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Research Article

Splitting Vertices of Bipartite Graphs Preserves de Bruijn–Erdős Property

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Abstract

In this article, we prove that every graph obtained from a bipartite graph by iteratively splitting vertices into two adjacent twins has the de Bruijn–Erdős property.

Keywords: Chen–Chvátal conjecture; metric spaces; bipartite graphs.

2020 Mathematics Subject Classification: 68R10.

1. Introduction

Given a metric space (X, dist), we follow the natural definition of betweenness introduced by Menger [12] in 1928: an element b is *between* elements a and c if $\text{dist}(a, b) + \text{dist}(b, c) = \text{dist}(a, c)$. We say that elements a, b and c are *collinear* if they are pairwise distinct and one of them is between the other two. Eight decades after Menger's work, Chen and Chvátal [5] introduce the notion of line for general metric spaces. Namely, given two points a and b of X, the *line generated by a and b*, denoted $\overline{ab}$, is the set $\{a, b\}$ augmented by all points forming a collinear triple with a and b. In their paper, they formulate a question which has since earned the title of conjecture.

Chen–Chvátal conjecture (Question 1 in [5])**.** *Every finite metric space $(X, dist)$ where no line consists of the entire ground set X determines at least $|X|$ distinct lines.*

Their original goal is to extend some results from the Euclidean setting to the more general setting of metric spaces. In this specific case, the authors refer to the original work of de Bruijn and Erdős [7]. Therefore, we say that a finite metric space (X, dist) has the *de Bruijn–Erdős Property* if it has a *universal line* (that is, a line containing all points of X) or at least $|X|$ lines.

This conjecture has attracted the attention of researchers in diverse corners of the world. For a broad view of clusters working on the topic, we may enumerate: the Czech connection [9, 10], the Chilean connection [3, 11], the Chinese connection [8] and the French connection [1, 2].

The Chen–Chvátal conjecture remains open in the restricted case of metric spaces arising from graphs where X is the vertex set and dist is the usual shortest path distance. Up to the writing of this note, the best general lower bound on the number of lines in a graph on n vertices with no universal line is $\mathcal{O}(n^{2/3})$ obtained by Huang [8]. In his extensive survey published in 2018, Chvátal [6] lists no less than twenty-nine open problems related to the Chen-Chvátal conjecture. Some of these have been solved since [1, 4]. We focus ourselves on the eighth problem of the survey.

Problem 1.1 (Problem 8 in [6])**.** *Prove that all graphs obtained from bipartite graphs by repeatedly splitting of vertices into adjacent twins have the de Bruijn–Erdős property.*

In this note, we answer positively to this problem via Theorem 1.1.

Theorem 1.1. *Let G be a graph on n vertices (with $n \geq 2$) obtained from a bipartite graph by repeated splitting of vertices into adjacent twins, then G admits a universal line or has at least n distinct lines.*

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In order to prove Theorem 1.1 we use a few known results that we recall here.

Lemma 1.1. *If G is a disconnected graph on two vertices or more, then G has the de Bruijn–Erdős property.*

Proof. Let G be a disconnected graph on n vertices, then there exists a non trivial partition of its vertex set into two sets V_1 of n_1 vertices and V_2 of n_2 vertices such that no edge goes between V_1 and V_2. Therefore any pair of vertices (u, v) in $V_1 \times V_2$ generates a line of cardinality 2: $\overline{uv} = \{u, v\}$. We thus have $n_1 n_2$ different lines. If both n_1 and n_2 are greater than or equal to 2, $n_1 n_2 \geq n$. If exactly one of them (say n_1) is 1, then we get $n_2 = n - 1$, another line is obtained by taking any pair of vertices in V_2. If both of them are 1, then the graph has two vertices and the line they define is universal. □

Lemma 1.2 (Theorem 6 in [6]). *If G is a bipartite connected graph, then any pair of adjacent vertices generates a universal line.*

Proof. In a connected graph G, the line defined by a pair of adjacent vertices contains every vertex except those equidistant to both vertices. Since G is connected, this enforces the existence of an odd cycle. □

2. Proof of Theorem 1.1

Let G be a graph on n vertices obtained from a bipartite graph H by repeated splitting of vertices into adjacent twins. In other words, there is a mapping $f : V(G) \to V(H)$ which assigns to each vertex of G the original vertex in H from which it has been derived. Note that this mapping is surjective (we only split and never delete so every vertex of H has at least one vertex in G being mapped to it). For a given vertex x of H, the *blob* of x is its preimage by f, denoted by $f^{-1}(x)$. Observe that a blob forms a clique in G. A blob is *trivial* if it is a singleton, and *rich* otherwise. Figure 2.1 depicts a transformation from a bipartite graph H to a graph G with five rich blobs and two trivial blobs.

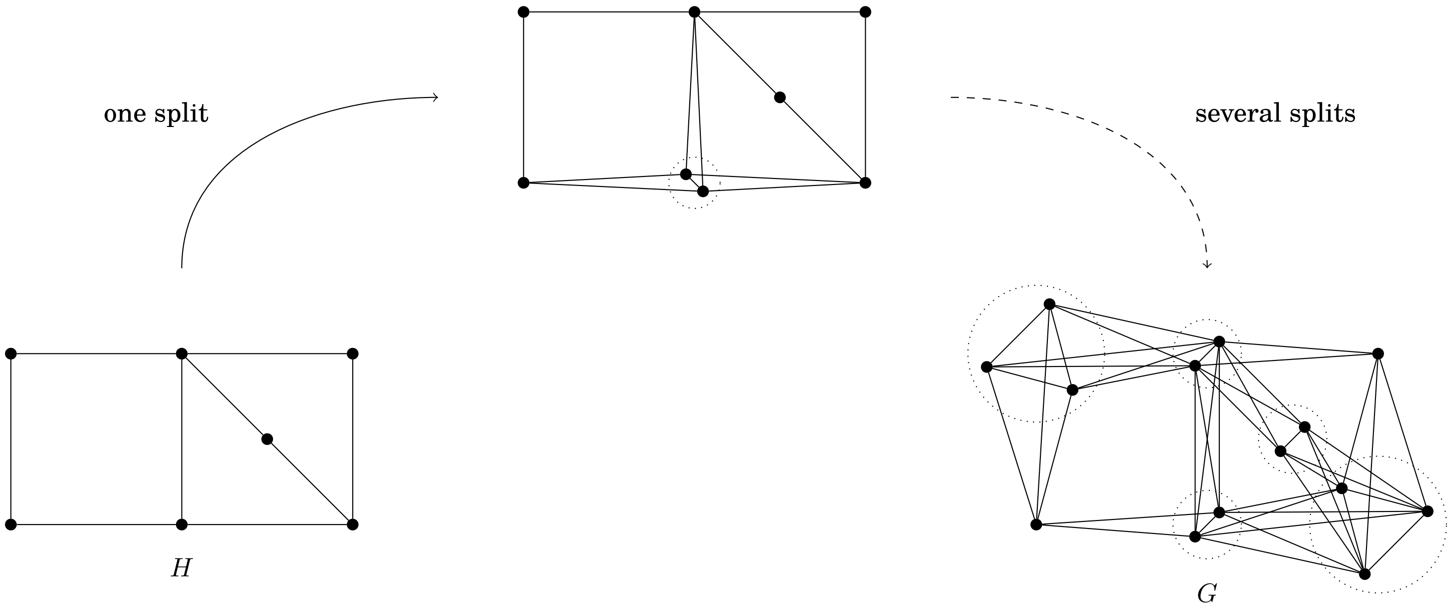

Figure 2.1: Splitting vertices from some graph H to some graph G.

If G is disconnected, then by Lemma 1.1, we are done. So, we now assume that G is connected.

Claim 2.1. *We may assume that G is connected.*

Let us now focus on the line generated by any two vertices u and v of G.

- If u and v are in the same blob, then note that $\overline{uv} = \{u, v\}$, so it is uniquely generated.
- If u and v are in different blobs (meaning $f(u) \neq f(v)$), consider L the line generated by $f(u)$ and $f(v)$ in H. Then, the line $\overline{uv}$ in G consists of u, v and all points z such that $f(z) \in L$ and $f(z) \notin \{f(u), f(v)\}$: indeed, for any vertex z such that $f(z) \notin \{f(u), f(v)\}$, the triplet of distances between u, v and z is exactly the same as between $f(u)$, $f(v)$ and $f(z)$ and thus so is the presence in the lines L and $\overline{uv}$; whereas for vertices z in the blobs of u or v, the offset of 1 prevents z to be in the line $\overline{uv}$.

The second case allows us to conclude the following: If two trivial blobs u and v are adjacent, then by Lemma 1.2 and the previous discussion, line $\overline{uv}$ is universal. So, we may further assume the following:

Claim 2.2. *We may assume that trivial blobs form an independent set.*

As a direct corollary of Claims 2.1 and 2.2, we can further our assumptions to the following:

Claim 2.3. *Every trivial blob has a rich blob neighbor.*

We may now define a pair of parameters that will allow us to lower bound the number of lines in G:

- let p be the number of vertices in G belonging to a rich blob (in Figure 2.1, $p = 12$),
- let k be the number of rich blobs having at least one trivial blob as a neighbor (in Figure 2.1, $k = 4$).

Observe that since each rich blob has at least two vertices and there are at least k rich blobs, then

$$p \geq 2k. \tag{1}$$

Any line generated by a pair of vertices among rich blobs is uniquely defined (if both u and v are in a common blob, the line has cardinality 2, if in different blobs, the line intersects exactly two rich blobs on singletons identifying u and v). Thus, we get $\binom{p}{2}$ distinct lines. Therefore if $p = n$, we obtain $\binom{n}{2}$ lines and G has the de Bruijn–Erdős property. Moreover, if no splitting has been made, then $p = 0$ and G is bipartite and we know that it has the de Bruijn–Erdős property. So, we may assume the following:

Claim 2.4. *We may assume that* $2 \leq p \leq n-1$.

Whenever $p < n$, there are some trivial blobs (exactly $n - p$). Thus by Claim 2.3, k cannot be 0. Then, there exists a rich blob B which has at least $\lceil \frac{n-p}{k} \rceil$ trivial blobs as neighbors. Consider the set of lines generated by pairs of trivial blobs adjacent to B. These trivial blobs are all at distance 2 one from each other and so any such lines takes only two of them. We get a set of $\binom{\lceil \frac{n-p}{k} \rceil}{2}$ different lines. These lines either take a blob completely or do not intersect it. Thus they are distinct from the $\binom{p}{2}$ lines previously defined.

Finally, for each vertex u in one of the k blobs having a trivial neighbor, we consider a line $\overline{uv_u}$ where v_u is the vertex in one of the trivial blobs adjacent to u. These lines intersect exactly one rich blob on a singleton (identifying u). Therefore they are distinct from each other and new with respect to both previous families of lines. Since every rich blob has at least two vertices, they form a family of at least $2k$ lines. In conclusion, the number of lines in G is at least,

$$\binom{p}{2} + \binom{\lceil \frac{n-p}{k} \rceil}{2} + 2k. \tag{2}$$

The end of the proof amounts to proving that this quantity is greater than or equal to n for any choice of p and k such that $2 \leq p \leq n-1$ and $1 \leq k \leq p/2$. It turns out that the additive term of $2k$ is needed only for very specific cases when n is small. Cases for low values (namely when $n \leq 39$) are thus examined through a basic computer program (see Appendix A). In general the first two terms of (2) are enough.

Let $\epsilon = 1.531$ and $n_0 = 40$. These numbers have been chosen in order to minimize n_0 while satisfying the following implications for any $n \geq n_0$ and any real number $x \geq 1$. The proofs are not difficult but require paper space. So we present them in Appendix B.

$$x \geq \epsilon\sqrt{n} \Rightarrow \frac{x^2 - x}{2} \geq n, \tag{3}$$

$$x \geq \epsilon\sqrt{\frac{n}{2}} \Rightarrow \frac{x^2 - x}{2} \geq \frac{n}{2}, \tag{4}$$

$$x \leq \epsilon\sqrt{n} \Rightarrow \frac{2n}{x} - 2 \geq \epsilon\sqrt{\frac{n}{2}}, \tag{5}$$

$$\frac{2\sqrt{n}}{\epsilon + \frac{2}{\sqrt{n}}} \geq \epsilon\sqrt{\frac{n}{2}}. \tag{6}$$

The basic sketch is the following: if p is sufficiently large (greater than $\epsilon\sqrt{n}$), then the first term is greater than n. If p is sufficiently small (less than $\frac{2\sqrt{n}}{\epsilon + 2/\sqrt{n}}$), then the second term is greater than n. If p is in between, each term is at least $n/2$ summing up to n.

Case 1: $p \geq \epsilon\sqrt{n}$**.** When p is large enough we consider only the first term $\binom{p}{2}$. By (3),

$$\binom{p}{2} = \frac{p^2 - p}{2} \geq n.$$

Case 2: $p \leq \frac{2\sqrt{n}}{\epsilon + \frac{2}{\sqrt{n}}}$**.** When p is small enough, we consider only the second term. First we observe that k being at most $p/2$, we have

$$\left\lceil \frac{n-p}{k} \right\rceil \geq \frac{n-p}{k} \geq \frac{2n}{p} - 2.$$

By the upper bound on p, we obtain:

$$\frac{2n}{p} - 2 \geq \epsilon\sqrt{n};$$

and using (3), we conclude

$$\binom{\left\lceil \frac{n-p}{k} \right\rceil}{2} \geq \binom{\frac{2n}{p} - 2}{2} \geq n.$$

Case 3: $\frac{2\sqrt{n}}{\epsilon + \frac{2}{\sqrt{n}}} < p < \epsilon\sqrt{n}$**.** In this last case, we prove that both first terms of the sum (2) provide at least $n/2$ lines. Observe that by (5) and (6),

$$p < \epsilon\sqrt{n} \text{ thus } \frac{2n}{p} - 2 \geq \epsilon\sqrt{\frac{n}{2}},$$
$$p > \frac{2\sqrt{n}}{\epsilon + \frac{2}{\sqrt{n}}} \text{ thus } p \geq \epsilon\sqrt{\frac{n}{2}}.$$

Using (4), we complete the proof of Theorem 1.1.

3. An Open Question

As a conclusion, we have proven that a graph obtained from a bipartite graph by repeated splitting of vertices into adjacent twins has the de Bruijn–Erdős property. Still, the greedy version of the conjecture (mentionned as Conjecture 3 in the survey of Chvátal [6]) is that for graphs we can reach not only a linear number of line but a little bit extra.

Conjecture 3.1 (Conjecture 3 in [6])**.** *All graphs G without a universal line have $\Omega\left(|G|^{\frac{4}{3}}\right)$ distinct lines.*

Our approach fails to that matter since when p is roughly $\sqrt{n}$, and k is $p/2$, the lower bound we obtain is only $\Omega(n)$. Thus it remains open whether this class of graphs has the expected number of lines.

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Appendix A. Python Code Checking Triples up to $n = 39$

This script can be downloaded from the GitLab repository[†] of the first author.

```
from numpy import ceil, floor
from math import comb

def nbLines(n,p,k):
''' Returns a lower bound on the number of lines in a graph G
obtained by iteratively splitting the vertices of a bipartite
graph H into true twins.

Input: n, number of vertices of G
p, number of vertices of G in rich blobs
k, number of rich blobs with trivial neighbors

Output: an integer lower bounding the number of lines in G.
'''
assert 2 <= p and p <= n-1 and 1<= k and k <= p/2
first = comb(p,2)
second = comb((int(ceil((n-p)/k))),2)
third = 2*k
return first + second + third

def main(n0):
''' Main routine, enumerates all legal triples (n,p,k) for n
between 3 and n0. And checks that the number of lines is at least
n. If some triple fails, it raises an Exception.
'''
for n in range(3,n0+1):
for p in range(2,n):
for k in range(1,int(floor(p/2))+1):
if nbLines(n,p,k) < n:
raise Exception("We have a problem.")
print("Checked.")

if __name__=="__main__":
main(39)
```

[†] https://gitlab.limos.fr/labeaudo/check_triples_lines

Appendix B. Proofs for ϵ and n_0

We want to prove that for $\epsilon = 1.531$ and $n_0 = 40$. The following are true for any $n \geq n_0$ and any real number $x \geq 1$.

$$x \geq \epsilon\sqrt{n} \Rightarrow \frac{x^2 - x}{2} \geq n, \tag{3}$$

$$x \geq \epsilon\sqrt{\frac{n}{2}} \Rightarrow \frac{x^2 - x}{2} \geq \frac{n}{2}, \tag{4}$$

$$x \leq \epsilon\sqrt{n} \Rightarrow \frac{2n}{x} - 2 \geq \epsilon\sqrt{\frac{n}{2}}, \tag{5}$$

$$\frac{2\sqrt{n}}{\epsilon + \frac{2}{\sqrt{n}}} \geq \epsilon\sqrt{\frac{n}{2}}. \tag{6}$$

First, observe that (4) implies (3) since for any $n \geq n_0$, we can let n' be $2n$ which is also greater than n_0 and then (4) on n' reads exactly as (3) on n. Now let us prove (4). Since $\epsilon\sqrt{\frac{n}{2}} > 1$ by our choice of n_0 and ϵ, and as function $f : x \mapsto x^2 - x$ is monotone and increasing on $[1/2, +\infty[$, we have

$$\begin{aligned}
\frac{x^2 - x}{2} &\geq \frac{\epsilon^2 \frac{n}{2} - \epsilon\sqrt{\frac{n}{2}}}{2} \\
&\geq \epsilon^2 \frac{n}{4} - \epsilon\sqrt{\frac{n}{8}} \\
&\geq \frac{n}{2} + (\epsilon^2 - 2)\frac{n}{4} - \epsilon\sqrt{\frac{n}{8}} \\
&\geq \frac{n}{2} + \frac{\sqrt{n}}{2}\left((\epsilon^2 - 2)\frac{\sqrt{n}}{2} - \frac{\epsilon}{\sqrt{2}}\right).
\end{aligned}$$

The last multiplicative term is increasing with n since $\epsilon > \sqrt{2}$ when $n = 40$ it is strictly positive ($\simeq 0.005$) thus validating implications (3) and (4).

Concerning implication (5),

$$\begin{aligned}
\frac{2n}{x} - 2 &\geq \frac{2n}{\epsilon\sqrt{n}} - 2 \\
&\geq \frac{2\sqrt{2}}{\epsilon}\sqrt{\frac{n}{2}} - 2 \\
&\geq \epsilon\sqrt{\frac{n}{2}} + \left(\frac{2\sqrt{2}}{\epsilon} - 1\right)\sqrt{\frac{n}{2}} - 2.
\end{aligned}$$

Since $\epsilon < 2\sqrt{2}$, the additive term is increasing with n. It happens to be positive for n_0 ($\simeq 0.68$), proving implication (5).

Finally, we check the last inequality (6):

$$\begin{aligned}
\frac{2\sqrt{n}}{\epsilon + \frac{2}{\sqrt{n}}} \geq \epsilon\sqrt{\frac{n}{2}} &\Leftrightarrow 2\sqrt{n} \geq \epsilon^2\sqrt{\frac{n}{2}} + \epsilon\sqrt{2} \\
&\Leftrightarrow \sqrt{n} \geq \frac{\epsilon\sqrt{2}}{2 - \epsilon^2/\sqrt{2}}
\end{aligned}$$

The right hand side term is strictly less than 6.321 while the square root of 40 is strictly greater than 6.324.