

Research Article

# On Two Conjectures Concerning the Hyperbolic Sombor Index and Its Counterpart

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## Abstract

The hyperbolic Sombor index (HSO index) and the complementary diminished Sombor index (CDSO index) of a graph  $G$  with edge set  $E(G)$  are defined, respectively, as  $HSO(G) = \sum_{uv \in E(G)} (\min\{d_G(u), d_G(v)\})^{-1} \sqrt{d_G(u)^2 + d_G(v)^2}$  and  $CDSO(G) = \sum_{uv \in E(G)} (\max\{d_G(u), d_G(v)\})^{-1} \sqrt{d_G(u)^2 + d_G(v)^2}$ , where  $d_G(u)$  and  $d_G(v)$  are the degrees of the vertices  $u$  and  $v$ , respectively. Recently, two conjectures concerning the extremum values of the HSO and CDSO indices of connected graphs of fixed order and cyclomatic number were posed in [A. M. Albalahi, S. Das, A. Ali, J. Barman, A. E. Hamza, *Discrete Math. Lett.* **16** (2025) 108–115]. In this paper, both of these conjectures are settled under almost all of the given constraints.

**Keywords:** hyperbolic Sombor index; complementary diminished Sombor index; cyclomatic number; maximum degree; minimum degree.

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## 1. Introduction

All graphs considered in this paper are simple, connected and finite. Standard graph-theoretic terminology not defined here can be found in standard books, like [6, 7]. The vertex set and edge set of a graph  $G$  are denoted by  $V(G)$  and  $E(G)$ , respectively. For a vertex  $u \in V(G)$ , let  $d_G(u)$  denote its degree. A vertex of degree one is called a pendant vertex. An edge incident with a pendant vertex is known as a pendant edge. A graph in which every vertex has degree 3 is known as a cubic graph.

The Sombor index of  $G$ , introduced by Gutman [11], is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

Although introduced only about five years ago, the Sombor index has attracted considerable research attention, and its mathematical properties have been investigated extensively; for example, see the survey [15].

Motivated by the interest in the Sombor index and its geometric interpretation, researchers have introduced and studied numerous related indices, and this family continues to expand; for instance, see the recent survey paper [12]. The present paper concerns two such variants: the hyperbolic Sombor index (HSO index), introduced in [5], and the complementary diminished Sombor index (CDSO index), introduced in [2]. For a graph  $G$ , these indices are defined by

$$HSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_G(u)^2 + d_G(v)^2}}{\min\{d_G(u), d_G(v)\}} \quad \text{and} \quad CDSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_G(u)^2 + d_G(v)^2}}{\max\{d_G(u), d_G(v)\}},$$

respectively. The CDSO index arises from applying the framework of complementary topological indices [10] to the diminished Sombor index [1, 16], with the constant factor  $1/\sqrt{2}$  omitted [2]. In addition to the results obtained in [2], the CDSO index was discussed in [3], where the graphs minimizing this index among unicyclic graphs of order at most 10 were determined computationally; see Remark 2.3 therein. Several known properties and applications of the HSO index can be found in the recent studies [2, 4, 5, 8, 9, 13, 14].

A graph containing no cycle is called an acyclic graph. The cyclomatic number of a graph  $G$  is the minimum number of edges whose removal makes  $G$  acyclic. For a connected graph  $G$ , its cyclomatic number is  $|E(G)| - |V(G)| + 1$ . A graph  $G$  with  $|V(G)| = n$  is called an  $n$ -vertex graph. The minimum and maximum degrees of a graph  $G$  are denoted by  $\delta(G)$  and  $\Delta(G)$ , respectively.

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The present paper addresses the following two conjectures.

**Conjecture 1.1** (see [2]). *Let  $G$  be a graph maximizing (respectively, minimizing) the HSO index (respectively, the CDSO index) among  $n$ -vertex connected graphs with cyclomatic number  $\ell \geq 1$ . Then,  $\Delta(G) = n - 1$ .*

**Conjecture 1.2** (see [2]). *Let  $G$  be a graph minimizing (respectively, maximizing) the HSO index (respectively, the CDSO index) among  $n$ -vertex connected graphs with cyclomatic number  $\ell \geq 2$  provided that  $n > 5(\ell - 1)$ . Then,  $\delta(G), \Delta(G) \in \{2, 3\}$ .*

In the present paper, it is shown that the maximal part of Conjecture 1.1 is true (see Theorem 2.1), whereas its minimal part is disproved for every  $\ell \geq 1$  (see Proposition 2.1; see also Remark 2.3 in [3] for  $\ell = 1$ ). For  $\ell = 2$ , the minimal part of Conjecture 1.2 has already been proved in [8], whereas its maximal part is not true because the 6-vertex graph  $H_6$  provides a counterexample, where  $H_6$  is the graph obtained from the unique 4-vertex graph of size 5 by attaching one pendant vertex to each of its two vertices of degree 2. However, it is shown that Conjecture 1.2 is true for every  $\ell \geq 3$  (see Theorems 2.2 and 2.3).

## 2. Main Results

Throughout this section, let

$$\phi(x, y) = \frac{\sqrt{x^2 + y^2}}{\min\{x, y\}}, \quad x, y > 0.$$

**Lemma 2.1.** *For every fixed  $y > 0$ , the function  $\phi(x, y)$  is strictly decreasing on  $(0, y]$  and strictly increasing on  $[y, \infty)$ .*

**Lemma 2.2.** *For every fixed  $y > 0$ , it holds that*

$$\phi(a + q, y) + \phi(b - q, y) > \phi(a, y) + \phi(b, y) \quad \text{for } a \geq b > q > 0. \quad (1)$$

**Proof.** For fixed  $y > 0$ , let  $F_y(x) = \phi(x, y)$ . Then, we have

$$F_y(x) = \begin{cases} \frac{\sqrt{x^2 + y^2}}{x}, & x \leq y, \\ \frac{\sqrt{x^2 + y^2}}{y}, & x \geq y. \end{cases}$$

A direct calculation gives

$$F_y''(x) = \begin{cases} \frac{y^2(3x^2 + 2y^2)}{x^3(x^2 + y^2)^{3/2}}, & x < y, \\ \frac{y}{(x^2 + y^2)^{3/2}}, & x > y, \end{cases}$$

while the one-sided derivatives at  $x = y$  satisfy

$$F_y'(y^-) = -\frac{1}{\sqrt{2}y} < \frac{1}{\sqrt{2}y} = F_y'(y^+).$$

Thus,  $F_y$  is strictly convex on  $(0, \infty)$ .

Let  $\lambda = q/(a - b + 2q)$ . Then, we have  $0 < \lambda < 1$ . Since  $a = (1 - \lambda)(a + q) + \lambda(b - q)$  and  $b = \lambda(a + q) + (1 - \lambda)(b - q)$ , by using the strict convexity of  $F_y$ , we have  $F_y(a) + F_y(b) < F_y(a + q) + F_y(b - q)$ , which implies (1).  $\square$

For a vertex  $u$  in a graph  $G$ , let  $N_G(u) = \{x \in V(G) : xu \in E(G)\}$  and  $N_G[u] = N_G(u) \cup \{u\}$ . The elements of  $N_G(u)$  are called neighbors of  $u$ . Let  $\overline{G}$  denote the complement of  $G$ . For  $A \subseteq E(G)$  and  $B \subseteq E(\overline{G})$ , let  $G - A + B$  denote the graph obtained from  $G$  by removing the edges of  $G$  belonging to  $A$  and adding the edges belonging to  $B$ .

The following result proves the maximal part of Conjecture 1.1.

**Theorem 2.1.** *Let  $G$  be a graph maximizing the HSO index among  $n$ -vertex connected graphs with cyclomatic number  $\ell \geq 1$ . Then,  $\Delta(G) = n - 1$ .*

**Proof.** Since  $\ell \geq 1$ , we have  $n \geq 3$ . Suppose, to the contrary, that  $\Delta := \Delta(G) \leq n - 2$ . Throughout the proof, we use  $d(\alpha) = d_G(\alpha)$  for  $\alpha \in V(G)$ . Let  $u \in V(G)$  such that  $d(u) = \Delta$ . Let  $X = V(G) \setminus N_G[u]$ . Then,  $X \neq \emptyset$ . Let  $A = \{w \in V(G) \setminus \{u\} : N_G(w) \cap X \neq \emptyset\}$ ; for example, see Figure 2.1. Since  $G$  is connected, there is an edge connecting a vertex in  $N_G(u)$  to a vertex in  $X$ , and hence  $A \neq \emptyset$ . Choose  $v \in A$  such that  $d(w) \leq d(v)$  for every  $w \in A$ . Thus, we have

$$d(w) \leq d(v) \leq \Delta \quad \forall w \in A. \quad (2)$$

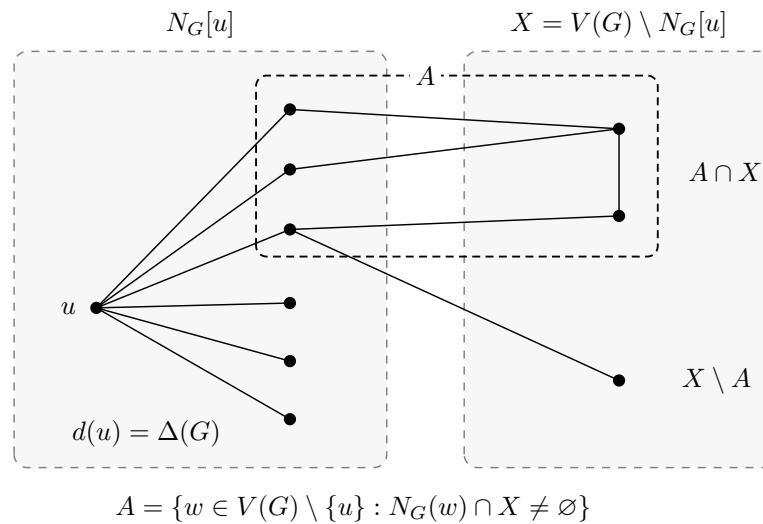


**Figure 2.1:** An illustrative example of  $G$  in the proof of Theorem 2.1.

We note that either  $v \in X$  or  $v \in N_G(u)$ . In each of the graph transformations used in the rest of the proof, we remark that the resulting graph  $G'$  remains an  $n$ -vertex connected graph with cyclomatic number  $\ell$ .

**Case 1.**  $v \in X$  and  $N_G(v) \cap N_G(u) \neq \emptyset$ .

Let  $q = |N_G(v) \cap X| \geq 1$ . Let  $G' = G - \{vx : x \in N_G(v) \cap X\} + \{ux : x \in N_G(v) \cap X\}$ . Then, we have

$$\begin{aligned} HSO(G) - HSO(G') &= \sum_{z \in N_G(u) \setminus N_G(v)} [\phi(\Delta, d(z)) - \phi(\Delta + q, d(z))] + \sum_{x \in N_G(v) \cap X} [\phi(d(v), d(x)) - \phi(\Delta + q, d(x))] \\ &\quad + \sum_{z \in N_G(u) \cap N_G(v)} [\phi(\Delta, d(z)) + \phi(d(v), d(z)) - \phi(\Delta + q, d(z)) - \phi(d(v) - q, d(z))], \end{aligned} \quad (3)$$

Since  $v \in X$ , we note that  $N_G(v) \cap X$  is a subset of  $A$ , and hence, (2) implies that  $d(x) \leq d(v) \leq \Delta$  for every  $x \in N_G(v) \cap X$ . Thus, by Lemma 2.1, the second sum of (3) is negative (it cannot be zero because  $N_G(v) \cap X \neq \emptyset$ ). The first sum of (3) is nonpositive because  $d(z) \leq \Delta$ , and its last sum is nonpositive by (1). Thus, (3) gives  $HSO(G) - HSO(G') < 0$ , which contradicts the maximality of  $G$ .

**Case 2.**  $v \in X$  and  $N_G(v) \cap N_G(u) = \emptyset$ .

Let  $y$  be the neighbor of  $v$  on a shortest  $v$ - $u$  path. Since the distance between  $v$  and  $u$  is at least three,  $y \in X$  and  $d(y) \geq 2$ . Also,  $y \in A$ , since  $v \in X$ . Thus, (2) gives  $2 \leq d(y) \leq d(v) \leq \Delta$ . Let  $G' = G - \{vx : x \in N_G(v) \setminus \{y\}\} + \{ux : x \in N_G(v) \setminus \{y\}\}$ . Each  $x \in N_G(v) \setminus \{y\}$  belongs to  $A$ . Thus, by (2),  $d(x) \leq d(v)$ , and hence by Lemma 2.1, we have

$$\begin{aligned} HSO(G) - HSO(G') &= \sum_{a \in N_G(u)} [\phi(\Delta, d(a)) - \phi(\Delta + d(v) - 1, d(a))] + \sum_{x \in N_G(v) \setminus \{y\}} [\phi(d(v), d(x)) - \phi(\Delta + d(v) - 1, d(x))] \\ &\quad + \phi(d(v), d(y)) - \phi(1, d(y)) \\ &\leq \sum_{a \in N_G(u)} [\phi(\Delta, d(a)) - \phi(\Delta + d(v) - 1, d(a))] + \phi(d(v), d(y)) - \phi(1, d(y)). \end{aligned} \quad (4)$$

For  $a \in N_G(u)$ , we have  $d(a) \leq \Delta$ . Thus, by the mean value theorem, there exists  $\xi \in (\Delta, \Delta + d(v) - 1)$  such that

$$\phi(\Delta + d(v) - 1, d(a)) - \phi(\Delta, d(a)) = (d(v) - 1) \frac{\xi}{d(a) \sqrt{\xi^2 + d(a)^2}} \geq \frac{d(v) - 1}{\sqrt{2} d(a)} \geq \frac{d(v) - 1}{\sqrt{2} \Delta},$$

which gives

$$\sum_{a \in N_G(u)} [\phi(\Delta, d(a)) - \phi(\Delta + d(v) - 1, d(a))] \leq -\frac{d(v) - 1}{\sqrt{2}}. \quad (5)$$

Since  $2 \leq d(y) \leq d(v)$ , we have  $\phi(d(v), d(y)) < d(v)/d(y) + 1$  and  $\phi(1, d(y)) > d(y)$ . As  $d(v)/d(y) + 1 - d(y)$  is decreasing in  $d(y)$ , it follows that

$$\phi(d(v), d(y)) - \phi(1, d(y)) < \frac{d(v)}{d(y)} + 1 - d(y) \leq \frac{d(v) - 2}{2} < \frac{d(v) - 1}{\sqrt{2}}. \quad (6)$$

Using (5) and (6) in (4), we obtain  $HSO(G) - HSO(G') < 0$ , a contradiction.

**Case 3.**  $v \in N_G(u)$ .

Let  $B = N_G(v) \cap X$ ,  $q = |B| \geq 1$ ,  $C = N_G(u) \cap N_G(v)$ , and  $K = N_G(u) \setminus (C \cup \{v\})$ . Since  $N_G(v)$  is the disjoint union of  $\{u\}$ ,  $C$ , and  $B$ , the numbers  $p = 1 + |C|$  and  $q$  satisfy  $d(v) = p + q$  and  $p \geq 1$ . Let  $G' = G - \{vx : x \in B\} + \{ux : x \in B\}$ . Then, we have

$$\begin{aligned} HSO(G) - HSO(G') &= \phi(\Delta, d(v)) - \phi(\Delta + q, p) + \sum_{z \in C} [\phi(\Delta, d(z)) + \phi(d(v), d(z)) - \phi(\Delta + q, d(z)) - \phi(p, d(z))] \\ &\quad + \sum_{y \in K} [\phi(\Delta, d(y)) - \phi(\Delta + q, d(y))] + \sum_{x \in B} [\phi(d(v), d(x)) - \phi(\Delta + q, d(x))]. \end{aligned} \quad (7)$$

Since  $(\Delta + q)d(v) - \Delta p = q(\Delta + d(v)) > 0$ , we have  $(\Delta + q)/p > \Delta/d(v)$ . Consequently,

$$\phi(\Delta, d(v)) - \phi(\Delta + q, p) < 0. \quad (8)$$

For every  $z \in C$ , from (1) it follows that

$$\phi(\Delta, d(z)) + \phi(d(v), d(z)) - \phi(\Delta + q, d(z)) - \phi(p, d(z)) \leq 0. \quad (9)$$

Thus, using (8) and (9) in (7), we have

$$HSO(G) - HSO(G') < \sum_{y \in K} [\phi(\Delta, d(y)) - \phi(\Delta + q, d(y))] + \sum_{x \in B^-} [\phi(d(v), d(x)) - \phi(\Delta + q, d(x))], \quad (10)$$

where  $B^- = \{x \in B : \phi(d(v), d(x)) > \phi(\Delta + q, d(x))\}$ .

For every  $x \in B^-$ , we have  $d(x) > d(v)$ ; otherwise, from the chain of inequalities  $d(x) \leq d(v) \leq \Delta < \Delta + q$  and Lemma 2.1, it follows that  $\phi(d(v), d(x)) < \phi(\Delta + q, d(x))$ , which implies  $x \notin B^-$ , a contradiction. Thus, by (2),  $x \notin A$ , and hence,  $N_G(x) \cap X = \emptyset$ . Since  $x \in X$ , it follows that  $N_G(x) \subseteq N_G(u)$ , and hence,

$$|N_G(x) \cap K| \geq d(x) - p \geq d(x) - d(v). \quad (11)$$

If  $y \in K$  has a neighbor in  $B^-$ , then  $y \in A$ , and hence,  $d(y) \leq d(v)$ . By the mean value theorem and Lemma 2.1, we have

$$\phi(\Delta + q, d(y)) - \phi(\Delta, d(y)) \geq \begin{cases} \frac{q\Delta}{d(v)\sqrt{\Delta^2 + d(v)^2}} \geq \frac{\Delta}{d(v)\sqrt{\Delta^2 + d(v)^2}} |N_G(y) \cap B^-| & \text{if } N_G(y) \cap B^- \neq \emptyset \\ 0 = \frac{\Delta}{d(v)\sqrt{\Delta^2 + d(v)^2}} |N_G(y) \cap B^-| & \text{if } N_G(y) \cap B^- = \emptyset. \end{cases} \quad (12)$$

From (11) and (12), we obtain

$$\begin{aligned} \sum_{y \in K} [\phi(\Delta + q, d(y)) - \phi(\Delta, d(y))] &\geq \frac{\Delta}{d(v)\sqrt{\Delta^2 + d(v)^2}} \sum_{y \in K} |N_G(y) \cap B^-| = \frac{\Delta}{d(v)\sqrt{\Delta^2 + d(v)^2}} \sum_{x \in B^-} |N_G(x) \cap K| \\ &\geq \frac{\Delta}{d(v)\sqrt{\Delta^2 + d(v)^2}} \sum_{x \in B^-} (d(x) - d(v)). \end{aligned} \quad (13)$$

For  $x \in B^-$ , we have  $d(v) < d(x) \leq \Delta < \Delta + q$ , and hence by the definition of  $B^-$ , it holds that

$$\sqrt{1 + \left(\frac{d(x)}{d(v)}\right)^2} = \phi(d(v), d(x)) > \phi(\Delta + q, d(x)) = \sqrt{1 + \left(\frac{\Delta + q}{d(x)}\right)^2},$$

which yields  $d(x)/d(v) > (\Delta + q)/d(x)$ . Therefore, by applying the mean value theorem to  $f(s) = \sqrt{1 + s^2}$ , we obtain some  $\xi_x \in ((\Delta + q)/d(x), d(x)/d(v))$  such that

$$\begin{aligned} \sum_{x \in B^-} [\phi(d(v), d(x)) - \phi(\Delta + q, d(x))] &= \sum_{x \in B^-} \frac{\xi_x}{\sqrt{1 + \xi_x^2}} \left( \frac{d(x)}{d(v)} - \frac{\Delta + q}{d(x)} \right) \leq \frac{\Delta}{\sqrt{\Delta^2 + d(v)^2}} \sum_{x \in B^-} \left( \frac{d(x)}{d(v)} - \frac{\Delta + q}{d(x)} \right) \\ &\leq \frac{\Delta}{d(v)\sqrt{\Delta^2 + d(v)^2}} \sum_{x \in B^-} (d(x) - d(v)), \end{aligned} \quad (14)$$

where the first inequality in (14) follows from  $\xi_x < d(x)/d(v) \leq \Delta/d(v)$  and the monotonicity of  $s/\sqrt{1 + s^2}$ , while the last inequality in (14) follows from  $\Delta + q > d(x)$ . Now, (13) and (14) yield

$$\sum_{x \in B^-} [\phi(d(v), d(x)) - \phi(\Delta + q, d(x))] \leq \sum_{y \in K} [\phi(\Delta + q, d(y)) - \phi(\Delta, d(y))], \quad (15)$$

which also holds when  $B^- = \emptyset$ . Now, from (15) and (10), we obtain  $HSO(G) - HSO(G') < 0$ , again a contradiction.

In each of the three cases, we arrive at a contradiction. Therefore,  $X = \emptyset$ , and hence,  $d(u) = n - 1 = \Delta$ .  $\square$

The following result disproves the minimal part of Conjecture 1.1 for every  $\ell \geq 1$ .

**Proposition 2.1.** *No graph minimizing the CDSO index among connected graphs of order  $5\ell + 7$  and cyclomatic number  $\ell$  has maximum degree  $5\ell + 6$ , where  $\ell \geq 1$ .*

**Proof.** Let  $G_\ell$  be the graph obtained from the complete bipartite graph  $K_{2,\ell+1}$ , with partite sets  $\{u_1, u_2\}$  and  $\{v_1, \dots, v_{\ell+1}\}$ , by attaching  $2(\ell + 1)$  pendant vertices to each of  $u_1$  and  $u_2$ . Then, we have  $|V(G_\ell)| = 5\ell + 7$  and  $|E(G_\ell)| = 6(\ell + 1)$ . Hence, the cyclomatic number of  $G_\ell$  is  $\ell$  and its maximum degree is  $3(\ell + 1)$ , which is less than  $|V(G_\ell)| - 1$ . Also, we have

$$CDSO(G_\ell) = \frac{4}{3}\sqrt{9(\ell+1)^2+1} + \frac{2}{3}\sqrt{9(\ell+1)^2+4}. \quad (16)$$

Using  $\sqrt{1+t} < 1 + t/2$  for  $t > 0$ , we obtain

$$\begin{aligned} CDSO(G_\ell) &= 4(\ell+1)\sqrt{1+\frac{1}{9(\ell+1)^2}} + 2(\ell+1)\sqrt{1+\frac{4}{9(\ell+1)^2}} \\ &< 4(\ell+1)\left(1+\frac{1}{18(\ell+1)^2}\right) + 2(\ell+1)\left(1+\frac{2}{9(\ell+1)^2}\right) \\ &= 6(\ell+1) + \frac{2}{3(\ell+1)}. \end{aligned} \quad (17)$$

Now, let  $H$  be any connected graph of order  $5\ell + 7$  and cyclomatic number  $\ell$  having a vertex  $u$  of maximum degree  $5\ell + 6$ . Then,  $|E(H)| = (5\ell + 7) - 1 + \ell = 6(\ell + 1)$ , and hence,  $|E(H - u)| = 6(\ell + 1) - (5\ell + 6) = \ell$ , where  $H - u$  is the graph obtained from  $H$  by deleting  $u$  and all its incident edges. Consequently, every  $w \neq u$  satisfies  $d_H(w) = 1 + d_{H-u}(w) \leq \ell + 1$ . Particularly, for every  $xy \in E(H - u)$ , it holds that  $2 \leq \min\{d_H(x), d_H(y)\} \leq \max\{d_H(x), d_H(y)\} \leq \ell + 1$ .

$$\begin{aligned} CDSO(H) &= \sum_{w \in V(H-u)} \sqrt{1 + \left(\frac{d_H(w)}{5\ell+6}\right)^2} + \sum_{xy \in E(H-u)} \sqrt{1 + \left(\frac{\min\{d_H(x), d_H(y)\}}{\max\{d_H(x), d_H(y)\}}\right)^2} \\ &> |V(H-u)| + |E(H-u)|\sqrt{1 + \frac{4}{(\ell+1)^2}} = 5\ell + 6 + \ell\sqrt{1 + \frac{4}{(\ell+1)^2}}. \end{aligned} \quad (18)$$

Since  $\ell \geq 1$ , from (17) and (18), we have

$$CDSO(H) - CDSO(G_\ell) > \ell \left( \sqrt{1 + \frac{4}{(\ell+1)^2}} - 1 \right) - \frac{2}{3(\ell+1)} > 0, \quad (19)$$

as required.  $\square$

The following two theorems settle Conjecture 1.2 for every  $\ell \geq 3$ . It needs to be noted here that Conjecture 1.2 remains true for every  $\ell \geq 3$  if the condition  $n > 5(\ell - 1)$  is replaced with  $n \geq 2\ell + 1$ .

**Theorem 2.2.** *Every graph maximizing the CDSO index among  $n$ -vertex connected graphs with cyclomatic number  $\ell$  has minimum degree 2, maximum degree 3 and exactly two edges with different end vertex degrees, where  $\ell \geq 3$  and  $n \geq 2\ell + 1$ . Also, this maximum value of the CDSO index is*

$$(n + \ell - 3)\sqrt{2} + \frac{2\sqrt{13}}{3}.$$

**Proof.** Let  $\psi(r, s) = \frac{\sqrt{r^2+s^2}}{\max\{r,s\}}$  for positive integers  $r$  and  $s$ . If  $r \leq s$ , then

$$\psi(r, s) = \frac{\sqrt{r^2+s^2}}{s} = \sqrt{1 + \left(\frac{r}{s}\right)^2} \leq \sqrt{2}, \quad (20)$$

where equality in the final inequality of (20) holds if and only if  $r = s$ . Also, if  $s \geq 2$ , then

$$\psi(1, s) \leq \frac{\sqrt{5}}{2}. \quad (21)$$

Moreover, if  $s \geq 3$ , then

$$\psi(2, s) \leq \frac{\sqrt{13}}{3}, \quad (22)$$

where the equality holds if and only if  $s = 3$ .

Let  $Q$  be the cubic graph obtained from the cycle  $C_{2\ell-2}$  of order  $2\ell-2$  by adding  $\ell-1$  pairwise non-adjacent edges between nonadjacent vertices of  $C_{2\ell-2}$ . Then,  $|E(Q)| = \frac{3(2\ell-2)}{2} = 3\ell-3$ , and hence, the cyclomatic number of  $Q$  is  $(3\ell-3) - (2\ell-2) + 1 = \ell$ . Now, we insert  $n - (2\ell-2)$  new vertices into one edge of  $Q$ , and denote the resulting graph by  $Q_n$ . Then, we have

$$CDSO(Q_n) = (n + \ell - 3)\sqrt{2} + \frac{2\sqrt{13}}{3}. \quad (23)$$

Now, let  $G$  be a graph maximizing the CDSO index among  $n$ -vertex connected graphs with cyclomatic number  $\ell$ . Let  $m$  denote the size of  $G$ . Then,  $m = n + \ell - 1$ . By (23), we have

$$CDSO(G) \geq (m - 2)\sqrt{2} + \frac{2\sqrt{13}}{3}. \quad (24)$$

Let  $n_1$  be the number of pendant vertices of  $G$ . If  $u_1u_2, v_1v_2 \in E(G)$  are two different pendant edges with  $d_G(u_1) = d_G(v_1) = 1$ , then using (20) and (21), we have

$$CDSO(G) \leq (m - 2)\sqrt{2} + \psi(1, d_G(u_2)) + \psi(1, d_G(v_2)) \leq (m - 2)\sqrt{2} + \sqrt{5} < (m - 2)\sqrt{2} + \frac{2\sqrt{13}}{3},$$

which contradicts (24). Hence,  $n_1 \leq 1$ .

Next, let  $S = \{w \in V(G) : d_G(w) = 2\}$ . We claim that  $S \neq \emptyset$ ; otherwise, every non-pendant vertex has degree at least 3, and hence,

$$2n + 2\ell - 2 = 2m = \sum_{w \in V(G)} d_G(w) \geq n_1 + 3(n - n_1) \geq 3n - 2,$$

which contradicts  $n \geq 2\ell + 1$ .

We claim that  $n_1 = 0$ . Suppose to the contrary that  $n_1 = 1$ . Since  $G$  is connected, has a vertex of degree 2, and is not a path or a cycle, it contains an edge joining a vertex of degree 2 to a vertex of degree  $\beta \geq 3$ . Also, the unique pendant vertex is adjacent to a vertex of degree  $\alpha \geq 2$ . Now, using (20), (21) and (22), we have

$$CDSO(G) \leq (m - 2)\sqrt{2} + \psi(1, \alpha) + \psi(2, \beta) \leq (m - 2)\sqrt{2} + \frac{\sqrt{13}}{3} + \frac{\sqrt{5}}{2} < (m - 2)\sqrt{2} + \frac{2\sqrt{13}}{3},$$

which contradicts (24). Thus,  $n_1 = 0$ . Consequently, we have  $\delta(G) = 2$ .

Let  $n_2 = |S|$ . Let  $m_{2,2}$  denote the number of edges of  $G$  joining two vertices of degree 2. Let  $k$  be the number of edges of  $G$  joining a vertex of degree 2 to a vertex of degree at least 3. Then,  $2n_2 = 2m_{2,2} + k$ , which implies that  $k$  is even. Since  $G$  is connected and is not the cycle graph, and since  $S \neq \emptyset$ , we have  $k \geq 2$ . Now, we choose two distinct edges  $x_1y_1, x_2y_2 \in E(G)$  such that  $d_G(x_i) = 2$  and  $d_G(y_i) \geq 3$  for  $i = 1, 2$ . Then, using (20) and (22), we have

$$CDSO(G) \leq (m - 2)\sqrt{2} + \psi(2, d_G(y_1)) + \psi(2, d_G(y_2)) \leq (m - 2)\sqrt{2} + \frac{2\sqrt{13}}{3}, \quad (25)$$

where the equality throughout in (25) holds if and only if  $d_G(y_1) = d_G(y_2) = 3$  and every edge other than  $x_1y_1$  and  $x_2y_2$  has end vertices of equal degrees; however, (24) forces equality throughout in (25).

Finally, we have to show that  $\Delta(G) = 3$ . Suppose contrarily that  $\Delta(G) \geq 4$ . We choose a path  $P$  from a vertex of degree  $\Delta(G)$  to  $y_1$ . Since  $d_G(y_1) = 3$ , the path  $P$  contains an edge whose end vertices have different degrees. At the first such edge, one end vertex still has degree  $\Delta(G) \geq 4$ , which contradicts the equality conditions of (25) because the only edges with unequal end vertex degrees are  $x_1y_1$  and  $x_2y_2$ , both with end vertex degrees 2 and 3. Therefore,  $\Delta(G) \leq 3$ . Since  $d_G(y_1) = 3$ , we conclude that  $\Delta(G) = 3$ , and hence

$$CDSO(G) = (m - 2)\sqrt{2} + \frac{2\sqrt{13}}{3} = (n + \ell - 3)\sqrt{2} + \frac{2\sqrt{13}}{3},$$

as required. □

**Theorem 2.3.** Every graph minimizing the HSO index among connected  $n$ -vertex graphs with cyclomatic number  $\ell$  has minimum degree 2, maximum degree 3 and exactly two edges with different end vertex degrees, where  $\ell \geq 3$  and  $n \geq 2\ell + 1$ . Also, this minimum value of the HSO index is  $(n + \ell - 3)\sqrt{2} + \sqrt{13}$ .

**Proof.** For positive integers  $r \leq s$ , it holds that

$$\phi(r, s) = \frac{\sqrt{r^2 + s^2}}{r} = \sqrt{1 + \left(\frac{s}{r}\right)^2} \geq \sqrt{2}, \quad (26)$$

with equality if and only if  $r = s$ , where the function  $\phi$  is defined at the start of Section 2. Also, if  $s \geq 2$ , then

$$\phi(1, s) \geq \sqrt{5}. \quad (27)$$

Moreover, if  $s \geq 3$ , then

$$\phi(2, s) \geq \frac{\sqrt{13}}{2}, \quad (28)$$

with equality if and only if  $s = 3$ . Let  $Q_n$  be the graph defined in the proof of Theorem 2.2. Then, we have

$$HSO(Q_n) = (n + \ell - 3)\sqrt{2} + \sqrt{13}. \quad (29)$$

Now, let  $G$  be a graph minimizing the HSO index among connected  $n$ -vertex graphs with cyclomatic number  $\ell$ . Let  $m = |E(G)|$ . Then,  $m = n + \ell - 1$ . By (29), we have

$$HSO(G) \leq (m - 2)\sqrt{2} + \sqrt{13}. \quad (30)$$

Let  $n_1$  be the number of pendant vertices of  $G$ . If  $G$  has two distinct pendant edges, then (26) and (27) give

$$HSO(G) \geq (m - 2)\sqrt{2} + 2\sqrt{5} > (m - 2)\sqrt{2} + \sqrt{13},$$

which contradicts (30). Hence,  $n_1 \leq 1$ .

Let  $S = \{w \in V(G) : d_G(w) = 2\}$ . Then, by the proof of Theorem 2.2, we have  $S \neq \emptyset$ .

We next show that  $n_1 = 0$ . Suppose that  $G$  has a pendant vertex, whose neighbor has degree  $\alpha \geq 2$ . Since  $G$  is connected, has a vertex of degree 2, and is not a path or a cycle, it contains an edge joining a vertex of degree 2 to a vertex of degree  $\beta \geq 3$ . Using (26), (27), and (28), we obtain

$$HSO(G) \geq (m - 2)\sqrt{2} + \phi(1, \alpha) + \phi(2, \beta) \geq (m - 2)\sqrt{2} + \sqrt{5} + \frac{\sqrt{13}}{2} > (m - 2)\sqrt{2} + \sqrt{13},$$

which contradicts (30). Thus, it holds that  $n_1 = 0$ , and consequently,  $\delta(G) = 2$ .

Let  $k$  be the number of edges joining a vertex of degree 2 to a vertex of degree at least 3. Then, by the proof of Theorem 2.2, we have  $k \geq 2$ . We choose two distinct edges  $x_1y_1, x_2y_2 \in E(G)$  such that  $d_G(x_i) = 2$  and  $d_G(y_i) \geq 3$  for  $i = 1, 2$ . By (26) and (28), we obtain

$$HSO(G) \geq (m - 2)\sqrt{2} + \phi(2, d_G(y_1)) + \phi(2, d_G(y_2)) \geq (m - 2)\sqrt{2} + \sqrt{13}. \quad (31)$$

Now, (30) forces equality throughout in (31); that is,  $d_G(y_1) = d_G(y_2) = 3$  and every edge other than  $x_1y_1$  and  $x_2y_2$  has end vertices of equal degrees.

Finally, by using an argument similar to the one provided in the last paragraph of the proof of Theorem 2.2, we conclude that  $\Delta(G) = 3$ . Therefore,  $HSO(G) = (n + \ell - 3)\sqrt{2} + \sqrt{13}$ , as required.  $\square$

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