Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **18** (2026) 107–114
DOI: 10.47443/dml.2026.206

Research Article

On the k-Distance Chromatic Number of Unicyclic Graphs

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(Received: 3 July 2026. Received in revised form: 30 August 2026. Accepted: 20 October 2026. Published online: 26 October 2026.)

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Abstract

We study the k-distance chromatic number of unicyclic graphs, providing exact values for some families and bounds for others, with particular emphasis on the case $k = 3$.

Keywords: k-distance coloring; k-distance chromatic number; unicyclic graph.

2020 Mathematics Subject Classification: 05C15, 05C38.

1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For definitions and notation in graph theory, see [2, 5].

Let r and k be positive integers. A k-distance r-coloring of G assigns r colors so that vertices at distance at most k receive distinct colors (see [8]). The minimum number r so that there exists a k-distance r-coloring is the k-distance chromatic number, $\chi_k(G)$.

The k-th graph power G^k of G is the graph whose vertex set is $V(G)$ and in which two distinct vertices are adjacent if and only if their distance in G is at most k. Consequently, $\chi_k(G) = \chi_1(G^k)$.

Wegner's conjecture [13], concerning the 2-distance chromatic number of planar graphs, remains an open problem and has been studied in recent years (see [3, 14]). Special attention has been paid to the bounding of $\chi_k(G)$ in terms of other parameters and for specific graphs; see [1, 4, 6, 7, 9–11].

If the diameter of G, $diam(G)$, is less than or equal to k, then all pairs of vertices of G^k will be adjacent. Therefore, G^k is the complete graph and so $\chi_k(G) = |V(G)|$. Throughout the paper, all graphs have a diameter greater than k.

Let s be a nonnegative integer, $v \in V(G)$. The open s-neighbors of v are the elements of the set

$$N_s(v) = \{w \in V(G) : \ 0 < dist_G(w, v) \leq s\},$$

where $dist_G(w, v)$ is the distance between w and v in G. The closed s-neighbors of v are the elements of the set $N_s(v) \cup \{v\}$, denoted by $N_s[v]$.

In 2019, the authors of [11] improved the known lower bound for the k-distance chromatic number deduced in [12].

Theorem 1.1. [11] *Let k be a positive integer and G be a graph with $diam(G) > k$. Then*

$$\chi_k(G) \geq \begin{cases} \max\limits_{v \in V(G)} \left| N_{\frac{k}{2}}[v] \right|, & \text{if } k \text{ is even} \\ \max\limits_{uv \in E(G)} \left| N_{\frac{k-1}{2}}[u] \cup N_{\frac{k-1}{2}}[v] \right|, & \text{if } k \text{ is odd.} \end{cases}$$

Moreover, if G is a tree, the inequalities become equalities.

The same authors described the k-distance chromatic number of any cycle with n vertices. In what follows, given a real number x, $\lceil x \rceil$ and $\lfloor x \rfloor$ are, respectively, the smallest integer greater than or equal to x and the biggest integer smaller than or equal to x.

Theorem 1.2. [11] *Let n, k be positive integers, with $n \geq 3$. Let r and l be integers, with $l \geq 2$, such that $n = l(k+1) + r$, $0 \leq r < k+1$. Then $\chi_k(C_n) = k + 1 + \lceil r/l \rceil$.*

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They also suggested that, since unicyclic graphs can be obtained from a tree by adding a single edge, determining the k-distance chromatic number of a unicyclic graph might be achieved through a modest extension of their result for the k-distance chromatic number of a tree. In this paper we show that, in general, this is not true.

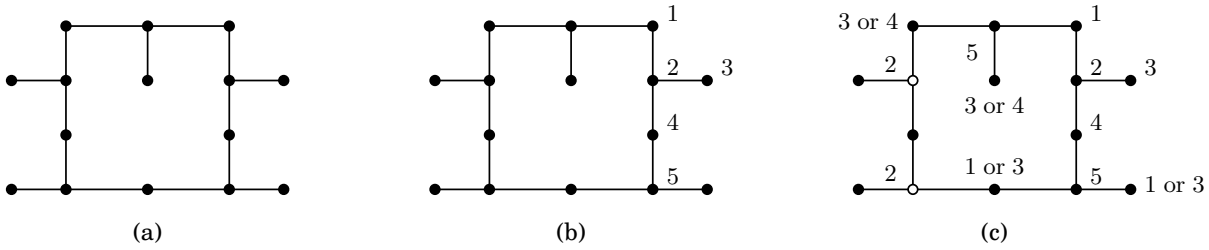


Figure 1.1: Three pictures used in Example 1.1.

Example 1.1. *Let G be the unicyclic graph, shown in Figure 1.1(a), with 15 vertices and containing the cycle C_{10}. By Theorem 1.1, $\chi_3(G) \geq 5$ and by Theorem 1.2, $\chi_3(C_{10}) = 5$. However, $\chi_3(G) \neq 5$. Suppose that $\chi_3(G) = 5$. Since there is a subgraph of G induced by 5 vertices and with diameter equal to 3, each of these 5 vertices must have a different color in a 3-distance 5-coloring of G. This implies the two pictures shown in Figures 1.1(b) and 1.1(c). In Figure 1.1(b), we assigned different colors to 5 vertices, and in Figure 1.1(c), we assigned the possible colors to each of the other vertices. In Figure 1.1(c), there are two white vertices colored with the color 2 whose distance between them is less than 3. So, $\chi_3(G) \geq 6$.*

In [11], the authors determined the 2-distance chromatic number of any connected graph in which every pair of cycles is edge disjoint. This naturally includes all unicyclic graphs with any number of vertices; therefore, this particular case is already solved.

While $\chi_k(G)$ is well understood for trees and cycles, extending results to unicyclic graphs is nontrivial due to the interaction between the cycle and attached trees. We show that such extensions are not straightforward and derive bounds and exact values for several cases.

This paper is organized as follows. Section 2 presents known results and a new definition relevant to this work. In Section 3 we characterize the k-distance chromatic number of some unicyclic graphs and provide lower bounds for others. The goal of Section 4 is to compute the 3-distance chromatic number of some unicyclic graphs.

2. Preliminaries

Throughout the paper, G is a unicyclic graph with girth $r \geq 3$. We denote its cycle by its vertices $v_1, \ldots, v_r, v_1$ or by the subgraph of G induced by the vertices of the cycle C_r.

Remark 2.1. *Let $u, v \in V(G)$. If $s = 0$, then $N_0(v) = \emptyset$ and $N_0[v] = \{v\}$ for every $v \in V(G)$. If $k \geq r$ and $dist_G(u, v) = r$, then $N_{k-r}[u] \subseteq N_k[v]$ (if $z \in N_{k-r}[u]$, then $dist_G(z, v) \leq dist_G(z, u) + dist_G(u, v) \leq k - r + r = k$).*

Definition 2.1. *Let G be a unicyclic graph and $v_1, \ldots, v_r, v_1$, with $r \geq 3$, be its cycle C_r. For $1 \leq i \leq r$, let T_{v_i} be the connected component of $G - E(C_r)$ containing v_i. Let $x \in V(T_{v_i})$ and $p = dist_G(x, v_i)$. For a nonnegative integer s, the s-exclusive neighborhood of x, denoted by $N_s^E(x)$, is $N_s^E(x) = \{y \in V(T_{v_i}) : dist_G(x, y) = s \ \wedge \ dist_G(y, v_i) = p + s\}$.*

Remark 2.2. *Let $1 \leq i, j \leq r$, $i \neq j$, then*

(i). $N_0^E(x) = \{x\}$.

(ii). If $x \in N_s^E(v_i)$, then $dist_G(x, v_j) = dist_G(x, v_i) + dist_G(v_i, v_j)$.

(iii). If $s, l \geq 0$, then $N_s^E(v_i) \cap N_l^E(v_j) = \emptyset$.

(iv). If $s, l \geq 0$, $s \neq l$, then $N_s^E(v_j) \cap N_l^E(v_j) = \emptyset$.

For convenience, with $k \geq 2$, define:

$$L_G^k = \max_{v \in V(G)} \left| N_{\frac{k}{2}}[v] \right| \quad (k \text{ even}) \qquad \text{and} \qquad M_G^k = \max_{uv \in E(G)} \left| N_{\frac{k-1}{2}}[u] \cup N_{\frac{k-1}{2}}[v] \right| \quad (k \text{ odd}).$$

3. k-Distance Coloring of Some Unicyclic Graphs

First, we derive lower bounds for the case where k is even.

For convenience, with $r \geq 6$, k an even integer, $\left\lfloor \frac{r}{2} \right\rfloor \leq k$ and $s = \left\lceil \frac{\left\lfloor \frac{r}{2} \right\rfloor}{2} \right\rceil$, define:

$$B_{k,r} = \left\{ \sum_{v \in S} \left| N_{\frac{k}{2}-s+1}^E(v) \right| : S \subseteq V(C_r), |S| = \left\lfloor \frac{r}{2} \right\rfloor, \forall\, x, y \in S, dist_G(x, y) \leq \left\lfloor \frac{r}{2} \right\rfloor - 1 \right\} \text{ and } Q_{k,r} = \max\{a : \ a \in B_{k,r}\}.$$

Lemma 3.1. *Let G be a unicyclic graph and $v_1, \ldots, v_r, v_1$, with $r \geq 3$, be the cycle C_r of G. Let k be an even integer, $\lfloor \frac{r}{2} \rfloor \leq k$ and $s = \lceil \lfloor \frac{r}{2} \rfloor / 2 \rceil$.*

1. *Then the subgraph of G induced by $\bigcup_{i=1}^{r} N_{\frac{k}{2}-s}[v_i]$ has a diameter of at most k.*
2. *If $r \geq 6$, the integer $\lfloor \frac{r}{2} \rfloor$ is odd, and H is a subset of vertices of the cycle that maximize the cardinality of*

$$\bigcup_{\substack{v \in S,\ S \subseteq V(C_r),\ |S| = \lfloor \frac{r}{2} \rfloor, \\ \forall\ x,y \in S,\ dist_G(x,y) \leq \lfloor \frac{r}{2} \rfloor - 1}} N^E_{\frac{k}{2}-s+1}(v),$$

then the subgraph of G induced by $\bigcup_{i=1}^{r} N_{\frac{k}{2}-s}[v_i] \cup \bigcup_{v \in H} N^E_{\frac{k}{2}-s+1}(v)$ has a diameter of at most k.

Proof.

1. Let $i, j \in \{1, \ldots, r\}$ and $x \in N_{\frac{k}{2}-s}[v_i], y \in N_{\frac{k}{2}-s}[v_j]$. Since $dist_G(v_i, v_j) \leq \lfloor \frac{r}{2} \rfloor$, we have
$$dist_G(x, y) \leq dist_G(x, v_i) + dist_G(v_i, v_j) + dist_G(v_j, y) \leq k.$$
2. Let $x, y \in \left(\bigcup_{i=1}^{r} N_{\frac{k}{2}-s}[v_i] \cup \bigcup_{v \in H} N^E_{\frac{k}{2}-s+1}(v) \right)$. Since $\lfloor \frac{r}{2} \rfloor$ is odd, we have $2s = \lfloor \frac{r}{2} \rfloor + 1$.
   (a) If x and y belong to $\bigcup_{i=1}^{r} N_{\frac{k}{2}-s}[v_i]$, then by part 1, we get $dist_G(x, y) \leq k$.
   (b) If $x \in \bigcup_{i=1}^{r} N_{\frac{k}{2}-s}[v_i]$ and $y \in \bigcup_{v \in H} N^E_{\frac{k}{2}-s+1}(v)$, then there exist $m \in \{1, \ldots, r\}$ and $v_n \in H$ such that $x \in N_{\frac{k}{2}-s}[v_m]$ and $y \in N^E_{\frac{k}{2}-s+1}(v_n)$. Then, $dist_G(x, y) \leq dist_G(x, v_m) + dist_G(v_m, v_n) + dist_G(v_n, y) \leq k$.
   (c) If $x, y \in \bigcup_{v \in H} N^E_{\frac{k}{2}-s+1}(v)$, then there exist $v_n, v_m \in H$ such that $x \in N^E_{\frac{k}{2}-s+1}(v_n), y \in N^E_{\frac{k}{2}-s+1}(v_m)$. Since $dist_G(v_n, v_m) \leq \lfloor \frac{r}{2} \rfloor - 1$, we get $dist_G(x, y) \leq dist_G(x, v_n) + dist_G(v_n, v_m) + dist_G(v_m, y) \leq k$. □

Proposition 3.1. *Let G be a unicyclic graph and $v_1, \ldots, v_r, v_1$, with $r \geq 3$, be the cycle C_r of G. Let k be an even integer, $\lfloor \frac{r}{2} \rfloor \leq k$ and $s = \left\lceil \frac{\lfloor \frac{r}{2} \rfloor}{2} \right\rceil$.*

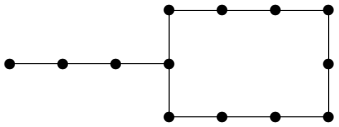
1. *Then $\chi_k(G) \geq \max \left\{ L^k_G, \left| \bigcup_{i=1}^{r} N_{\frac{k}{2}-s}[v_i] \right| \right\}$.*
2. *If $r \geq 6$ and the integer $\lfloor \frac{r}{2} \rfloor$ is odd, then $\chi_k(G) \geq \max \left\{ L^k_G, \left| \bigcup_{i=1}^{r} N_{\frac{k}{2}-s}[v_i] \right| + Q_{k,r} \right\}$.*

Proof.

1. This follows from Lemma 3.1 and Theorem 1.1.
2. Let H be a set such that $H \subseteq V(C_r),\ |H| = \lfloor \frac{r}{2} \rfloor,\ \forall\ x, y \in H,\ dist_G(x, y) \leq \lfloor \frac{r}{2} \rfloor - 1$ and $\sum_{v \in H} \left| N^E_{\frac{k}{2}-s+1}(v) \right| = Q_{k,r}$. By Remark 2.2, we get
$$\left| \bigcup_{v \in H} N^E_{\frac{k}{2}-s+1}(v) \right| = \sum_{v \in H} |N^E_{\frac{k}{2}-s+1}(v)| = Q_{k,r}.$$
Using the fact that $\forall\ i \in \{1, \ldots, r\},\ \forall\ v \in H,\ \ N_{\frac{k}{2}-s}[v_i] \cap N^E_{\frac{k}{2}-s+1}(v) = \emptyset$, Lemma 3.1 and Theorem 1.1, the result follows. □

The last result provides a lower bound for $\chi_k(G)$ that, in certain cases, enables us to obtain a better lower bound than that given in Theorem 1.1.

Example 3.1. *Let G be the following unicyclic graph with 13 vertices and containing the cycle C_{10}.*



If $k = 6$ then $s = \lceil \frac{5}{2} \rceil = 3$. Therefore, $L^k_G = 10$ and $\left| \bigcup_{i=1}^{10} N_{\frac{6}{2}-3}[v_i] \right| + Q_{6,10} = 10 + 1 = 11$. By Proposition 3.1, $\chi_6(G) \geq 11$.

Proposition 3.2. *Let k and p be positive integers, with $k \geq 2$. Let G be a unicyclic graph and $v_1, \ldots, v_r, v_1$, with $r \geq 3$, be its cycle. If k is even and the subgraph of G induced by $\bigcup_{i=1}^{r} N_{\frac{k}{2}-1}[v_i]$ is k-distance p-colored, then $\chi_k(G) \leq \max\{L^k_G, p\}$.*

Proof. Consider a k-distance p-coloring of the subgraph of G induced by $\bigcup_{i=1}^{r} N_{\frac{k}{2}-1}[v_i]$. If $V(G) = \bigcup_{i=1}^{r} N_{\frac{k}{2}-1}[v_i]$, then $\chi_k(G) \leq p$.

If $V(G) \setminus \bigcup_{i=1}^{r} N_{\frac{k}{2}-1}[v_i] \neq \emptyset$, then there exists $1 \leq i \leq r$ such that $N_{\frac{k}{2}}[v_i]$ has uncolored vertices. Let

$$A_{v_i} = \{y \in N_{\frac{k}{2}}[v_i] : y \text{ is uncolored}\}.$$

Note that, $A_{v_i} = N^E_{\frac{k}{2}}(v_i)$. Since $\left|N_{\frac{k}{2}}[v_i]\right| \leq L^k_G \leq \max\{L^k_G, p\}$, using $\max\{L^k_G, p\}$ colors, assign to each vertex of A_{v_i} a different color and distinct from the colors assigned to the vertices of $N_{\frac{k}{2}}[v_i] \setminus A_{v_i}$. Note that, if $y \in A_{v_i}$, $z \notin N_{\frac{k}{2}}[v_i]$ and z already has a color assigned, then $dist_G(y,z) = dist_G(y,v_i) + dist_G(v_i,z) > \frac{k}{2} + \frac{k}{2} = k$. Repeat this process for every vertex in the cycle with uncolored vertices in its $\frac{k}{2}$-neighborhood. In this way, we assign colors to all vertices of $\bigcup_{i=1}^{r} N_{\frac{k}{2}}[v_i]$.

If $V(G) = \bigcup_{i=1}^{r} N_{\frac{k}{2}}[v_i]$, then $\chi_k(G) \leq \max\{L^k_G, p\}$. Otherwise, there exists a vertex x of G such that $dist_G(x,v) = 1$, for some vertex v in the cycle of G and $N_{\frac{k}{2}}[x]$ has uncolored vertices.

Consider the set $A_x = \{y \in N_{\frac{k}{2}}[x] : y \text{ is uncolored}\}$ and repeat the previous process using x instead of v_i. Do the same for every vertex under the same conditions as x. Therefore, we assign colors to all the vertices of $\bigcup_{i=1}^{r} N_{\frac{k}{2}+1}[v_i]$.

If $V(G) = \bigcup_{i=1}^{r} N_{\frac{k}{2}+1}[v_i]$, then $\chi_k(G) \leq \max\{L^k_G, p\}$. Otherwise, there exists a vertex z such that $dist_G(z,v) = 2$, for some vertex v in the cycle of G and $N_{\frac{k}{2}}[z]$ has uncolored vertices.

If we repeat this process until we find an integer ℓ such that $V(G) = \bigcup_{i=1}^{r} N_{\frac{k}{2}+\ell}[v_i]$ and all these vertices have a color assigned from the $\max\{L^k_G, p\}$ colors, then $\chi_k(G) \leq \max\{L^k_G, p\}$. □

Theorem 3.1. *Let G be a unicyclic graph and $v_1, \ldots, v_r, v_1$, with $3 \leq r \leq 5$, be its cycle. If k is an even integer, $k \geq 2$, then* $\chi_k(G) = \max\{L^k_G, \left|\bigcup_{i=1}^{r} N_{\frac{k}{2}-1}[v_i]\right|\}$.

Proof. Let $p = \left|\bigcup_{i=1}^{r} N_{\frac{k}{2}-1}[v_i]\right|$. Using Proposition 3.1, with $3 \leq r \leq 5$, we get $s = 1$ and $\chi_k(G) \geq \max\{L^k_G, p\}$. Assign a different color to every vertex of $\bigcup_{i=1}^{r} N_{\frac{k}{2}-1}[v_i]$. Then we have a k-distance p-coloring of $\bigcup_{i=1}^{r} N_{\frac{k}{2}-1}[v_i]$. By Proposition 3.2, we have $\chi_k(G) \leq \max\{L^k_G, p\}$. Thus, the result follows. □

Corollary 3.1. *Let G be a unicyclic graph and v_1, v_2, v_3, v_1 be its cycle. If k is an even integer, $k \geq 2$, then $\chi_k(G) = L^k_G$.*

Proof. If $x \in \bigcup_{i=1}^{3} N_{\frac{k}{2}-1}[v_i]$, then there exists $i \in \{1,2,3\}$ such that $dist_G(x,v_i) \leq \frac{k}{2} - 1$. So, for $j \in \{1,2,3\}$,

$$dist_G(x,v_j) \leq dist_G(x,v_i) + dist_G(v_i,v_j) \leq \frac{k}{2} \quad \text{and} \quad \bigcup_{i=1}^{3} N_{\frac{k}{2}-1}[v_i] \subseteq N_{\frac{k}{2}}[v_j].$$

Consequently, $\left|\bigcup_{i=1}^{3} N_{\frac{k}{2}-1}[v_i]\right| \leq \max_{j\in\{1,2,3\}} |N_{\frac{k}{2}}[v_j]| \leq L^k_G$. By Theorem 3.1 with $r = 3$, we get $\chi_k(G) = L^k_G$. □

Theorem 3.2. *Let G be a unicyclic graph and $v_1, v_2, \ldots, v_6, v_1$ be its cycle. Let k be an even integer, $k \geq 4$. Then*

$$\chi_k(G) = \max\left\{L^k_G, \left|\bigcup_{i=1}^{6} N_{\frac{k}{2}-2}[v_i]\right| + Q_{k,6}\right\}.$$

Proof. By Proposition 3.1, we have $s = 2$ and $\chi_k(G) \geq \max\left\{L^k_G, \left|\bigcup_{i=1}^{6} N_{\frac{k}{2}-2}[v_i]\right| + Q_{k,6}\right\}$.

Let $H = \{v_j, v_t, v_l\} \subseteq V(C_6)$ be a set of vertices such that $|H| = 3$, $\forall\, x, y \in H$, $dist_G(x,y) \leq 2$ and $\sum_{v\in H} \left|N^E_{\frac{k}{2}-1}(v)\right| = Q_{k,6}$. Assign to each vertex in

$$\bigcup_{i=1}^{6} N_{\frac{k}{2}-2}[v_i] \cup \bigcup_{v\in\{v_j,v_t,v_l\}} N^E_{\frac{k}{2}-1}(v)$$

a different color.

For each $v \in \{v_j, v_t, v_l\}$ denote by $o(v)$ the vertex of C_6 such that $dist_G(o(v), v) = 3$. Since

$$Q_{k,6} = \sum_{z\in H} \left|N^E_{\frac{k}{2}-1}(z)\right| \geq \sum_{z\in H\setminus\{v\}} \left|N^E_{\frac{k}{2}-1}(z)\right| + \left|N^E_{\frac{k}{2}-1}(o(v))\right|,$$

we have $|N^E_{\frac{k}{2}-1}(v)| \geq |N^E_{\frac{k}{2}-1}(o(v))|$. This implies that, for each $v \in \{v_j, v_t, v_l\}$, we can use the colors of the vertices of $N^E_{\frac{k}{2}-1}(v)$ to assign a different color to each vertex of $N^E_{\frac{k}{2}-1}(o(v))$. So, $\bigcup_{i=1}^{6} N_{\frac{k}{2}-1}[v_i]$ is k-distance p-colored with $p = \left|\bigcup_{i=1}^{6} N_{\frac{k}{2}-2}[v_i]\right| + Q_{k,6}$. By Proposition 3.2, $\chi_k(G) \leq \max\{L^k_G, p\}$. Then, $\chi_k(G) = \max\left\{L^k_G, \left|\bigcup_{i=1}^{6} N_{\frac{k}{2}-2}[v_i]\right| + Q_{k,6}\right\}$. □

As we will see, the results are more complicated when k is odd.

For convenience, with $r \geq 3$, k an odd integer, $k \geq 3$, and $w = 2\left\lfloor \frac{r}{2} \right\rfloor - 1 \leq k$, we define:

$$R_{k,r} = \max\left\{ \sum_{v \in S} \left| N^E_{\frac{k-w+2}{2}}(v) \right| : \ S = \{v_i, v_j\} \subseteq V(C_r), \ v_i v_j \in E(G) \right\}.$$

Lemma 3.2. *Let G be a unicyclic graph and $v_1, \ldots, v_r, v_1$, with $r \geq 3$, be the cycle C_r of G. Let k be an odd integer, $k \geq 3$ and $w = 2\left\lfloor \frac{r}{2} \right\rfloor - 1 \leq k$.*

1. *Then the subgraph of G induced by $\bigcup_{i=1}^{r} N_{\frac{k-w}{2}}[v_i]$ has a diameter of at most k.*
2. *If $r \geq 4$, then the subgraph of G induced by $\bigcup_{i=1}^{r} N_{\frac{k-w}{2}}[v_i] \cup N^E_{\frac{k-w+2}{2}}(v_j) \cup N^E_{\frac{k-w+2}{2}}(v_t)$ has a diameter of at most k, where v_j and v_t are vertices of C_r such that $v_j v_t \in E(G)$ and*

$$\left| N^E_{\frac{k-w+2}{2}}(v_j) \cup N^E_{\frac{k-w+2}{2}}(v_t) \right| = \max_{uv \in E(C_r)} \left| N^E_{\frac{k-w+2}{2}}(v) \cup N^E_{\frac{k-w+2}{2}}(u) \right|.$$

Proof.

1. Let $x, y \in \bigcup_{i=1}^{r} N_{\frac{k-w}{2}}[v_i]$. Then, there exist $l, m \in \{1, \ldots, r\}$ such that $x \in N_{\frac{k-w}{2}}[v_l]$ and $y \in N_{\frac{k-w}{2}}[v_m]$. Consequently, $dist_G(x, y) \leq dist_G(x, v_l) + dist_G(v_l, v_m) + dist_G(v_m, y) \leq \frac{k-w}{2} + \left\lfloor \frac{r}{2} \right\rfloor + \frac{k-w}{2} \leq k$.
2. Let $x, y \in \bigcup_{i=1}^{r} N_{\frac{k-w}{2}}[v_i] \cup N^E_{\frac{k-w+2}{2}}(v_j) \cup N^E_{\frac{k-w+2}{2}}(v_t)$. We have three cases to consider:
   - If $x, y \in \bigcup_{i=1}^{r} N_{\frac{k-w}{2}}[v_i]$, then by 1., $dist_G(x, y) \leq k$.
   - If $x \in \bigcup_{i=1}^{r} N_{\frac{k-w}{2}}[v_i]$ and $y \in N^E_{\frac{k-w+2}{2}}(v_m)$, with $m \in \{j, t\}$, then there exists $l \in \{1, \ldots, r\}$ such that $x \in N_{\frac{k-w}{2}}[v_l]$. Thus, $dist_G(x, y) \leq dist_G(x, v_l) + dist_G(v_l, v_m) + dist_G(v_m, y) \leq \frac{k-w}{2} + \left\lfloor \frac{r}{2} \right\rfloor + \frac{k-w+2}{2} = k - \left\lfloor \frac{r}{2} \right\rfloor + 2 \leq k$.
   - If $x \in N^E_{\frac{k-w+2}{2}}(v_m)$ and $y \in N^E_{\frac{k-w+2}{2}}(v_n)$, with $m, n \in \{j, t\}$, then

$$dist_G(x, y) \leq dist_G(x, v_m) + dist_G(v_m, v_n) + dist_G(v_n, y) \leq k.$$

□

Proposition 3.3. *Let G be a unicyclic graph and $v_1, \ldots, v_r, v_1$, with $r \geq 3$, be the cycle C_r of G. Let k be an odd integer, $k \geq 3$, and $w = 2\left\lfloor \frac{r}{2} \right\rfloor - 1 \leq k$.*

1. *Then $\chi_k(G) \geq \max\left\{ M_G^k, \left| \bigcup_{i=1}^{r} N_{\frac{k-w}{2}}[v_i] \right| \right\}$.*
2. *If $r \geq 4$ then $\chi_k(G) \geq \max\left\{ M_G^k, \left| \bigcup_{i=1}^{r} N_{\frac{k-w}{2}}[v_i] \right| + R_{k,r} \right\}$.*

Proof.

1. This result follows immediately from Lemma 3.2 and Theorem 1.1.
2. Let H be a set of vertices such that $H = \{v_i, v_j\} \subseteq V(C_r), \ v_i v_j \in E(G)$, $\sum_{v \in H} \left| N^E_{\frac{k-w+2}{2}}(v) \right| = R_{k,r}$. By Remark 2.2,

$$\left| \bigcup_{v \in H} N^E_{\frac{k-w+2}{2}}(v) \right| = \sum_{v \in H} \left| N^E_{\frac{k-w+2}{2}}(v) \right| = R_{k,r}.$$

Using the fact that $\forall\, i \in \{1, \ldots, r\}, \ \forall\, v \in H, \ \ N_{\frac{k-w}{2}}[v_i] \cap N^E_{\frac{k-w+2}{2}}(v) = \emptyset$, Lemma 3.2 and Theorem 1.1, we get the result.

□

Proposition 3.4. *Let k and p be positive integers, with $k \geq 2$. Let G be a unicyclic graph and $v_1, v_2, \ldots, v_r, v_1$ be its cycle. If k is odd and the subgraph of G induced by $\bigcup_{i=1}^{r} N_{\frac{k-1}{2}}[v_i]$ is k-distance p-colored, then $\chi_k(G) \leq \max\{M_G^k, p\}$.*

Proof. Consider a k-distance p-coloring of the subgraph of G induced by $\bigcup_{i=1}^{r} N_{\frac{k-1}{2}}[v_i]$. If $V(G) = \bigcup_{i=1}^{r} N_{\frac{k-1}{2}}[v_i]$, then $\chi_k(G) \leq p$.

If $V(G) \setminus \bigcup_{i=1}^{r} N_{\frac{k-1}{2}}[v_i] \neq \emptyset$, there exists $1 \leq i \leq r$ such that $N_{\frac{k+1}{2}}[v_i]$ has uncolored vertices. Let $y_1, \ldots, y_h$ be the vertices adjacent to v_i with uncolored vertices in their $\frac{k-1}{2}$-neighborhood.

For each $1 \leq j \leq h$, let $A_{y_j} = \{y \in N_{\frac{k-1}{2}}[y_j] : y \text{ is uncolored}\}$. Note that $A_{y_j} = N^E_{\frac{k-1}{2}}(y_j)$. Since

$$\left|N_{\frac{k-1}{2}}[y_j] \cup N_{\frac{k-1}{2}}[v_i]\right| \leq M^k_G \leq \max\{M^k_G, p\},$$

using the $\max\{M^k_G, p\}$ colors, assign to each vertex of A_{y_j} a different color and distinct from the colors assigned to the vertices of $\left(N_{\frac{k-1}{2}}[y_j] \cup N_{\frac{k-1}{2}}[v_i]\right) \setminus A_{y_j}$.

Notice that if $z_1 \in N^E_{\frac{k-1}{2}}(y_j)$ and $z_2 \in N^E_{\frac{k-1}{2}}(y_l)$, with $1 \leq j, l \leq h, j \neq l$, then

$$dist_G(z_1, z_2) = dist_G(z_1, y_j) + dist_G(y_j, y_l) + dist_G(y_l, z_2) = k + 1.$$

Repeat this process for each vertex in the cycle that has adjacent vertices with uncolored vertices in its $\frac{k-1}{2}$-neighborhood. In this way, using the $\max\{M^k_G, p\}$ colors, we assign colors to all the vertices of $\bigcup_{i=1}^r N_{\frac{k+1}{2}}[v_i]$.

If $V(G) = \bigcup_{i=1}^r N_{\frac{k+1}{2}}[v_i]$, then $\chi_k(G) \leq \max\{M^k_G, p\}$. Otherwise, there exists a vertex x such that $dist_G(x, v) = 1$, for some vertex v in the cycle of G and there exist vertices $y'_1, \ldots, y'_s$ adjacent to x with uncolored vertices in their $\frac{k-1}{2}$-neighborhoods.

For each $1 \leq u \leq s$, consider the set $A_{y'_u} = \{y \in N_{\frac{k-1}{2}}[y'_u] : y \text{ is uncolored}\}$ and repeat the previous process with y'_u instead of y_j and with x instead of v_i. In this way, using the $\max\{M^k_G, p\}$ colors, we assign colors to all the vertices of $\bigcup_{i=1}^r N_{\frac{k+3}{2}}[v_i]$.

If $V(G) = \bigcup_{i=1}^r N_{\frac{k+3}{2}}[v_i]$, then $\chi_k(G) \leq \max\{M^k_G, p\}$. Otherwise, we repeat this process until we find an integer t such that $V(G) = \bigcup_{i=1}^r N_{\frac{k-1}{2}+t}[v_i]$ and, using the $\max\{M^k_G, p\}$ colors, all these vertices have a color assigned. Thus, $\chi_k(G) \leq \max\{M^k_G, p\}$. □

Theorem 3.3. *Let G be a unicyclic graph and v_1, v_2, v_3, v_1 be its cycle. If k is an odd integer such that $k \geq 3$, then* $\chi_k(G) = \max\left\{M^k_G, \left|\bigcup_{i=1}^3 N_{\frac{k-1}{2}}[v_i]\right|\right\}$.

Proof. Let $p = \left|\bigcup_{i=1}^3 N_{\frac{k-1}{2}}[v_i]\right|$. By Proposition 3.3 with $r = 3$ and $w = 1$, we have $\chi_k(G) \geq \max\{M^k_G, p\}$. Assign a different color to each vertex of $\bigcup_{i=1}^3 N_{\frac{k-1}{2}}[v_i]$. By Proposition 3.4, we get $\chi_k(G) \leq \max\{M^k_G, p\}$. Therefore, the result follows. □

Theorem 3.4. *Let G be a unicyclic graph and v_1, v_2, v_3, v_4, v_1 be its cycle. If k is an odd integer, $k \geq 3$, then*

$$\chi_k(G) = \max\left\{M^k_G, \left|\bigcup_{i=1}^4 N_{\frac{k-3}{2}}[v_i]\right| + R_{k,4}\right\}.$$

Proof. Since $w = 3$, consider $p = |\bigcup_{i=1}^4 N_{\frac{k-3}{2}}[v_i]| + R_{k,4}$. Using Proposition 3.3, we get $\chi_k(G) \geq \max\{M^k_G, p\}$. Let $H = \{v_j, v_t\}$ be a set of vertices such that $H \subseteq V(C_4)$, $v_jv_t \in E(G)$, $R_{k,4} = \sum_{v \in H}\left|N^E_{\frac{k-1}{2}}(v)\right|$. Assign a different color to each vertex of $\bigcup_{i=1}^4 N_{\frac{k-3}{2}}[v_i] \cup N^E_{\frac{k-1}{2}}(v_j) \cup N^E_{\frac{k-1}{2}}(v_t)$. Thus, we use p colors.

For each $v \in \{v_j, v_t\}$ denote by $o(v)$ the vertex of C_4 such that $dist_G(o(v), v) = 2$. Since

$$R_{k,4} = \sum_{z \in H}\left|N^E_{\frac{k-1}{2}}(z)\right| \geq \sum_{z \in H \setminus \{v\}}\left|N^E_{\frac{k-1}{2}}(z)\right| + \left|N^E_{\frac{k-1}{2}}(o(v))\right|,$$

we conclude that

$$|N^E_{\frac{k-1}{2}}(v)| \geq |N^E_{\frac{k-1}{2}}(o(v))|.$$

This implies that, for each $v \in \{v_j, v_t\}$, we can use the colors of the vertices of $N^E_{\frac{k-1}{2}}(v)$ to assign a different color to each vertex of $N^E_{\frac{k-1}{2}}(o(v))$. By Proposition 3.4, with

$$p = \left|\bigcup_{i=1}^4 N_{\frac{k-3}{2}}[v_i]\right| + R_{k,4},$$

we get $\chi_k(G) \leq \max\{M^k_G, p\}$. Then,

$$\chi_k(G) = \max\left\{M^k_G, \left|\bigcup_{i=1}^4 N_{\frac{k-3}{2}}[v_i]\right| + R_{k,4}\right\}.$$

□

4. The 3-Distance Chromatic Number of a Class of Unicyclic Graphs

We now focus on $k = 3$.

Lemma 4.1. *Let G be a unicyclic graph and $v_1, \ldots, v_{4l+s}, v_1$ be its cycle, C_{4l+s}, with $0 \leq s \leq 3$ and $l \geq 2$. Let $\mathcal{G}$ be the subgraph of G induced by $\bigcup_{i=1}^{4l+s} N_1[v_i]$. Then,*

$$\min\{M_G^3, \chi_3(\mathcal{G})\} \geq M_{\mathcal{G}}^3 = \max_{v_iv_j \in E(C_{4l+s})} |N_1[v_i] \cup N_1[v_j]| = \max_{v_iv_j \in E(C_{4l+s})} (|N_1(v_i)| + |N_1(v_j)|)\,.$$

Proof. Since $\mathcal{G}$ is a subgraph of G, $M_G^3 \geq M_{\mathcal{G}}^3$. By Theorem 1.1, $\chi_3(\mathcal{G}) \geq M_{\mathcal{G}}^3$. On the other hand, if v is a vertex of $\mathcal{G}$ but not of the cycle then v is a leaf of $\mathcal{G}$. So, $M_{\mathcal{G}}^3$ is obtained with an edge on the cycle. Moreover, if $v_iv_j \in E(C_{4l+s})$ then $v_i \in N_1(v_j)$ and $v_j \in N_1(v_i)$. Thus, we get the result. □

Lemma 4.2. *Let G be a unicyclic graph and $v_1, \ldots, v_{4l+s}, v_1$ be its cycle, with $l \geq 2$ and $s \in \{0, 2\}$. Let $\mathcal{G}$ be the subgraph of G induced by $\bigcup_{i=1}^{4l+s} N_1[v_i]$. Then $\chi_3(\mathcal{G}) \leq \frac{s}{2} + M_{\mathcal{G}}^3$.*

Proof. Using Lemma 4.1 we may assume that $H = \{v_1, v_2\}$ and $\sum_{v \in H} |N_1(v)| = M_{\mathcal{G}}^3$.

By Theorem 1.2, $\chi_3(C_{4l+s}) = 4 + \frac{s}{2}$. For $1 \leq i \leq 4l-8$, color the vertex v_i with $(i-1)(mod\ 4)+1$, for $4l-7 \leq i \leq 4l-4+\frac{s}{2}$, color the vertex v_i with $i - 4l + 8$, for $4l-3+\frac{s}{2} \leq i \leq 4l+s$, color the vertex v_i with $i - 4l + 4 - \frac{s}{2}$ and assign to each vertex of $N_1^E(v_1) \cup N_1^E(v_2)$ a different color and distinct from the $4 + \frac{s}{2}$ colors used to color the vertices of C_{4l+s}. Thus, we use $4 + |N_1^E(v_1) \cup N_1^E(v_2)| + \frac{s}{2} = M_{\mathcal{G}}^3 + \frac{s}{2}$ colors.

Label the colors of $N_1^E(v_1) \cup N_1^E(v_2)$ with numbers. For $3 \leq i \leq 4l+s$, assign distinct colors to the vertices of $N_1^E(v_i)$ from among the labeled colors assigned to the vertices of $N_1^E(v_j)$, where $j \in \{1, 2\}$ and $i \equiv j \pmod 2$, using first the colors with the smallest labels.

Since $|N_1^E(v_1) \cup N_1^E(v_{4l+s})| \leq |N_1^E(v_1) \cup N_1^E(v_2)| = M_{\mathcal{G}}^3 - 4$, we conclude that $|N_1^E(v_{4l+s})| \leq |N_1^E(v_2)|$. Thus, all vertices of $N_1^E(v_{4l+s})$ have been assigned a color. If there exists $3 \leq \ell \leq 4l+s-1$ such that $N_1^E(v_\ell)$ has uncolored vertices, since $|N_1^E(v_\ell) \cup N_1^E(v_{\ell+1})| \leq M_{\mathcal{G}}^3 - 4$ and $|N_1^E(v_\ell) \cup N_1^E(v_{\ell-1})| \leq M_{\mathcal{G}}^3 - 4$, then assign to each uncolored vertex of $N_1^E(v_\ell)$ a different color from the labeled colors of the vertices of $N_1^E(v_j)$, with $j \in \{1, 2\}$ and $\ell \equiv (j-1) \pmod 2$, not used in the vertices of $N_1^E(v_{\ell-1}) \cup N_1^E(v_{\ell+1})$.

Because

- if $x \in N_1^E(v_i)$ and $y \in N_1^E(v_j)$, with $i, j \in \{1, \ldots, 4l+s\}$, $i \neq j$ and $i \equiv j \pmod 2$, then $dist_G(x, y) > 3$ and
- if $s = 2$ then only the vertices v_{4l+2} and v_{4l-3} are colored with the color 5,

we get a 3-distance $(\frac{s}{2} + M_{\mathcal{G}}^3)$-coloring of $\mathcal{G}$. Consequently, $\chi_3(\mathcal{G}) \leq \frac{s}{2} + M_{\mathcal{G}}^3$. □

Theorem 4.1. *Let G be a unicyclic graph and C_{4l} be its cycle, with $l \geq 2$. Then $\chi_3(G) = M_G^3$.*

Proof. By Theorem 1.1, we have $\chi_3(G) \geq M_G^3$. Let $v_1, \ldots, v_{4l}, v_1$ be the cycle C_{4l} of G. Let $\mathcal{G}$ be the subgraph of G induced by $\bigcup_{i=1}^{4l} N_1[v_i]$. By Lemma 4.2, $\chi_3(\mathcal{G}) \leq M_{\mathcal{G}}^3$. Using Lemma 4.1 we get $\chi_3(\mathcal{G}) = M_{\mathcal{G}}^3 \leq M_G^3$. By Proposition 3.4 we have $\chi_3(G) \leq M_G^3$. Therefore, $\chi_3(G) = M_G^3$. □

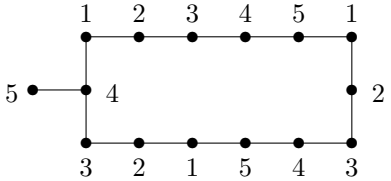
Theorem 4.2. *Let G be a unicyclic graph and $v_1, \ldots, v_{4l+2}, v_1$ be its cycle, with $l \geq 2$. Let $\mathcal{G}$ be the subgraph of G induced by $\bigcup_{i=1}^{4l+2} N_1[v_i]$. Then, $M_G^3 \leq \chi_3(G) \leq \max\{M_G^3, M_{\mathcal{G}}^3 + 1\}$.*

Proof. By Lemma 4.2 we have $\chi_3(\mathcal{G}) \leq M_{\mathcal{G}}^3 + 1$. Using Proposition 3.4 we get the result. □

The bounds given in Theorem 4.2 are sharp:

Example 4.1.

- *Let G be the following unicyclic graph with 15 vertices, where the numbers represent the color assigned to the corresponding vertex.*



In this case, $\chi_3(G) = M_G^3 = 5$.

- *Let G be the unicyclic graph of Example 1.1, where we showed that $\chi_3(G) \geq 6$. By Theorem 4.2, $\chi_3(G) \leq M_G^3 + 1 = 5 + 1 = 6$. Consequently, $\chi_3(G) = M_G^3 + 1 = 6$.*

Theorem 4.3. *Let G be a unicyclic graph and $v_1, \ldots, v_{4l+s}, v_1$ be its cycle, with $s \in \{1,3\}$ and $l \geq \max\{2,s\}$. Let $\mathcal{G}$ be the subgraph of G induced by $\bigcup_{i=1}^{4l+s} N_1[v_i]$. Let $\rho = \min_{1\leq i\leq 4l+s} \left\{\left|N_1^E(v_i)\right|\right\}$. Then,*

$$M_{\mathcal{G}}^3 \leq \chi_3(\mathcal{G}) \leq M_{\mathcal{G}}^3 + \max\{\rho, 1\} \quad \textit{and} \quad M_G^3 \leq \chi_3(G) \leq \max\{M_{\mathcal{G}}^3 + \max\{\rho,1\}, M_G^3\}.$$

Proof. By Lemma 4.1, we get $\chi_3(\mathcal{G}) \geq M_{\mathcal{G}}^3$. Now we will describe a 3-distance $(M_{\mathcal{G}}^3 + \max\{\rho,1\})$-coloring of $\mathcal{G}$. Suppose that $H = \{v_1, v_2\}$ and $\sum_{v\in H} |N_1(v)| = M_{\mathcal{G}}^3$. Using the definition of $M_{\mathcal{G}}^3$ and ρ, there is $j \in \{3, \ldots, 4l+s\}$ such that $|N_1^E(v_j)| = \rho$.

- If $s = 1$, let $h \in \{3, \ldots, 4l+1\}$ such that $dist_G(v_j, v_h) = 3$. Assign to the vertex v_h the color 5, for $1 \leq i \leq h-1$, assign to the vertex v_i the color $(i-1)(mod\ 4)+1$ and for $h+1 \leq i \leq 4l+1$, assign to the vertex v_i the color $(i-2)(mod\ 4)+1$.
- If $s = 3$, let $\{h, f, \ell\} \subseteq \{1, \ldots, 4l+3\}$, $|\{h,f,\ell\}| = 3$ such that $dist_G(v_j, v_h) = dist_G(v_j, v_f) = 3$, $dist_G(v_f, v_\ell) = 4$ and $dist_G(v_j, v_\ell) = 7$. Let $\{x, y, z\} = \{f, \ell, h\}$ be such that $x < y < z$. Assign to the vertices v_x, v_y, v_z the color 5, for $1 \leq i \leq x-1$, assign to the vertex v_i the color $(i-1)(mod\ 4)+1$, for $x+1 \leq i \leq y-1$, assign to the vertex v_i the color $(i-2)(mod\ 4)+1$, for $y+1 \leq i \leq z-1$, assign to the vertex v_i the color $(i-3)(mod\ 4)+1$ and for $z+1 \leq i \leq 4l+3$, assign to the vertex v_i the color $(i-4)(mod\ 4)+1$.

In this way, we obtain a 3-distance 5-coloring of C_{4l+s}. Assign to each vertex of $N_1^E(v_1) \cup N_1^E(v_2)$ a different color and distinct from the 5 colors used to color the vertices of C_{4l+s}.

Since $M_{\mathcal{G}}^3 = |N_1(v_1)| + |N_1(v_2)| = |N_1^E(v_1)| + |N_1^E(v_2)| + 4$ and for each $3 \leq i \leq j-1$, $|N_1^E(v_{i-1})| + |N_1^E(v_i)| \leq M_{\mathcal{G}}^3 - 4$, assign to each vertex of $N_1^E(v_i)$ a different color from the $M_{\mathcal{G}}^3 - 4$ colors not assigned to the vertices of $N_1^E(v_{i-1})$. If $j \neq 4l+s$, since $|N_1^E(v_{4l+s})| + |N_1^E(v_1)| \leq M_{\mathcal{G}}^3 - 4$, assign to each vertex of $N_1^E(v_{4l+s})$ a different color from the $M_{\mathcal{G}}^3 - 4$ colors not assigned to the vertices of $N_1^E(v_1)$. For each $j+1 \leq i \leq 4l+s-1$, since

$$|N_1^E(v_{(4l+s-i+j)})| + |N_1^E(v_{(4l+s-i+j+1)})| \leq M_{\mathcal{G}}^3 - 4,$$

assign to each vertex of $N_1^E(v_{(4l+s-i+j)})$ a different color from the $M_{\mathcal{G}}^3 - 4$ colors not assigned to the vertices of $N_1^E(v_{(4l+s-i+j+1)})$.

Now, if $\rho = 0$, then $\bigcup_{i=1}^{4l+s} N_1[v_i]$ is 3-distance colored with $5 + M_{\mathcal{G}}^3 - 4 = M_{\mathcal{G}}^3 + 1$ colors. If $\rho = 1$, assign the color 5 to the unique vertex of $N_1^E(v_j)$ and we get a 3-distance coloring of $\bigcup_{i=1}^{4l+s} N_1[v_i]$ with $5 + M_{\mathcal{G}}^3 - 4 = M_{\mathcal{G}}^3 + 1$ colors. Finally, if $\rho \geq 2$, assign the color 5 to one of the leaves adjacent to v_j in $\mathcal{G}$ and assign $\rho - 1$ new colors to the remaining uncolored vertices that are adjacent to v_j in $\mathcal{G}$. So, we have a 3-distance coloring of $\bigcup_{i=1}^{4l+s} N_1[v_i]$ with $5 + M_{\mathcal{G}}^3 - 4 + \rho - 1 = M_{\mathcal{G}}^3 + \rho$ colors. Consequently, $\chi_3(\mathcal{G}) \leq M_{\mathcal{G}}^3 + \max\{\rho, 1\}$. □

Acknowledgment

This work is funded by national funds through the FCT- Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UID/297/2025 and UID/PRR/297/2025 (Center for Mathematics and Applications-Nova Math).

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