

Thus sigma irregularity is maximized by near 0-star graphs when $0 \leq m < n-1$. Ali et al. [3] maximized sigma irregularity and found extremal graphs for graphs of order n and size m, where $n-1 \leq m \leq 2n-3$. The extremal graphs are near k-star graphs for $1 \leq k \leq 2$. It is not clear whether near k-star graphs are the extremal graphs for all graphs of order n and size m.

Note that graphs with maximum sigma irregularity for fixed order n are shown in [1] to be a particular class of k-stars ($K_s + \overline{K}_{n-s}$, where $s \approx \frac{n}{4}$).

4. Maximum First Zagreb Index

Some analogous results hold for the first Zagreb index.

Definition 4.1. *The **first Zagreb index** is* $M_1(G) = \sum_{v \in V(G)} (d(v))^2 = \sum_{uv \in E(G)} (d(u) + d(v))$.

The first Zagreb index is another tool for measuring how close to regular a graph is. In particular, for given order n and size m, $M_1(G) \geq \frac{4m^2}{n}$, with equality if and only if G is a regular graph. When there is no regular graph with this n and m, the graphs with the smallest first Zagreb index will have two consecutive degrees.

There are many bounds on the first Zagreb index, which are surveyed in [8, 10]. We have an upper bound on first Zagreb index that is analogous to previous lemmas on irregularity.

Theorem 4.1. *Let G be a graph with n vertices and m edges. Then $M_1(G) \leq m(m+1)$. Equality holds if and only if $G = K_{1,m} \cup (n-m-1)K_1$, or $G = K_3 \cup \overline{K}_{n-3}$.*

Proof. We use induction on the number of edges. The result is obvious for empty graphs. Assume the result holds for graphs with m edges, and let G have $m+1 > 0$ edges. Let $e = uv$ be an edge of G. Then $M_1(G-e) \leq m(m+1)$ and $d(u) + d(v) \leq m$ (all degrees refer to $G-e$). Note that $(d(u)+1)^2 = (d(u))^2 + 2d(u) + 1$. Thus

$$\begin{aligned} M_1(G) &= M_1(G-e) + 2d(u) + 1 + 2d(v) + 1 \\ &\leq m(m+1) + 2m + 2 \\ &= (m+1)(m+2) \end{aligned}$$

Equality requires that $d(u) + d(v) = m$ for every edge uv. Equivalently, G is $2K_2$-free, so it has one nontrivial component that is a star or K_3. It is clear that equality holds for these graphs. □

The upper bound is due to Das [9]. The extremal graphs are misstated in [14] (omitting $K_3 \cup \overline{K}_{n-3}$) and no proof is provided. This incorrect result is repeated in [3].

Some extremal graphs for first Zagreb index are near k-stars, but not all. For example, $M_1(K_{1,3}) = M_1(K_3 \cup K_1) = 12$. Also, if $G_1 = NS(5,6)$ and $G_2 = K_4 \cup K_1$, then $M_1(G_1) = 34 < 36 = M_1(G_2)$. Similar examples occur for larger n.

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