Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **18** (2026) 100–106
DOI: 10.47443/dml.2026.164

Research Article

Bicyclic Graphs with Maximum Complementary Diminished Sombor Index

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(Received: 2 June 2026. Received in revised form: 3 August 2026. Accepted: 23 October 2026. Published online: 26 October 2026.)

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Abstract

Albalahi et al. [*Discrete Math. Lett.* **16** (2025) 108–115] conjectured that graphs maximizing the complementary diminished Sombor index have minimum degree at least two, provided their order is sufficiently large relative to the cyclomatic number. We settle the bicyclic case by finding the unique maximizer at every order $n \geq 4$. It is $K_4 - e$ when $n = 4$ and two triangles sharing an edge, with a leaf attached to the remaining vertex of one triangle, when $n = 5$. At $n = 6$, the maximizer comes from attaching a leaf to each degree-2 vertex of $K_4 - e$; for $n \geq 7$, it comes from subdividing a nonshared edge in the order-5 graph. All these maximizers have a leaf when $n \geq 6$, contrary to the conjectured minimum-degree condition. The proof separates graphs of maximum degree at most three from those with a vertex of higher degree.

Keywords: complementary diminished Sombor index; bicyclic graph; cyclomatic number; extremal graph.

2020 Mathematics Subject Classification: 05C07, 05C09, 05C35.

1. Introduction

For a finite simple connected graph G, denote the degree of $v \in V(G)$ by $d_G(v)$. The hyperbolic Sombor index and its complementary diminished counterpart are

$$\mathcal{HSO}(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_G(u)^2 + d_G(v)^2}}{\min\{d_G(u), d_G(v)\}}, \qquad {}^{c}\mathcal{DSO}(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_G(u)^2 + d_G(v)^2}}{\max\{d_G(u), d_G(v)\}}.$$

Recall that the cyclomatic number of G is $\ell(G) = |E(G)| - |V(G)| + 1$. A connected graph is unicyclic if $\ell(G) = 1$ and bicyclic if $\ell(G) = 2$.

Albalahi et al. [1] introduced the complementary diminished Sombor index in their work on the hyperbolic Sombor index. Both belong to the family of degree-based indices built from the degrees at the ends of an edge. For comparison, the diminished Sombor index uses their sum in the denominator; bounds and further properties appear in [5, 6]. Barman and Das [3] discuss the geometric background of the hyperbolic index, while Das and Ahmad [4] establish bounds for this index on several graph classes, including bicyclic graphs.

There has also been work on extremal graphs for these indices. In particular, Ali, Das, and Attiya [2] study bond incident degree indices of unicyclic graphs. Their Remark 2.3 concerns the *minimum*, rather than the maximum, of the complementary diminished Sombor index: they establish a comparison for $n \geq 7$ and report computations through order 10. Su and Tang [7] investigate extremal unicyclic and bicyclic graphs for another variant, the Euler Sombor index. Our question is the maximum of the complementary diminished Sombor index among bicyclic graphs.

The starting point is Conjecture 4.2 of Albalahi et al. [1].

Conjecture 1.1 (see [1])**.** *Let G be a connected graph of order n and cyclomatic number ℓ, where $\ell \geq 2$ and $n > 5(\ell - 1)$. Among all connected graphs with this order and cyclomatic number, any graph maximizing ${}^{c}\mathcal{DSO}$, or minimizing $\mathcal{HSO}$, has both minimum and maximum degree in $\{2, 3\}$.*

We consider only the assertion about maximizing ${}^{c}\mathcal{DSO}$. When $\ell = 2$, the condition $n > 5(\ell - 1)$ becomes $n \geq 6$. As we shall see, the unique bicyclic maximizer has a leaf at every such order, so its minimum degree is 1, not 2 or 3. This does not contradict the conjecture's assertion about maximum degree. We also determine the maximizers at $n = 4$ and $n = 5$, the two bicyclic orders outside its range.

The key observation is that each edge contributes at most $\sqrt{2}$ to ${}^{c}\mathcal{DSO}$, with equality when its endpoints have the same degree. Instead of comparing the index directly, we keep track of the difference from this upper bound. For bicyclic graphs the number of edges is fixed by the order, which makes this comparison particularly convenient.

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2. Extremal Candidates and the Defect Formulation

We call an edge an (i,j)-edge when its endpoints have degrees i and j (in either order). The shorthand $(i,j)^k$ means that the graph has k such edges.

There are two constructions to consider. First, form A_6 from $K_4 - e$ by attaching a leaf at each of the two degree-2 vertices. The resulting graph has degree sequence $(3,3,3,3,1,1)$, with five $(3,3)$-edges and two $(1,3)$-edges. Thus

$$^c\mathcal{DSO}(A_6) = 5\sqrt{2} + \frac{2\sqrt{10}}{3}. \tag{1}$$

For the second construction, start with a triangle xyz, attach a leaf at x, and add a path from y to z with $n-4$ internal vertices, none belonging to the triangle. Call this graph B_n ($n \geq 5$). In B_5 the two cycles are triangles sharing the edge yz. The graphs of larger order are obtained by subdividing an edge of the second triangle. There are three vertices of degree 3, one leaf, and $n-4$ vertices of degree 2. More precisely, the edge types are

$$(3,3)^3, (1,3), (2,3)^2, (2,2)^{n-5}.$$

Therefore

$$^c\mathcal{DSO}(B_n) = 3\sqrt{2} + \frac{\sqrt{10}}{3} + \frac{2\sqrt{13}}{3} + (n-5)\sqrt{2} = (n-2)\sqrt{2} + \frac{\sqrt{10}+2\sqrt{13}}{3}. \tag{2}$$

Theorem 2.1. *Let G be a connected simple bicyclic graph of order n.*

- *If $n = 4$, then $K_4 - e$ is the unique maximizer of $^c\mathcal{DSO}$.*
- *If $n = 5$, then B_5 is the unique maximizer of $^c\mathcal{DSO}$.*
- *If $n = 6$, then A_6 is the unique maximizer of $^c\mathcal{DSO}$.*
- *If $n \geq 7$, then B_n is the unique maximizer of $^c\mathcal{DSO}$.*

For positive integers a and b, put

$$q(a,b) = \sqrt{2} - \frac{\sqrt{a^2+b^2}}{\max\{a,b\}}.$$

This is the amount by which an (a,b)-edge falls short of $\sqrt{2}$. In particular, $q(a,b) = q(b,a) \geq 0$, with equality exactly when $a = b$. Write

$$\mathfrak{D}(G) = \sum_{uv \in E(G)} q(d_G(u), d_G(v)).$$

Lemma 2.1. *If G is a connected bicyclic graph of order n, then*

$$^c\mathcal{DSO}(G) = (n+1)\sqrt{2} - \mathfrak{D}(G).$$

Consequently, at fixed order, maximizing $^c\mathcal{DSO}$ is equivalent to minimizing $\mathfrak{D}$.

Proof. Since G is connected and bicyclic, $|E(G)| = n+1$. Therefore

$$^c\mathcal{DSO}(G) = \sum_{uv \in E(G)} \big(\sqrt{2} - q(d_G(u), d_G(v))\big) = |E(G)|\sqrt{2} - \mathfrak{D}(G) = (n+1)\sqrt{2} - \mathfrak{D}(G).$$

At fixed n, only $\mathfrak{D}(G)$ varies. □

Set

$$\alpha = q(1,2) = \sqrt{2} - \frac{\sqrt{5}}{2}, \qquad \beta = q(1,3) = \sqrt{2} - \frac{\sqrt{10}}{3}, \qquad \gamma = q(2,3) = \sqrt{2} - \frac{\sqrt{13}}{3}. \tag{3}$$

Edges joining vertices of equal degree contribute nothing to $\mathfrak{D}$. For our two constructions this gives

$$\mathfrak{D}(A_6) = 2\beta, \qquad \mathfrak{D}(B_n) = \beta + 2\gamma \quad (n \geq 5). \tag{4}$$

For fixed $a \geq 1$ and $b \geq a$, we have

$$q(a,b) = \sqrt{2} - \sqrt{1 + (a/b)^2}.$$

Hence, with a fixed, the defect increases strictly as b runs from a upwards. We shall use, in particular,

$$q(1,d) \geq \alpha \qquad (d \geq 2). \tag{5}$$

The following inequalities will be useful:

$$\alpha > \gamma, \qquad 2\gamma > \beta, \qquad 2\alpha > \beta + \gamma. \tag{6}$$

We set $D_* = \beta + 2\gamma$ for the rest of the paper. These inequalities imply

$$4\gamma,\ \alpha + 3\gamma,\ 3\alpha,\ 2\beta + 2\gamma,\ \alpha + \beta + \gamma,\ 2\alpha + \gamma > D_*, \tag{7}$$

as well as $D_* > 2\beta$.

3. The Case $n \geq 7$

3.1. Subcubic Bicyclic Graphs

Let n_i count the vertices of degree i and x_{ij} the edges with endpoint degrees i and j, where $i \leq j$.

Lemma 3.1. *Let G be a connected bicyclic graph of order $n \geq 7$ with $\Delta(G) \leq 3$. Then $\mathfrak{D}(G) \geq \beta + 2\gamma$, with equality if and only if $G \cong B_n$.*

Proof. Since $|E(G)| = n + 1$,

$$\sum_{v \in V(G)} (d_G(v) - 2) = 2|E(G)| - 2|V(G)| = 2.$$

Under the assumption $\Delta(G) \leq 3$, the identity reduces to

$$-n_1 + n_3 = 2. \tag{8}$$

In particular, $n_3 = n_1 + 2$. The only edges that can contribute a nonzero defect are $(1,2)$-, $(1,3)$-, and $(2,3)$-edges. Therefore

$$\mathfrak{D}(G) = \alpha x_{12} + \beta x_{13} + \gamma x_{23}. \tag{9}$$

The number of leaves now determines the argument.

(i) $n_1 = 0$. Then $n_3 = 2$. There are exactly two degree-3 vertices. They can have at most one edge between them, leaving at least four incidences with degree-2 vertices. Thus $x_{23} \geq 4$, and (6) yields

$$\mathfrak{D}(G) \geq 4\gamma > \beta + 2\gamma.$$

(ii) $n_1 = 1$. Then $n_3 = 3$. Write x_{33} for the number of edges joining two degree-3 vertices. Counting incidences at these three vertices gives

$$x_{13} + x_{23} = 9 - 2x_{33} \geq 3, \tag{10}$$

because $x_{33} \leq \binom{3}{2} = 3$.

If the leaf meets a degree-3 vertex, then $x_{13} = 1$, and (10) forces $x_{23} \geq 2$. Consequently,

$$\mathfrak{D}(G) \geq \beta + 2\gamma.$$

Equality requires $x_{23} = 2$ and $x_{33} = 3$, with no further edges joining different degrees. The three degree-3 vertices must therefore form a triangle. The one adjacent to the leaf has all three neighbors accounted for; each of the other two has exactly one neighbor of degree 2.

The degree-2 vertices cannot contain a cycle component, since it would be disconnected from the triangle. They therefore lie on paths, including possibly single vertices. As exactly two edges run from the degree-2 vertices to the triangle, connectedness leaves only one possibility: a path between the two triangle vertices not adjacent to the leaf. This is B_n, which attains the bound by (4).

If the leaf instead meets a degree-2 vertex, then $x_{12} = 1$ and $x_{13} = 0$. Now (10) gives $x_{23} \geq 3$, so

$$\mathfrak{D}(G) \geq \alpha + 3\gamma > \beta + 2\gamma$$

by the comparisons above.

(iii) $n_1 = 2$. Here $n_3 = 4$ and $n_2 = n - 6 \geq 1$. Counting the two leaf edges gives

$$x_{12} + x_{13} = 2. \tag{11}$$

Because G is connected, the subgraph on the degree-2 vertices consists of paths and has no cycle component. It is nonempty, and every such path has two edges leading outside it. Hence

$$x_{12} + x_{23} \geq 2. \tag{12}$$

Finally, the degree-3 vertices cannot be isolated from the remaining vertices, so

$$x_{13} + x_{23} \geq 1. \tag{13}$$

If $x_{12} = 0$, then $x_{13} = 2$ and (12) gives $x_{23} \geq 2$, so

$$\mathfrak{D}(G) \geq 2\beta + 2\gamma.$$

If $x_{12} = 1$, then $x_{13} = 1$ and (12) gives $x_{23} \geq 1$, so

$$\mathfrak{D}(G) \geq \alpha + \beta + \gamma.$$

If $x_{12} = 2$, then $x_{13} = 0$ and (13) gives $x_{23} \geq 1$, so

$$\mathfrak{D}(G) \geq 2\alpha + \gamma.$$

Each of these three bounds exceeds $\beta + 2\gamma$ by (7).

(iv) $n_1 \geq 3$. Each leaf edge contributes at least α, so (5) gives

$$\mathfrak{D}(G) \geq n_1\alpha \geq 3\alpha > \beta + 2\gamma$$

by the comparisons above. This completes the proof. □

3.2. Graphs Having Maximum Degree at Least Four

For graphs with $\Delta(G) \geq 4$, it is enough to examine the vertices whose degrees differ from 2. The remaining vertices will enter the calculation only through the edges incident with them.

Lemma 3.2. *Let G be a connected bicyclic graph with L leaves and $\Delta(G) \geq 4$. If $L \leq 2$, the multiset of degrees of the vertices of degree at least 3 is one of the following:*

L	*multiset of degrees at least* 3
0	$\{4\}$
1	$\{5\},\ \{4,3\}$
2	$\{6\},\ \{5,3\},\ \{4,4\},\ \{4,3,3\}$.

Proof. Summing $d_G(v) - 2$ over the vertices gives 2, so

$$\sum_{d_G(v)\geq 3} (d_G(v) - 2) = L + 2. \tag{14}$$

Each term on the left is a positive integer, and at least one is at least 2 because $\Delta(G) \geq 4$. Listing the partitions of $L + 2$ for $L = 0, 1, 2$ gives precisely the seven degree multisets above. □

Let S be the set of vertices of degree 1 or at least 3, and take $F = G[S]$. Thus each vertex outside S has degree 2.

An edge with both ends outside S has zero defect. The edges with both ends in S are precisely those of F, while a vertex $v \in S$ has $d_G(v) - d_F(v)$ neighbors of degree 2. We obtain

$$\mathfrak{D}(G) = \sum_{uv\in E(F)} q(d_G(u), d_G(v)) + \sum_{v\in S} \big(d_G(v) - d_F(v)\big) q(d_G(v), 2). \tag{15}$$

We now fix the degrees of the vertices in S and allow F to be *any* simple graph on this vertex set satisfying $d_F(v) \leq d_G(v)$. This is deliberately a larger class than the induced subgraphs that can come from connected bicyclic graphs. Minimizing the right-hand side of (15) over the larger class still gives a lower bound for $\mathfrak{D}(G)$, and that is all we need. In particular, an F giving the minimum in Table 3.1 need not arise from any connected bicyclic graph.

Table 3.1: Minima of (15) over simple graphs F satisfying $d_F(v) \le d_G(v)$. The multisets include all degrees in S, including those of leaves. The minima give lower bounds; the corresponding graphs F need not be realizable within a connected bicyclic graph.

L	Degree multiset on S	Minimum value
0	$\{4\}$	$4q(2,4)$
1	$\{1,5\}$	$q(1,5)+4q(2,5)$
1	$\{1,4,3\}$	$q(1,4)+q(3,4)+2q(2,4)+2q(2,3)$
2	$\{1,1,6\}$	$6q(2,6)$
2	$\{1,1,5,3\}$	$q(3,5)+4q(2,5)+2q(2,3)$
2	$\{1,1,4,4\}$	$6q(2,4)$
2	$\{1,1,4,3,3\}$	$2q(3,4)+2q(2,4)+2q(2,3)$

Lemma 3.3. *The minimum of the expression in* (15)*, with F ranging over the simple graphs described above, is as listed in Table 3.1 for each of the seven degree multisets.*

Proof. Put $b_r = q(r,2)$. We can begin with the edgeless graph F. The expression in (15) then equals $\sum_{v\in S} d_G(v) b_{d_G(v)}$. Adding an edge whose endpoints have prescribed degrees r and t changes the expression by the negative of

$$s(r,t) = b_r + b_t - q(r,t). \tag{16}$$

The reductions add independently as long as the constraint $d_F(v) \le d_G(v)$ is respected. Since $|S| \le 5$, there are only a few possible edges to compare.

For $d \ge 2$, direct simplification gives

$$s(1,d) = \alpha - \frac{3}{d\left(\sqrt{d^2+1}+\sqrt{d^2+4}\right)}. \tag{17}$$

The right-hand side increases with d, and $s(1,d) > 0$ for $d \ge 3$. Indeed,

$$\alpha = \frac{3}{2(2\sqrt{2}+\sqrt{5})} > \frac{1}{\sqrt{10}+\sqrt{13}}.$$

Edges joining vertices of degree at least 3 are also favorable in the cases we need:

$$s(3,3) = 2\gamma, \quad s(4,4) = 2\alpha, \quad s(3,4) = \alpha + \frac{5}{4} - \frac{\sqrt{13}}{3} > 0, \quad s(3,5) = \gamma + \frac{\sqrt{34}-\sqrt{29}}{5} > 0.$$

The only remaining choice is how to place two leaves. Since $s(1,1) = 2\alpha$, direct calculation gives

$$\begin{aligned} s(1,1) - 2s(1,6) &= \frac{2\sqrt{10}-\sqrt{37}}{3} > 0, \\ s(1,1) - 2s(1,5) &= \frac{2(\sqrt{29}-\sqrt{26})}{5} > 0, \\ s(1,1) - 2s(1,4) &= \sqrt{5} - \frac{\sqrt{17}}{2} > 0. \end{aligned} \tag{18}$$

The first three rows are immediate from these observations. With degree multiset $\{4\}$, F has no possible edge. With $\{1,5\}$, its single possible edge is beneficial. For $\{1,4,3\}$, we include the $(3,4)$-edge and attach the leaf to the degree-4 vertex, since $s(1,4) > s(1,3)$.

Consider next $\{1,1,6\}$. Either the leaves are joined to each other or they use two separate edges to the degree-6 vertex. The first comparison in (18) favors the edge between the leaves. With $\{1,1,5,3\}$, we also include the $(3,5)$-edge. The two leaves, if kept separate, can improve the expression by at most $2s(1,5)$, so the second comparison again favors joining them. This gives the fourth and fifth rows.

For $\{1,1,4,4\}$, we include the edge between the degree-4 vertices. The third comparison in (18) then shows that the leaves should be joined to each other. In the last case, $\{1,1,4,3,3\}$, all three edges among the vertices of degree at least 3 decrease the expression, and the same leaf comparison applies: keeping the leaves separate can save at most $2s(1,4)$. These choices yield the final two entries.

The leaf-to-leaf edges used in the last four rows are not meant as edges of a connected bicyclic graph: the two leaves would form a separate component. They are permitted in this calculation precisely because we are computing a lower bound over a larger class of graphs F. □

Lemma 3.4. *If G is a connected bicyclic graph with $\Delta(G) \geq 4$, then $\mathfrak{D}(G) > \beta + 2\gamma$.*

Proof. Write L for the number of leaves. The case $L \geq 3$ needs no further analysis: by (5), $\mathfrak{D}(G) \geq 3\alpha > D_*$. When $L \leq 2$, Lemmas 3.2 and 3.3 leave the seven bounds in Table 3.1. None is as small as D_*:

$$\begin{aligned} 4q(2,4) = 4\alpha &> D_*, \\ q(1,5) + 4q(2,5) > \beta + 4\gamma &> D_*, \\ q(1,4) + q(3,4) + 2q(2,4) + 2\gamma &> D_*, \\ 6q(2,6) > 6\gamma &> D_*, \\ q(3,5) + 4q(2,5) + 2\gamma &> D_*, \\ 6q(2,4) = 6\alpha &> D_*, \\ 2q(3,4) + 2\alpha + 2\gamma &> D_*. \end{aligned}$$

Here we have used the monotonicity of q, its positivity for unequal degrees, and (6). We use only the inequalities, not the existence of bicyclic graphs realizing any row of the table. □

Proof of Theorem 2.1 for $n \geq 7$. Let $n \geq 7$. By Lemma 2.1, we may work with $\mathfrak{D}(G)$ instead of the index. For $\Delta(G) \leq 3$, Lemma 3.1 gives $\mathfrak{D}(G) \geq \mathfrak{D}(B_n)$, with equality if and only if $G \cong B_n$. If $\Delta(G) \geq 4$, Lemma 3.4 gives $\mathfrak{D}(G) > \mathfrak{D}(B_n)$. Thus equality is possible only for B_n, as claimed. □

4. The Small Orders

It remains to consider $n = 4, 5, 6$. The same defect estimates handle these orders directly.

At order 4 there is nothing to compare: a simple bicyclic graph has five edges and must be $K_4 - e$.

Lemma 4.1. *Among connected simple bicyclic graphs of order 5, the graph B_5 is the unique maximizer of ${}^c\mathcal{DSO}$.*

Proof. Take $|V(G)| = 5$. For $\Delta(G) \geq 4$, Lemma 3.4 gives $\mathfrak{D}(G) > \beta + 2\gamma = \mathfrak{D}(B_5)$. Suppose $\Delta(G) \leq 3$. Equation (8) holds, and $n_2 = 5 - n_1 - n_3 = 3 - 2n_1$, so $n_1 \in \{0, 1\}$.

With no leaves, case (i) of the proof of Lemma 3.1 yields $\mathfrak{D}(G) \geq 4\gamma > \beta + 2\gamma$. With one leaf, case (ii) yields $\mathfrak{D}(G) \geq \beta + 2\gamma$, with equality if and only if the three degree-3 vertices induce a triangle and the unique degree-2 vertex joins the two triangle vertices not incident with the leaf. This graph is precisely B_5. □

Lemma 4.2. *Among connected simple bicyclic graphs of order 6, the graph A_6 is the unique maximizer of ${}^c\mathcal{DSO}$.*

Proof. Take $|V(G)| = 6$. Lemma 3.4 rules out $\Delta(G) \geq 4$, since $D_* > 2\beta$ and $\mathfrak{D}(G) > \beta + 2\gamma > 2\beta = \mathfrak{D}(A_6)$. Suppose $\Delta(G) \leq 3$. Equation (8) holds, and $n_2 = 6 - n_1 - n_3 = 4 - 2n_1$, so $n_1 \in \{0, 1, 2\}$.

If $n_1 = 0$, then $\mathfrak{D}(G) \geq 4\gamma > 2\beta$. If $n_1 = 1$, then either the leaf is adjacent to a degree-3 vertex, giving $\mathfrak{D}(G) \geq \beta + 2\gamma > 2\beta$, or it is adjacent to a degree-2 vertex, giving $\mathfrak{D}(G) \geq \alpha + 3\gamma > 4\gamma > 2\beta$.

The only possibility left is $n_1 = 2$, giving $n_3 = 4$ and $n_2 = 0$. Both leaves meet degree-3 vertices, and these are the only edges with unequal endpoint degrees. It follows that $\mathfrak{D}(G) = 2\beta$. The leaves cannot have the same neighbor: deleting them would otherwise produce a graph with degree sequence $(3, 3, 3, 1)$, which is not graphical. Deleting leaves with distinct neighbors instead produces a four-vertex graph with degree sequence $(3, 3, 2, 2)$, necessarily $K_4 - e$. Reattaching the leaves gives exactly A_6. □

Together with the argument for $n \geq 7$, the three small orders establish Theorem 2.1.

Corollary 4.1. *For every $n \geq 6$, the unique maximizer of ${}^c\mathcal{DSO}$ among connected simple bicyclic graphs of order n has minimum degree 1. Consequently, the minimum-degree assertion of Conjecture 1.1 is false for every bicyclic order in its stated range.*

Proof. For $\ell = 2$, the hypothesis $n > 5(\ell - 1)$ is exactly $n \geq 6$. The graph A_6 and every B_n have a leaf. □

Acknowledgments

The author is grateful to one of the anonymous referees for suggesting the general bicyclic case and for helpful comments on the manuscript. Thanks are also due to Albalahi, Das, Ali, Barman, and Hamza for posing the conjecture considered here.

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