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Research Article

Digital Consequences of Legendre's Formula for Factorial Valuations

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Abstract

For a prime p, let $v_p(n!)$ denote the exponent of p in the prime factorization of $n!$. Starting from the digital identity $v_p(n!) = (n - s_p(n))/(p-1)$, where $s_p(n)$ is the sum of the base-p digits of n, we develop three results. First, we give two direct generating-function identities and derive an inhomogeneous Mahler-type functional equation for the ordinary generating function $H_p(z) = \sum_{n\geq 0} v_p(n!)z^n$, obtaining from it an explicit iterated series expansion. Second, on each complete base-p block $0 \leq n < p^m$, we determine an exact factorization for the generating polynomial $\sum_{0\leq n<p^m} y^{v_p(n!)}$ and deduce exact formulas for the mean and variance of $v_p(n!)$ on that block, together with a limiting distribution theorem for the normalized values $v_p(n!)/p^m$. Third, we prove the asymptotic formula $\sum_{n=p}^{N} 1/v_p(n!) = (p-1)\log N + C_p + O(\log N/N)$, where C_p is given by an explicit convergent series.

Keywords: p-adic valuation; factorial; sum of digits; generating function; generating polynomial; limiting distribution; reciprocal sums.

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1. Introduction and Preliminaries

For a prime p and an integer $n \geq 0$, Legendre's formula gives

$$v_p(n!) = \sum_{k\geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

A standard reformulation gives this valuation in terms of the base-p sum-of-digits function $s_p(n)$:

$$v_p(n!) = \frac{n - s_p(n)}{p-1}. \tag{1}$$

Background on digital expansions, automatic sequences, and sum-of-digits functions may be found in [1]. General references for elementary number theory and p-adic valuation methods include [5,6]. For expository treatments of Legendre's formula and related Kummer-type valuation identities, see [4, 7].

The sum-of-digits function is a classical digital, or q-additive, function; see, for example, [1–3]. Formulas (3) and (4), the product formula (10), and the moment formulas derived from it follow directly from Legendre's formula and standard properties of the base-p digit-sum function. The main contributions beyond these standard formulas are the iterated representation for $H_p(z)$ in Theorem 2.1, the limiting distribution on complete base-p blocks in Theorem 3.3, and the reciprocal-sum asymptotic with an explicit constant in Theorem 4.1.

The first result concerns the ordinary generating function

$$H_p(z) = \sum_{n\geq 0} v_p(n!)z^n.$$

By decomposing integers according to their residue classes modulo p, we derive an inhomogeneous Mahler-type functional equation for $H_p(z)$ that involves the generating function of the digit-sum sequence. We then iterate this equation to obtain an explicit series representation. We also give two direct formulas for $H_p(z)$, one from Legendre's formula and one from (1).

The second result concerns the behavior of $v_p(n!)$ on complete base-p blocks

$$B_m = \{0, 1, \ldots, p^m - 1\}.$$

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Using the digit expansion of integers in B_m, we obtain an exact factorization of the block generating polynomial

$$P_{p,m}(y) = \sum_{0 \le n < p^m} y^{v_p(n!)}.$$

This factorization gives exact formulas for the mean and variance of $v_p(n!)$ when n is chosen uniformly from B_m. We also show that, after normalization by p^m, these block values admit a simple limiting distribution. In this sense, the block structure makes the distribution of $v_p(n!)$ explicit at both the finite and asymptotic levels.

The third result concerns the reciprocal sum

$$\sum_{n=p}^{N} \frac{1}{v_p(n!)}.$$

Since $v_p(n!) \sim n/(p-1)$, one expects logarithmic growth. We prove that this is indeed the case.

These results also have some direct applications. The polynomial $P_{p,m}(y)$ gives the exact distribution of $v_p(n!)$ on the block B_m, while the limiting result describes the asymptotic behavior of these values as m increases. Since factorial valuations occur naturally in the p-adic valuations of binomial and multinomial coefficients, the results may also be useful in related divisibility questions.

The paper is organized as follows. The basic digit representation needed later is given below. In Section 2 we give generating-function representations for $H_p(z)$ and obtain an iterated expansion. Section 3 considers complete-block generating polynomials, exact block moments, and a limiting distribution theorem. In Section 4 we prove the asymptotic formula for the reciprocal sum.

We write $\mathbb{N} = \{0, 1, 2, \dots\}$. For $n \in \mathbb{N}$ with base-p expansion

$$n = a_0 + a_1 p + \cdots + a_{m-1} p^{m-1}, \ 0 \le a_i \le p-1,$$

we have $s_p(n) = a_0 + a_1 + \cdots + a_{m-1}$. Since

$$n - s_p(n) = \sum_{i=0}^{m-1} a_i (p^i - 1),$$

substituting into (1) gives the following result.

Lemma 1.1. *Let $n \in B_m$ with base-p digits $a_0, \dots, a_{m-1}$. Then*

$$v_p(n!) = \sum_{i=0}^{m-1} a_i \frac{p^i - 1}{p-1}. \tag{2}$$

2. Generating-Function Identities

We next consider the ordinary generating function of the sequence $v_p(n!)$. For $|z| < 1$, define

$$H_p(z) = \sum_{n \ge 0} v_p(n!) z^n, D_p(z) = \sum_{n \ge 0} s_p(n) z^n.$$

From Legendre's formula and absolute convergence for $|z| < 1$,

$$H_p(z) = \sum_{k \ge 1} \sum_{n \ge 0} \left\lfloor \frac{n}{p^k} \right\rfloor z^n = \sum_{k \ge 1} \frac{z^{p^k}}{(1-z)(1-z^{p^k})}. \tag{3}$$

On the other hand, the digit identity (1) gives

$$H_p(z) = \frac{1}{p-1} \left(\frac{z}{(1-z)^2} - D_p(z) \right). \tag{4}$$

Write $n = pm + r$ with $0 \le r < p$. Since $s_p(pm + r) = s_p(m) + r$, (1) gives

$$v_p((pm+r)!) = p\, v_p(m!) + s_p(m).$$

Therefore

$$H_p(z) = \frac{1 - z^p}{1 - z} \Big(p H_p(z^p) + D_p(z^p) \Big). \tag{5}$$

Equation (5) is an inhomogeneous Mahler-type functional equation, involving the digit-sum generating function $D_p(z)$. Its iteration yields a convergent series expansion.

By using equation (4) with z replaced by z^p to eliminate $D_p(z^p)$ from equation (5), we obtain the self-contained Mahler-type equation

$$H_p(z) = \frac{1 - z^p}{1 - z} \left(H_p(z^p) + \frac{z^p}{(1-z^p)^2} \right).$$

Theorem 2.1. *For $|z| < 1$, the series in (6) and (7) converge locally uniformly, and*

$$H_p(z) = \sum_{\ell=0}^{\infty} p^{\ell} \left(\prod_{k=0}^{\ell} \frac{1 - z^{p^{k+1}}}{1 - z^{p^k}} \right) D_p\left(z^{p^{\ell+1}}\right). \tag{6}$$

Equivalently,

$$H_p(z) = \sum_{\ell=0}^{\infty} p^{\ell} \frac{1 - z^{p^{\ell+1}}}{1 - z} D_p\left(z^{p^{\ell+1}}\right). \tag{7}$$

Proof. Set

$$A(w) := \sum_{r=0}^{p-1} w^r = \frac{1 - w^p}{1 - w}.$$

Then (5) may be written as

$$H_p(z) = A(z)\big(pH_p(z^p) + D_p(z^p)\big).$$

Iterating once gives

$$H_p(z) = A(z)D_p(z^p) + pA(z)A(z^p)D_p(z^{p^2}) + p^2A(z)A(z^p)H_p(z^{p^2}),$$

and by induction, for every $m \geq 1$,

$$H_p(z) = \sum_{\ell=0}^{m-1} p^{\ell} \left(\prod_{k=0}^{\ell} A(z^{p^k}) \right) D_p(z^{p^{\ell+1}}) + p^m \left(\prod_{k=0}^{m-1} A(z^{p^k}) \right) H_p(z^{p^m}). \tag{8}$$

Now

$$\prod_{k=0}^{\ell} A(z^{p^k}) = \prod_{k=0}^{\ell} \frac{1 - z^{p^{k+1}}}{1 - z^{p^k}} = \frac{1 - z^{p^{\ell+1}}}{1 - z}, \tag{9}$$

so (8) becomes

$$H_p(z) = \sum_{\ell=0}^{m-1} p^{\ell} \frac{1 - z^{p^{\ell+1}}}{1 - z} D_p(z^{p^{\ell+1}}) + p^m \frac{1 - z^{p^m}}{1 - z} H_p(z^{p^m}).$$

Since $v_p(n!) = 0$ for $0 \leq n < p$, we have $H_p(w) = O(w^p)$ as $w \to 0$. Hence

$$H_p(z^{p^m}) = O\left(|z|^{p^{m+1}}\right), |z| < 1,$$

and therefore

$$p^m \frac{1 - z^{p^m}}{1 - z} H_p(z^{p^m}) \to 0, (m \to \infty).$$

Letting $m \to \infty$ in (8) gives (6). The equivalence with (7) follows from (9). □

Example 2.1 ($p = 2$)**.** *For $p = 2$, since $A(w) = 1 + w$, Theorem 2.1 gives*

$$H_2(z) = \sum_{\ell=0}^{\infty} 2^{\ell} \frac{1 - z^{2^{\ell+1}}}{1 - z} D_2(z^{2^{\ell+1}}).$$

For comparison, (3) gives the direct representation

$$H_2(z) = \sum_{k \geq 1} \frac{z^{2^k}}{(1 - z)(1 - z^{2^k})}.$$

Both identities hold as power series with constant term zero, consistent with $H_2(0) = v_2(0!) = 0$.

3. Complete-Block Statistics and Limiting Distribution

For $m \geq 1$, define the polynomial

$$P_{p,m}(y) := \sum_{n=0}^{p^m-1} y^{v_p(n!)}.$$

Theorem 3.1. *For every prime p and every integer $m \geq 1$,*

$$P_{p,m}(y) = \prod_{i=0}^{m-1} \left(\sum_{a=0}^{p-1} y^{a\frac{p^i-1}{p-1}} \right). \tag{10}$$

Equivalently,

$$P_{p,m}(y) = p \prod_{i=1}^{m-1} \left(1 + y^{\frac{p^i-1}{p-1}} + \cdots + y^{(p-1)\frac{p^i-1}{p-1}} \right).$$

Here, when $m = 1$, the product is empty and is understood to be equal to 1.

Proof. Since

$$\frac{p^i-1}{p-1} = 1 + p + \cdots + p^{i-1} \in \mathbb{N}, (i \geq 1),$$

all exponents in (10) are nonnegative integers, so $P_{p,m}(y)$ is an ordinary polynomial in y.

Every integer $n \in B_m$ has a unique base-p expansion

$$n = \sum_{i=0}^{m-1} a_i p^i, 0 \leq a_i \leq p-1.$$

By Lemma 1.1,

$$y^{v_p(n!)} = \prod_{i=0}^{m-1} y^{a_i \frac{p^i-1}{p-1}}.$$

Summing over all independent choices of the digits $a_0, \ldots, a_{m-1}$ gives

$$P_{p,m}(y) = \sum_{a_0=0}^{p-1} \cdots \sum_{a_{m-1}=0}^{p-1} \prod_{i=0}^{m-1} y^{a_i \frac{p^i-1}{p-1}} = \prod_{i=0}^{m-1} \left(\sum_{a=0}^{p-1} y^{a\frac{p^i-1}{p-1}} \right),$$

which is (10). □

The product factorization gives exact block moments immediately.

Theorem 3.2. *Let X_m be the random variable obtained by choosing n uniformly from B_m and setting $X_m = v_p(n!)$. Then*

$$\mathbb{E}[X_m] = \frac{1}{2} \sum_{i=0}^{m-1} (p^i - 1) = \frac{1}{2} \left(\frac{p^m-1}{p-1} - m \right), \tag{11}$$

and

$$\mathrm{Var}(X_m) = \frac{p^2-1}{12(p-1)^2} \sum_{i=0}^{m-1} (p^i-1)^2. \tag{12}$$

Equivalently,

$$\mathrm{Var}(X_m) = \frac{p^2-1}{12(p-1)^2} \left(\frac{p^{2m}-1}{p^2-1} - \frac{2(p^m-1)}{p-1} + m \right). \tag{13}$$

In particular,

$$\sum_{n=0}^{p^m-1} v_p(n!) = \frac{p^m}{2(p-1)} \big(p^m - 1 - m(p-1) \big). \tag{14}$$

Proof. Let $A_0, \ldots, A_{m-1}$ be independent random variables, each uniformly distributed on $\{0, 1, \ldots, p-1\}$. By Lemma 1.1, the distribution of X_m is the same as that of

$$\sum_{i=0}^{m-1} c_i A_i, c_i := \frac{p^i-1}{p-1}.$$

Now

$$\mathbb{E}[A_i] = \frac{p-1}{2}, \mathrm{Var}(A_i) = \frac{p^2-1}{12}.$$

Therefore

$$\mathbb{E}[X_m] = \sum_{i=0}^{m-1} c_i \mathbb{E}[A_i] = \frac{p-1}{2} \sum_{i=0}^{m-1} \frac{p^i - 1}{p-1} = \frac{1}{2} \sum_{i=0}^{m-1} (p^i - 1),$$

which proves (11). Since the A_i are independent,

$$\mathrm{Var}(X_m) = \sum_{i=0}^{m-1} c_i^2 \mathrm{Var}(A_i) = \frac{p^2-1}{12} \sum_{i=0}^{m-1} \left(\frac{p^i - 1}{p-1} \right)^2,$$

which is (12). Summing the geometric series

$$\sum_{i=0}^{m-1} (p^i - 1)^2 = \sum_{i=0}^{m-1} p^{2i} - 2 \sum_{i=0}^{m-1} p^i + m = \frac{p^{2m}-1}{p^2-1} - \frac{2(p^m-1)}{p-1} + m$$

gives (13). Finally, (14) follows from

$$\sum_{n=0}^{p^m-1} v_p(n!) = p^m \mathbb{E}[X_m]$$

and (11). □

Example 3.1 ($p = 2$, $m = 4$)**.** *For $p = 2$ and $m = 4$, the coefficients in Lemma 1.1 are $0, 1, 3, 7$. Hence*

$$P_{2,4}(y) = 2(1+y)(1+y^3)(1+y^7) = 2(1 + y + y^3 + y^4 + y^7 + y^8 + y^{10} + y^{11}).$$

The leading factor 2 is exactly the contribution of the least significant binary digit, which does not change $v_p(n!)$. Equivalently, for $0 \le n < 16$, the values of $v_p(n!)$ are

$$0, 0, 1, 1, 3, 3, 4, 4, 7, 7, 8, 8, 10, 10, 11, 11.$$

The formulas in Theorem 3.2 give

$$\mathbb{E}[X_4] = \frac{1}{2}(15 - 4) = \frac{11}{2}, \mathrm{Var}(X_4) = \frac{1}{4}(0^2 + 1^2 + 3^2 + 7^2) = \frac{59}{4}.$$

We next describe the asymptotic distribution of $v_p(n!)$ on complete blocks under the natural scaling by p^m.

Theorem 3.3. *Let N_m be chosen uniformly from*

$$B_m = \{0, 1, \ldots, p^m - 1\},$$

and define

$$X_m = v_p(N_m!).$$

Then, as $m \to \infty$,

$$\frac{X_m}{p^m} \Longrightarrow \mathrm{Unif}\left[0, \frac{1}{p-1}\right].$$

Equivalently, for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{p^m} \sum_{n=0}^{p^m-1} f\left(\frac{v_p(n!)}{p^m} \right) \longrightarrow (p-1) \int_0^{1/(p-1)} f(x)\, dx. \tag{15}$$

Proof. By identity (1),

$$v_p(n!) = \frac{n - s_p(n)}{p-1}.$$

Hence, for $0 \le n < p^m$,

$$\frac{v_p(n!)}{p^m} = \frac{n}{(p-1)p^m} - \frac{s_p(n)}{(p-1)p^m}.$$

Since $0 \le s_p(n) \le (p-1)m$, we obtain the uniform bound

$$\sup_{0 \le n < p^m} \left| \frac{v_p(n!)}{p^m} - \frac{n}{(p-1)p^m} \right| \le \frac{m}{p^m} \to 0, (m \to \infty).$$

Thus X_m/p^m and $N_m/((p-1)p^m)$ differ by a uniformly vanishing error.

But, N_m/p^m is uniformly distributed on the equally spaced set

$$\left\{0, \frac{1}{p^m}, \frac{2}{p^m}, \ldots, \frac{p^m-1}{p^m}\right\}.$$

Hence, for every bounded continuous function f,

$$\frac{1}{p^m}\sum_{n=0}^{p^m-1} f\left(\frac{n}{p^m}\right) \longrightarrow \int_0^1 f(x)\,dx$$

by convergence of Riemann sums. Therefore

$$\frac{N_m}{p^m} \Longrightarrow \mathrm{Unif}[0,1],$$

and consequently by the continuous mapping theorem

$$\frac{N_m}{(p-1)p^m} \Longrightarrow \mathrm{Unif}\left[0, \frac{1}{p-1}\right].$$

Since X_m/p^m and $N_m/((p-1)p^m)$ differ by a uniformly vanishing error, the same limit holds for X_m/p^m. This also gives (15). □

An immediate consequence is the convergence of all normalized block moments.

Corollary 3.1. *For every integer $r \geq 1$,*

$$\frac{1}{p^m}\sum_{n=0}^{p^m-1}\left(\frac{v_p(n!)}{p^m}\right)^r \longrightarrow \frac{1}{(p-1)^r(r+1)}, (m \to \infty). \tag{16}$$

Proof. For $0 \leq n < p^m$, identity (1) gives

$$0 \leq \frac{v_p(n!)}{p^m} = \frac{n - s_p(n)}{(p-1)p^m} \leq \frac{n}{(p-1)p^m} < \frac{1}{p-1}.$$

Thus all values of $v_p(n!)/p^m$ lie in the fixed compact interval

$$I = \left[0, \frac{1}{p-1}\right].$$

Let $g : \mathbb{R} \to \mathbb{R}$ be any bounded continuous function such that $g(x) = x^r$ for all $x \in I$. Applying Theorem 3.3 to g, we obtain

$$\lim_{m\to\infty}\frac{1}{p^m}\sum_{n=0}^{p^m-1}\left(\frac{v_p(n!)}{p^m}\right)^r = (p-1)\int_0^{1/(p-1)} x^r\,dx = \frac{1}{(p-1)^r(r+1)}.$$

□

4. Reciprocal Sums

We now consider the reciprocal sum

$$S_p(N) := \sum_{n=p}^{N}\frac{1}{v_p(n!)}.$$

The sum begins at $n = p$ because $v_p(n!) = 0$ for $0 \leq n < p$, so the reciprocal is not defined there. Since $v_p(n!) \sim n/(p-1)$, one expects logarithmic growth. The next theorem gives the precise main term and constant.

Theorem 4.1. *For every fixed prime p, as $N \to \infty$,*

$$S_p(N) = (p-1)\log N + C_p + O\left(\frac{\log N}{N}\right), \tag{17}$$

$$C_p = (p-1)\left(\gamma - \sum_{n=1}^{p-1}\frac{1}{n}\right) + \sum_{n=p}^{\infty}\left(\frac{1}{v_p(n!)} - \frac{p-1}{n}\right) \tag{18}$$

where γ denotes the Euler–Mascheroni constant and C_p is finite.

Proof. From (1), it follows that

$$\frac{1}{v_p(n!)} = \frac{p-1}{n - s_p(n)} = \frac{p-1}{n} \cdot \frac{1}{1 - s_p(n)/n}.$$

Since $s_p(n) \le (p-1)(\lfloor \log_p n \rfloor + 1)$, we have $s_p(n) = O(\log n)$ and hence $s_p(n)/n \to 0$. For all sufficiently large n,

$$\frac{1}{1 - s_p(n)/n} = 1 + \frac{s_p(n)}{n} + O\left(\frac{s_p(n)^2}{n^2}\right),$$

and hence

$$\frac{1}{v_p(n!)} = \frac{p-1}{n} + O\left(\frac{\log n}{n^2}\right). \tag{19}$$

Therefore the series

$$\sum_{n=p}^{\infty} \left(\frac{1}{v_p(n!)} - \frac{p-1}{n} \right)$$

converges absolutely, proving that C_p in (18) is well defined.

Now we sum (19) from $n = p$ to N:

$$S_p(N) = (p-1) \sum_{n=p}^{N} \frac{1}{n} + \sum_{n=p}^{N} \left(\frac{1}{v_p(n!)} - \frac{p-1}{n} \right).$$

Since

$$\sum_{n>N} \frac{\log n}{n^2} = O\left(\frac{\log N}{N}\right),$$

we obtain

$$\sum_{n=p}^{N} \left(\frac{1}{v_p(n!)} - \frac{p-1}{n} \right) = \sum_{n=p}^{\infty} \left(\frac{1}{v_p(n!)} - \frac{p-1}{n} \right) + O\left(\frac{\log N}{N}\right).$$

Finally,

$$\sum_{n=p}^{N} \frac{1}{n} = \log N + \gamma - \sum_{n=1}^{p-1} \frac{1}{n} + O\left(\frac{1}{N}\right).$$

Combining this with the definition of C_p gives (17). □

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