

Research Article

Sum of Squares of Vertex Degrees of Simple Planar (n, m) -Graphs

Batmend Horoldagva¹, Tsend-Ayush Selenge^{2,*}, Ivan Gutman³

¹Department of Mathematics, Mongolian National University of Education, Ulaanbaatar, Mongolia

²Department of Mathematics, National University of Mongolia, Ulaanbaatar, Mongolia

³Faculty of Science, University of Kragujevac, Kragujevac, Serbia

(Received: 18 March 2026. Received in revised form: 28 May 2026. Accepted: 2 June 2026. Published online: 11 June 2026.)

© 2026 the authors. This is an open-access article under the CC BY (International 4.0) license (www.creativecommons.org/licenses/by/4.0/).

Abstract

A simple graph G with n vertices and m edges is called an (n, m) -graph. We study the sum of squares of the vertex degrees of planar (n, m) -graphs and characterize those attaining maximum value. Moreover, we determine the degree sequences of planar graphs with the minimum value of this sum.

Keywords: planar graph; degree (of vertex); degree sequence.

2020 Mathematics Subject Classification: 05C35, 05C07, 05C10.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The degree of vertex $v \in V(G)$ is denoted by d_v . For $v \in V(G)$, $N(v)$ denotes the set of neighbors of the vertex v . A simple graph G with n vertices and m edges is referred to as an (n, m) -graph. A connected graph is said to be planar if it can be embedded in the plane so that no two edges intersect at a point other than a vertex. The sum of squares of the vertex degrees of a graph G is denoted by $\Sigma(G)$.

Goodman [9] formulated the problem of determining the best possible upper bound for the number of triangles in a graph and its complement, in terms of the number of vertices and edges, which is equivalent to finding the maximum number of monochromatic triangles in the graph. This problem naturally leads to investigating an upper bound for the sum of the squares of the vertex degrees. Caen [8] provided such an upper bound, which was later improved by Cioabă [6] and Das [7].

Boesch et al. [2] studied the problem of characterizing graphs with maximum Σ -value among (n, m) -graphs and showed that every optimal graph is a threshold graph. They constructed two threshold graphs and proved that one of them is optimal. Olpp [14] proved an equivalent result, motivated by problems in Ramsey theory. Peled et al. [15] constructed four additional classes of threshold graphs to the two previously known classes, and showed that every optimal graph belongs to one of these six classes.

Gutman and Trinajstić [12] observed that $\Sigma(G)$ appears in an approximate expression for total π -electron energy on molecular structure (molecular graph G). Soon after that, it was recognized that $\Sigma(G)$ reflects the degree of branching of the underlying molecule [11]. Since then, this vertex-degree-based graph invariant has been studied under the name of the first Zagreb index and an unusually large number of theoretical results and applications have been established [10]. Details of its mathematical properties can be found in the survey [4] and the references therein.

A main problem for vertex-degree-based graph invariants is characterizing the extremal structures among (n, m) -graphs, which remains unsolved for most invariants. Ahlswede and Katona [1] showed that quasi-complete or quasi-star graphs maximize Σ . Later, Bollobás, Erdős, and Sarkar [3] solved the maximization of the second Zagreb index (the sum of the products of degrees of adjacent vertices) for general (n, m) -graphs. For connected (n, m) -graphs, the problem was studied by Xu, Das, and Balachandran [18], but their proposed conjecture is still open. Although degree-based graph invariants have been extensively studied, relatively little research has been conducted on planar graphs. Therefore, in this study, we focus on Σ for planar (n, m) -graphs.

Cook [6] proved that $\Sigma(G) \leq 2n^2 + O(n)$ for a simple planar graph G of order n . Later, Truszczynski [17] obtained a sharp upper bound. In the present paper, we obtain a sharp upper bound on Σ for simple planar (n, m) -graphs and characterize the corresponding extremal graphs. Moreover, the degree sequences of simple planar (n, m) -graphs with minimum Σ -value are determined.

*Corresponding author (selenge@num.edu.mn).

2. Sharp Upper Bounds on Σ

For any integers n and m satisfying $n - 1 \leq m \leq 2n - 3$, we denote by $N(n, m)$ a graph with n vertices and m edges in which the maximum degree is $n - 1$ and the second maximum degree is $m - n + 2$. Xu et al. [18] characterized the extremal graphs G with maximal $\Sigma(G)$ among all connected graphs of order n , where $n + 3 \leq m \leq 2n - 4$. Note that the proof of Theorem 2.10 in [18] remains valid in the case $2n - 3$. Thus, the result of [18] can be restated as follows:

Lemma 2.1. *Let G be a simple (n, m) -graph of order $n \geq 5$. If $n + 3 \leq m \leq 2n - 3$, then*

$$\Sigma(G) \leq n(n - 1) + (m - n + 1)(m - n + 6)$$

with equality if and only if $G \cong N(n, m)$.

For any integer $n \geq 5$, denote by $N(n)$ a graph with n vertices and $n + 1$ edges in which the maximum degree is $n - 1$ and the second maximum degree is 2. Let $Q(n, m)$ be the planar (n, m) -graph whose special case is depicted in Figure 2.1 such that it has 2 vertices of degrees $n - 1$, $m - 2n + 2$ vertices of degree 4, two vertices of degree 3, and $3n - m - 6$ vertices of degree 2. A planar graph is called maximal planar if no additional edge can be added without violating its planarity. Hence, we easily conclude that each face of a maximal planar graph is a triangle and the number of edges is at most $3n - 6$.

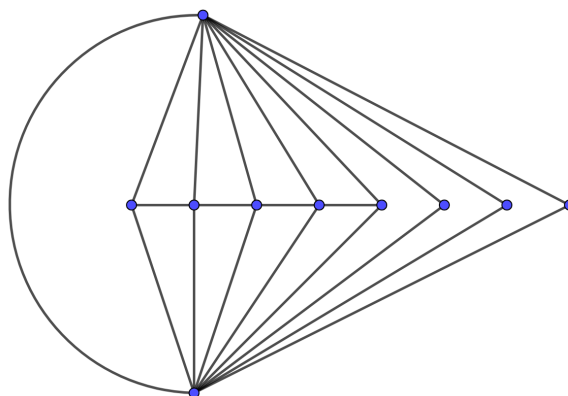


Figure 2.1: The planar graph $Q(10, 21)$.

Theorem 2.1. *Let G be a simple planar (n, m) -graph with $n \geq 5$.*

- (i) *If $m \leq 2n - 3$, then $\Sigma(G) \leq n(n - 1) + (m - n + 1)(m - n + 6)$ with equality if and only if G is isomorphic to $N(n, m)$ or $N(n)$.*
- (ii) *If $2n - 3 < m$, then $\Sigma(G) \leq 2n^2 + 12m - 24n + 28$ with equality if and only if G is isomorphic to $Q(n, m)$, or G_0 when $(n, m) = (6, 10)$, or G_1 when $(n, m) = (7, 15)$, or G_2 when $(n, m) = (8, 18)$.*

Proof. (i) Since G is connected, $n - 1 \leq m$. Hence, the proof directly follows from Lemma 2.1 along with Theorems 2.5 and 2.7 of [18] because the graphs $N(n, m)$ and $N(n)$ are planar.

(ii) Let v be a minimum degree vertex of G . Since G is planar, $d_v \leq 5$. Then $\Sigma(Q(n, m)) = 2n^2 + 12m - 24n + 28$ and we have to prove that

$$\Sigma(G) \leq \Sigma(Q(n, m)). \tag{1}$$

We proceed by induction on n . If $n = 5$ then $7 < m < 10$ since G is planar. If $m = 8$, then G is isomorphic to the wheel graph W_5 or $Q(5, 8)$, and we have $\Sigma(W_5) < \Sigma(Q(5, 8))$. If $m = 9$, then G is isomorphic to a graph $Q(5, 9)$. Therefore, the base case of the induction holds.

Case (1): $d_v \leq 3$. Let G' be the graph obtained by deleting the vertex v from G . Then

$$\Sigma(G) = \Sigma(G') + d_v^2 + \sum_{u \in N(v)} d_u^2 - \sum_{u \in N(v)} (d_u - 1)^2 = \Sigma(G') + d_v^2 - d_v + 2 \sum_{u \in N(v)} d_u \tag{2}$$

and the number of vertices and edges of G' are $n - 1$ and $m - d_v$, respectively. Obviously, G' is planar.

First, we let $d_v \leq 2$. Then $2(n - 1) - 3 < m - d_v$. Thus by induction hypothesis, we have $\Sigma(G') \leq \Sigma(Q(n - 1, m - d_v))$ and it follows that

$$\begin{aligned} \Sigma(G) &\leq \Sigma(Q(n - 1, m - d_v)) + d_v^2 - d_v + 2 \sum_{u \in N(v)} d_u \\ &= \Sigma(Q(n, m)) - 4n + d_v^2 - 13d_v + 26 + 2 \sum_{u \in N(v)} d_u \end{aligned} \tag{3}$$

from the relations (2). In order to get the inequality (1), it is sufficient to show that

$$d_v^2 - 13d_v + 2 \sum_{u \in N(v)} d_u \leq 4n - 26. \tag{4}$$

If $d_v = 1$, then the inequality in (4) is strict for $n > 6$. Now consider $n = 5$ and $n = 6$, which yield the edge ranges $7 < m \leq 9$ and $9 < m \leq 12$, respectively. Thus, it is easy to see that there is no planar (n, m) -graph G except in the specific case where $(n, m) = (6, 10)$ since

$$\sum_{u \in V(G')} = 2(m - 1).$$

If $(n, m) = (6, 10)$ then there is only one graph G' with degree sequence $4^3 3^2$ and it follows that $G \cong G_0$ and the equality holds in (1) for G_0 (see Figure 2.2).

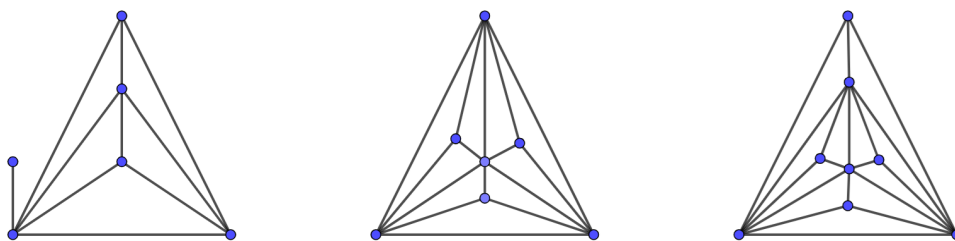


Figure 2.2: Extremal planar graphs G_0, G_1 and G_2 , respectively.

If $d_v = 2$, then we get the inequality (1) and the equality holds if and only if the neighbors of v have degrees $n - 1$. On the other hand the equality in (3) holds if and only if G' is isomorphic to $Q(n - 1, m - 2)$. Hence we conclude that the equality in (1) holds if and only if $G \cong Q(n, m)$.

Let now $d_v = 3$. Then from (2) and $2n - 3 < m$, we get

$$\Sigma(G) \leq \begin{cases} \Sigma(N(n - 1, 2n - 5)) + 6 + 2 \sum_{u \in N(v)} d_u, & \text{if } m = 2n - 2, \\ \Sigma(Q(n - 1, m - 3)) + 6 + 2 \sum_{u \in N(v)} d_u, & \text{if } m > 2n - 2 \end{cases} \tag{5}$$

by (i) and induction hypothesis, respectively. Let x, y, z be neighbor vertices of v . Then

$$|N(x) \cap N(y)| + |N(x) \cap N(z)| + |N(y) \cap N(z)| - 2|N(x) \cap N(y) \cap N(z)| \leq n.$$

On the other hand $|N(x) \cap N(y) \cap N(z)| \leq 2$ because G does not contain $K_{3,3}$ as a subgraph. Therefore by inclusion and exclusion criteria, we get $n \geq |N(x) \cup N(y) \cup N(z)| \geq |N(x)| + |N(y)| + |N(z)| - n - 2 \geq d_x + d_y + d_z - n - 2$, that is, $d_x + d_y + d_z \leq 2n + 2$.

Let $m = 2n - 2$. If $d_x + d_y + d_z = 2n + 2$, then

$$4n - 4 = \sum_{u \in V(G)} d_u \geq 2n + 5 + 3(n - 4),$$

which contradicts $n \geq 5$. Similarly, if $d_x + d_y + d_z = 2n + 1$, then we get a contradiction. Hence $d_x + d_y + d_z < 2n + 1$. Then from the first inequality in (5), we get $\Sigma(G) < \Sigma(Q(n, 2n - 2))$ since $\Sigma(N(n - 1, 2n - 5)) = 2n^2 - 4n - 4$ and $\Sigma(Q(n, 2n - 2)) = 2n^2 + 4$.

Consider now the case $m > 2n - 2$. Then from the second inequality in (5), we get $\Sigma(G) \leq \Sigma(Q(n, m))$ since

$$\Sigma(Q(n - 1, m - 3)) = \Sigma(Q(n, m)) - 4n - 10$$

and $d_x + d_y + d_z \leq 2n + 2$. Assume that $\Sigma(G) = \Sigma(Q(n, m))$. We must then have $d_x + d_y + d_z = 2n + 2$, hence, the minimum vertex degree of G' is still at least 3. Also, we have $\Sigma(G') = \Sigma(Q(n - 1, m - 3))$. Let $n = 6$. From the base case of the induction, we only have $G' \cong Q(5, 9)$, whose degree sequence is $4^3 3^2$. Thus, $G \cong Q(6, 12)$, which has the degree sequence is $5^2 4^2 3^2$ since the sum of the degrees of the neighbors of $v \in V(G')$ equals 11. Hence, if $n = 7$ then $G' \cong Q(6, 12)$. Since the sum of the degrees of the neighbors of $v \in V(G')$ equals 13, G has the degree sequence $6^2 4^3 3^2$ or $6^1 5^3 3^3$. Thus $G \cong Q(7, 15)$ or $G \cong G_1$ (see Figure 2.2). Similarly, we obtain $G \cong Q(8, 18)$ or $G \cong G_2$ when $n = 8$ (see Figure 2.2). Namely, the degree sequence of a graph G such that $\Sigma(G) = \Sigma(Q(n, m))$ can be constructed inductively as follows:

$$4^3 3^2 \rightarrow 5^2 4^2 3^2 \rightarrow \begin{cases} 6^2 4^3 3^2 \rightarrow 7^2 4^4 3^2 \rightarrow 8^2 4^5 3^2 \rightarrow \dots \rightarrow (n - 1)^2 4^{n-4} 3^2 \\ 6^1 5^3 3^3 \rightarrow 6^4 3^4. \end{cases}$$

Hence, we conclude that the equality in (1) holds if and only if G is isomorphic to $Q(n, m)$, or G_1 when $(n, m) = (7, 15)$, or G_2 when $(n, m) = (8, 18)$.

Case (2): $d_v = 4$. Let x, y, z and t be neighbor vertices of v . Since G is planar, it does not contain K_5 as a subgraph. Therefore, there exist two nonadjacent neighbors of v , say x and y . Let G' be the obtained graph from G by deleting the vertex v and adding the edge xy . Then the number of edges of G' is $m - 3$ and

$$\begin{aligned} \Sigma(G) &= \Sigma(G') + 2(d_z + d_t) + 14 \leq \begin{cases} \Sigma(N(n - 1, 2n - 5)) + 2(d_z + d_t) + 14, & \text{if } m = 2n - 2, \\ \Sigma(Q(n - 1, m - 3)) + 2(d_z + d_t) + 14, & \text{if } m > 2n - 2 \end{cases} \\ &= \begin{cases} \Sigma(Q(n, 2n - 2)) - 2(2n - 3 - d_z - d_t), & \text{if } m = 2n - 2, \\ \Sigma(Q(n, m)) - 2(2n - 2 - d_z - d_t), & \text{if } m > 2n - 2 \end{cases} \end{aligned} \tag{6}$$

by (i) and induction hypothesis, respectively.

Let $m = 2n - 2$ and $d_z + d_t \geq 2n - 3$. Then $4n - 4 = \sum_{u \in V(G)} d_u \geq 2n + 1 + 4(n - 3) \implies d_z \leq n - 2$ and but we have $n \geq 5$. Hence $d_z + d_t < 2n - 3$ and it follows that $\Sigma(G) < \Sigma(Q(n, 2n - 2))$.

Let now $m > 2n - 2$. Then from the second inequality in (6), we get $\Sigma(G) \leq \Sigma(Q(n, m))$ since $d_z \leq n - 1$ and $d_t \leq n - 1$.

Suppose now that equality holds in (6). Then $d_z = d_t = n - 1$ and it follows that the minimum degree of G' is again 4. On the other hand G' is isomorphic to $Q(n - 1, m - 3)$ and its minimum degree is 2 or 3. This is a contradiction, and therefore the inequality in (6) is strict.

Case (3): $d_v = 5$. Let x, y, z, t, u be the neighbors of v . Since the subgraph induced by $\{x, y, z, t, u\}$ is not isomorphic to K_5 , there exist two nonadjacent vertices among them, say x and y . If $xz, xt, xu \in E(G)$ and $yz, yt, yu \in E(G)$, then the subgraph induced by $\{x, y, z, t, u\}$ is isomorphic to $K_{2,3}$ and it follows that G contains $K_{3,3}$ as a subgraph induced by $\{v, x, y, z, t, u\}$. Suppose that $xz \notin E(G)$. Then consider the graph G' obtained from G by deleting the vertex v and adding the edges xy and xz . Then the number of edges of G' is $m - 3$ and

$$\begin{aligned} \Sigma(G) &= \Sigma(G') + 2(d_t + d_u - d_x) + 22 \\ &\leq \begin{cases} \Sigma(N(n - 1, 2n - 5)) + 2(d_t + d_u - d_x) + 22, & \text{if } m = 2n - 2, \\ \Sigma(Q(n - 1, m - 3)) + 2(d_t + d_u - d_x) + 22, & \text{if } m > 2n - 2 \end{cases} \\ &= \begin{cases} \Sigma(Q(n, 2n - 2)) - 2(2n - 7 - d_t - d_u + d_x), & \text{if } m = 2n - 2, \\ \Sigma(Q(n, m)) - 2(2n - 6 + d_x - d_u - d_t), & \text{if } m > 2n - 2 \end{cases} \end{aligned} \tag{7}$$

by (i) and induction hypothesis, respectively. Similarly as in the above cases, we easily show that the inequalities in (7) are strict. This completes the proof. \square

Any planar graph G can be transformed into a maximal planar graph by adding edges, which increases the sum of the squares of the vertex degrees. Hence, one of the main results in [17] follows directly from Theorem 2.1:

Corollary 2.1. *Let G be a simple planar graph of order $n \geq 5$. Then $\Sigma(G) \leq 2n^2 + 12n - 44$ with equality if and only if G is isomorphic to $Q(n, 3n - 6)$, or G_1 when $n = 7$, or G_2 when $n = 8$.*

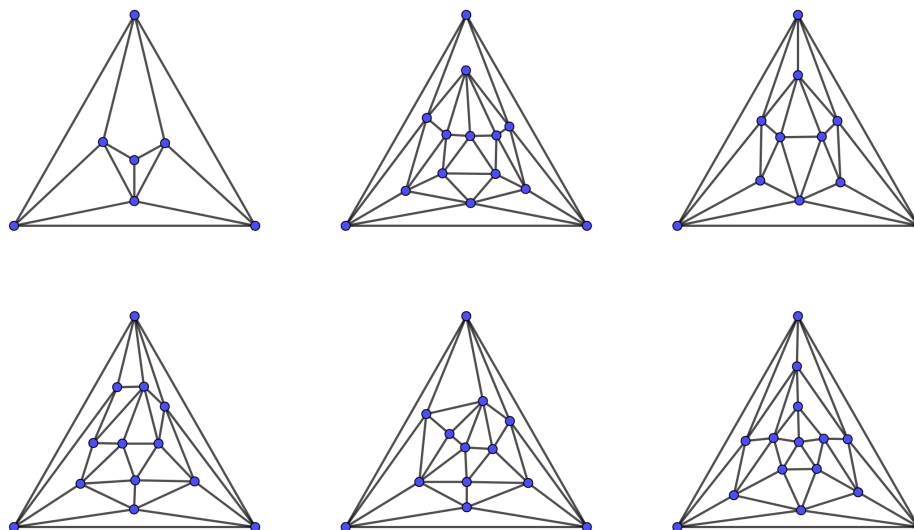


Figure 2.3: Planar graphs with degree sequences $5^1 4^5 3^1$, $6^1 5^{12} 4^1$, $6^1 5^8 4^2$, $6^1 5^{10} 4^2$, $6^2 5^{10} 4^1$ and $6^2 5^{12} 4^1$, respectively.

3. Degree Sequences of Planar Graphs With Minimum Σ -Value

Let $\pi = (d_1, d_2, \dots, d_n)$ be a non-increasing sequence of positive integers. If $d_1 - d_n = k$ then the sequence π is called k -sequence. If $\sum_{i=1}^n d_i \leq 6n - 12$, then π is called an Euler sequence. If there exists a simple (planar) graph having π as its vertex degree sequence, then π is called (planar) graphical. Hawkins et al [13] determined which graphical Euler 0-sequences are planar graphical.

Lemma 3.1. *Every graphical Euler 0-sequence is planar graphical except 4^7 and 5^{14} .*

Schmeichel and Hakimi [16] determined completely which graphical Euler 1-sequences are planar graphical. Their result is the following.

Lemma 3.2. *Every graphical Euler 1-sequence is planar graphical except for $5^{10}4^1$, $5^{12}4^1$, $6^{15}5^2$, and $6^{15}4^1$.*

Theorem 3.1. *Let G be a simple planar (n, m) -graph. Let $\Sigma(G)$ be minimum among all simple planar (n, m) -graphs.*

- (i) *If (n, m) is different from $(7, 14)$, $(14, 35)$, $(11, 27)$, $(13, 32)$, $(13, 33)$ and $(15, 38)$, then the degree sequence of G is a 0-sequence when $2m$ is divisible by n , otherwise a 1-sequence.*
- (ii) *If $(n, m) = (7, 14)$, $(14, 35)$, $(11, 27)$, $(13, 32)$, $(13, 33)$ and $(15, 38)$ then the degree sequence of G is $5^1 4^5 3^1$, $6^1 5^{12} 4^1$, $6^1 5^8 4^2$, $6^1 5^{10} 4^2$, $6^2 5^{10} 4^1$ and $6^2 5^{12} 4^1$, respectively.*

Proof. Let $\pi = (d_1, d_2, \dots, d_n)$ be the non-increasing degree sequence of G . Let q and r be positive integers such that $2m = nq + r$ and $0 \leq r < n$. Denote by π' the sequence (c_1, c_2, \dots, c_n) , where $c_1 = \dots = c_r = q + 1$ and $c_{r+1} = \dots = c_n = q$.

(i) Suppose that π is not 0-sequence and 1-sequence, i.e., π is different from π' . Then there exists a positive integer k such that

$$d_1 \geq d_2 \geq \dots \geq d_k \geq q + 1 > q \geq d_{k+1} \geq \dots \geq d_n \tag{8}$$

since $\sum_{i=1}^n d_i = \sum_{i=1}^n c_i = 2m$. Obviously, $d_1 + d_2 + \dots + d_i \geq c_1 + c_2 + \dots + c_i$ for all $1 \leq i \leq k$. Suppose that

$$d_1 + d_2 + \dots + d_j < c_1 + c_2 + \dots + c_j \tag{9}$$

for some $k + 1 \leq j \leq n$. Then from (8) and $c_t \geq q$ for all t , it follows that

$$d_{j+1} + d_{j+2} + \dots + d_n \leq q(n - j) \leq c_{j+1} + c_{j+2} + \dots + c_n. \tag{10}$$

From (9) and (10), we have $\sum_{i=1}^n d_i < \sum_{i=1}^n c_i$ and a contradiction. Hence π majorizes π' and it follows $\sum_{i=1}^n c_i^2 < \sum_{i=1}^n d_i^2$ by the Karamata's inequality and $d_1 - d_n \geq 2$. If π' is planar graphical, then it contradicts to the fact that $\Sigma(G)$ is minimum among all simple planar (n, m) -graphs. Thus $\pi = \pi'$. On the other hand, by Lemma 3.1 and Lemma 3.2, π' is planar graphical except for $(m, n) = (7, 14)$, $(14, 35)$, $(11, 27)$, $(13, 32)$, $(13, 33)$ and $(15, 38)$.

(ii) Let $S_{n,m}$ be the set of positive integer sequences of length n whose elements sum to $2m$. Then from the proof of part (i), we conclude that the sums of squares of the elements of $4^7, 5^{14}, 5^{10}4^1, 5^{12}4^1, 6^15^{12}$ and $6^{15}4^1$ are the smallest in $S_{n,m}$ when $(n, m) = (7, 14), (14, 35), (11, 27), (13, 32), (13, 33)$ and $(15, 38)$, respectively. By using Matlab we obtain the second smallest elements in $S_{n,m}$ for each case, namely $5^14^53^1, 6^15^{12}4^1, 6^15^84^2, 6^15^{10}4^2, 6^25^{10}4^1$ and $6^25^{12}4^1$. These sequences are all planar graphical (see Figure 2.3). This completes the proof. \square

A fullerene graph is a 3-connected cubic planar graph whose faces are only pentagons and hexagons, modeling carbon cage molecules. By Euler's formula, every fullerene contains exactly 12 pentagonal faces. Fullerenes exist for all even numbers of vertices $n \geq 24$ and also for $n = 20$. The dual graph of a fullerene with n vertices is maximal planar with $n/2 + 2$ vertices. Hence we have the following results from Theorem 3.1. For $5 \leq n \leq 13$ and $n \neq 12$, maximal planar graphs of order n with minimum Σ -value are illustrated in Table 1,

Table 1. Degree sequences of maximal planar graph of order n , where $5 \leq n \leq 13$ and $n \neq 12$.

$(n, 3n - 6)$	(5, 9)	(6, 12)	(7, 15)	(8, 18)	(9, 21)	(10, 24)	(11, 27)	(13, 33)
Degree sequence	4^33^2	4^6	5^24^5	5^44^4	5^64^3	5^84^2	$6^15^84^2$	$6^25^{10}4^1$

Corollary 3.1. *Let G be a maximal planar graph of order n with minimum $\Sigma(G)$, where $n \geq 12$ and $n \neq 13$. Then G is the dual graph of a fullerene.*

4. Conclusion

In this paper, we have completely characterized the extremal graphs maximizing the first Zagreb index within the class of simple planar (n, m) -graphs. The first and second Zagreb indices have been extensively studied together in the literature. Fully resolving the corresponding maximization problem for the second Zagreb index (as well as other vertex-degree-based topological indices) remains an intriguing open challenge.

Acknowledgments

The authors are very grateful to the anonymous referees for their valuable comments, which have significantly improved the presentation of the paper. This work has been completed within the framework of the project (P2025-5035) supported by the National University of Mongolia. The first author is grateful to the Mongolian National University of Education (Funder ID: 100020678, Grant No. MNUE2026F011).

References

- [1] R. Ahlswede, G. O. H. Katona, Graphs with maximal number of adjacent pairs of edges, *Acta Math. Hung.* **32** (1978) 97–120.
- [2] F. Boesch, R. Brigham, S. Burr, R. Dutton, R. Tindell, *Maximizing the Sum Of the Squares of the Degrees of a Graph*, Technical Report, Stevens Inst. Tech., Hoboken, 1990.
- [3] B. Bollobás, P. Erdős, A. Sarkar, Extremal graphs for weights. *Discrete Math.* **200** (1999) 5–19.
- [4] B. Borovičanić, K. C. Das, B. Furtula, I. Gutman, Bounds for Zagreb indices, *MATCH Commun. Math. Comput. Chem.* **78** (2017) 17–100.
- [5] S. M. Cioabă, Sums of powers of the degrees of a graph, *Discrete Math.* **306** (2006) 1959–1964.
- [6] R. J. Cook, Vertex degrees of planar graphs, *J. Combin. Theory Ser. B* **26** (1979) 337–345.
- [7] K. C. Das, Maximizing the sum of the squares of the degrees of a graph, *Discrete Math.* **285** (2004) 57–66.
- [8] D. de Caen, An upper bound on the sum of squares of degrees, *Discrete Math.* **185** (1998) 245–248.
- [9] A. W. Goodman, Triangles in a complete chromatic graph, *J. Austral. Math. Soc. Ser. A* **39** (1985) 86–93.
- [10] I. Gutman, K. C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.* **50** (2004) 83–92.
- [11] I. Gutman, B. Rušić, N. Trinajstić, C. F. Wilcox, Graph theory and molecular orbitals. XII. Acyclic polyenes, *J. Chem. Phys.* **62** (1975) 3399–3405.
- [12] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total π -electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.
- [13] A. Hawkins, A. Hill, J. Reeve, J. Tyrell, On certain polyhedra, *Math. Gaz.* **50** (1966) 140–144.
- [14] D. Olpp, A conjecture of Goodman and the multiplicities of graphs, *Austral. J. Combin.* **14** (1996) 267–282.
- [15] U. N. Peled, R. Petreschi, A. Sterbini, (n, e) -graphs with maximum sum of squares of degrees, *J. Graph Theory* **31** (1999) 283–295.
- [16] E. F. Schmeichel, S. L. Hakimi, On planar graphical degree sequences, *SIAM J. Appl. Math.* **32** (1977) 598–609.
- [17] M. Truszczynski, Note on vertex degrees of planar graphs, *J. Graph Theory* **8** (1984) 171–176.
- [18] K. Xu, K. C. Das, S. Balachandran, Maximizing the Zagreb indices of (n, m) -graphs, *MATCH Commun. Math. Comput. Chem.* **72** (2014) 641–654.