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Research Article

On the Sum of the Randić Index and the Reciprocal Randić Index of Uniform Hypergraphs

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Abstract

Let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H})$ and hyperedge set $E(\mathcal{H})$. If $|e| = k$ for every $e \in E(\mathcal{H})$, then $\mathcal{H}$ is called a k-uniform hypergraph. The sum of the Randić index and the reciprocal Randić index of $\mathcal{H}$ is defined as

$$(R + RR)(\mathcal{H}) = \sum_{e \in E(\mathcal{H})} \left(\frac{1}{\sqrt{\prod\limits_{v \in e} d_{\mathcal{H}}(v)}} + \sqrt{\prod_{v \in e} d_{\mathcal{H}}(v)} \right).$$

In this paper, an extremal problem for the sum of the Randić index and the reciprocal Randić index of k-uniform hypergraphs with $m \geq 3$ hyperedges is investigated. For k-uniform hypertrees with $m \geq 3$ hyperedges, the minimum and maximum values, along with the corresponding extremal hypertrees, are obtained. For connected k-uniform hypergraphs with $m \geq 3$ hyperedges, the minimum value and the corresponding extremal hypergraph are determined, and the hypergraphs that attain the maximum value are discussed.

Keywords: hypergraph; hypertree; k-uniform hypergraph; Randić index; reciprocal Randić index; extremal value.

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1. Introduction

A *hypergraph* $\mathcal{H}$ consists of a non-empty finite vertex set $V(\mathcal{H})$ and a hyperedge set $E(\mathcal{H})$, where each hyperedge $e \in E(\mathcal{H})$ is a subset of $V(\mathcal{H})$ with at least two vertices. For $v \in V(\mathcal{H})$ and $e \in E(\mathcal{H})$, the *degree* of v, denoted by $d_{\mathcal{H}}(v)$, is the number of hyperedges that contain the vertex v, and the *degree* of e, denoted by $|e|$, is the number of vertices that are contained in e. If $d_{\mathcal{H}}(v) = 1$, then v is called a *pendent vertex*. If e contains exactly $|e| - 1$ pendent vertices, then e is called a *pendent hyperedge*. Two hyperedges in $\mathcal{H}$ are said to be *adjacent* if they have at least one common vertex.

A *path* P between two vertices v_0 and v_t in $\mathcal{H}$ is a sequence of distinct vertices and hyperedges, $P = (v_0, e_1, v_1, e_2, v_2, \ldots, e_t, v_t)$, where $\{v_{i-1}, v_i\} \subseteq e_i$ for $i = 1, 2, \ldots, t$. t is called the *length* of P. A path P in $\mathcal{H}$ is called a *pendant path* at v_0 if $d_{\mathcal{H}}(v_0) \geq 2$, $d_{\mathcal{H}}(v_i) = 2$ for $i = 1, 2, \ldots, t-1$, $d_{\mathcal{H}}(v_t) = 1$, and $d_{\mathcal{H}}(v) = 1$ for $v \in e_i \setminus \{v_{i-1}, v_i\}$ with $1 \leq i \leq t$. A hypergraph $\mathcal{H}$ is *connected* if there exists a path between any two vertices in $\mathcal{H}$.

A hypergraph $\mathcal{H}$ is said to be *linear* if any two hyperedges in $\mathcal{H}$ have at most one vertex in common. A hypergraph with $|e| = k$ for every $e \in E(\mathcal{H})$ is called a *k-uniform* hypergraph. The 2-uniform hypergraph is the ordinary graph. A *complete* k-uniform hypergraph is a hypergraph in which each set of k vertices forms a hyperedge. A complete k-uniform hypergraph with n vertices is denoted by $\mathcal{K}_n^{(k)}$.

A *hypertree* $\mathcal{T}$ is a connected hypergraph where the removal of any hyperedge in $\mathcal{T}$ disconnects the hypergraph. Notably, a hypertree may contain cycles [7]. A *hyperpath* is a hypertree where each vertex has degree at most two, and each hyperedge is adjacent to at most two other hyperedges.

Hypergraphs find application in chemistry when modeling molecules or chemical reactions involving multiple atoms bonding simultaneously. Unlike graphs, hypergraphs can represent interactions involving more than two atoms, which is particularly relevant for reactions with complex bonding patterns or in capturing molecular properties that arise from multiple atom groupings. Hypergraphs offer a more accurate depiction of certain chemical scenarios, such as transition states in reactions, which involve multiple atoms simultaneously changing their bonding configurations [5]. Recently, the idea of topological indices is extended from graphs to hypergraphs, and the relevant research results can be found in [2–4,6,8–11].

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In 2025, Lu and Zhu [6] studied the product of the Wiener index and the Harary index for k-uniform hypergraphs, and determined the unique k-uniform hypergraphs with maximum, minimum and second minimum values, respectively. However, the sum of a topological index and its reciprocal topological index of hypergraphs remains largely unexplored. This paper is a new attempt to study on this topic and it is likely to attract further attention.

For a hypergraph $\mathcal{H}$, the *Randić index* and *reciprocal Randić index* of $\mathcal{H}$ are defined (see [8]) as

$$R(\mathcal{H}) = \sum_{e\in E(\mathcal{H})} \frac{1}{\sqrt{\prod\limits_{v\in e} d_{\mathcal{H}}(v)}} \quad \text{and} \quad RR(\mathcal{H}) = \sum_{e\in E(\mathcal{H})} \sqrt{\prod_{v\in e} d_{\mathcal{H}}(v)}, \tag{1}$$

respectively. The sum of the Randić index and the reciprocal Randić index of $\mathcal{H}$ is

$$R+RR = (R+RR)(\mathcal{H}) = \sum_{e\in E(\mathcal{H})} \left(\frac{1}{\sqrt{\prod\limits_{v\in e} d_{\mathcal{H}}(v)}} + \sqrt{\prod_{v\in e} d_{\mathcal{H}}(v)} \right). \tag{2}$$

In this paper, we study the sum $R+RR$ for k-uniform hypergraphs with $m \geq 3$ hyperedges. In Section 2, we present some key lemmas. In Section 3, the minimum and maximum values of $R+RR$ are obtained for k-uniform hypertrees with $m \geq 3$ hyperedges, and the extremal hypertrees that reach the minimum and maximum values are characterized, respectively. In Section 4, the minimum value of $R+RR$ is obtained for connected k-uniform hypergraphs with $m \geq 3$ hyperedges, and the extremal hypergraph that reach the minimum value is characterized. In Section 5, we discuss hypergraphs that reach the maximum value of $R+RR$ among all connected k-uniform hypergraphs with $m \geq 3$ hyperedges.

2. Some Lemmas

In this section, we present some lemmas used in later proofs, where the first two lemmas are obvious.

Lemma 2.1. *The function $f(x) = \frac{1}{x} + x$ is increasing on x for $x \geq 1$.*

Lemma 2.2. *Let $a > b \geq 1$. Then the function*

$$g(x) = \frac{1}{ax} + ax - \left(\frac{1}{bx} + bx\right)$$

is increasing on x for $x > 0$.

Lemma 2.3. *The following five functions are increasing on x for $x \geq 1$.*

(1) $F_1(x) = \sqrt{2(x+2)} + \frac{1}{\sqrt{2(x+2)}} - \sqrt{x+2} - \frac{1}{\sqrt{x+2}}$;

(2) $F_2(x) = x\left(\sqrt{x+2} + \frac{1}{\sqrt{x+2}} - \sqrt{x+1} - \frac{1}{\sqrt{x+1}}\right)$;

(3) $F_3(x) = 2\sqrt{x+2} + \frac{2}{\sqrt{x+2}} - \sqrt{2(x+1)} - \frac{1}{\sqrt{2(x+1)}}$;

(4) $F_4(x) = \sqrt{x+2} + \frac{1}{\sqrt{x+2}} + \sqrt{2(x+2)} + \frac{1}{\sqrt{2(x+2)}} - \sqrt{2(x+1)} - \frac{1}{\sqrt{2(x+1)}}$;

(5) $F_5(x) = 2\sqrt{2(x+2)} + \frac{2}{\sqrt{2(x+2)}} - \sqrt{2(x+1)} - \frac{1}{\sqrt{2(x+1)}}$.

Proof. By Lemma 2.2, $F_1(x)$ is increasing on x for $x \geq 1$. Note that $F_4(x) = F_1(x) + F_3(x)$ and $F_5(x) = 2F_1(x) + F_3(x)$. We only need to prove that both $F_2(x)$ and $F_3(x)$ are increasing on x for $x \geq 1$. For $F_2(x)$, note that

$$\begin{aligned} F_2'(x) &= \sqrt{x+2} + \frac{1}{\sqrt{x+2}} - \sqrt{x+1} - \frac{1}{\sqrt{x+1}} + \frac{x}{2}\left(\frac{1}{\sqrt{x+2}} - \frac{1}{(x+2)^{3/2}} - \frac{1}{\sqrt{x+1}} + \frac{1}{(x+1)^{3/2}}\right) \\ &= \frac{3\sqrt{x+1}\sqrt{x+2} - 3x - 4}{2\sqrt{x+1}} + \frac{x}{2}\left(\frac{1}{(x+1)^{3/2}} - \frac{1}{(x+2)^{3/2}}\right). \end{aligned}$$

Since

$$\left(3\sqrt{x+1}\sqrt{x+2}\right)^2 - (3x+4)^2 = 3x+2 > 0,$$

we have $F_2'(x) > 0$, that is, $F_2(x)$ is increasing on x for $x \geq 1$.

For $F_3(x)$, note that

$$F_3'(x) = \frac{1}{\sqrt{x+2}} - \frac{1}{(x+2)\sqrt{x+2}} - \frac{1}{\sqrt{2x+2}} + \frac{1}{(2x+2)\sqrt{2x+2}}$$
$$= \frac{1}{\sqrt{x+2}} \cdot \frac{x+1}{x+2} - \frac{1}{\sqrt{2x+2}} \cdot \frac{2x+1}{2x+2}.$$

Since

$$\left(\frac{1}{\sqrt{x+2}} \cdot \frac{x+1}{x+2}\right)^2 - \left(\frac{1}{\sqrt{2x+2}} \cdot \frac{2x+1}{2x+2}\right)^2 = \frac{4x^5+12x^4+7x^3-6x^2-4x}{8(x+1)^3(x+2)^3} > 0,$$

we have

$$\frac{1}{\sqrt{x+2}} \cdot \frac{x+1}{x+2} > \frac{1}{\sqrt{2x+2}} \cdot \frac{2x+1}{2x+2}.$$

Then $F_3'(x) > 0$, that is, $F_3(x)$ is increasing on x for $x \geq 1$. □

Let $\mathcal{H}$ be a hypergraph. For a hyperedge $e \in E(\mathcal{H})$, we denote

$$(R+RR)_{\mathcal{H}}(e) = \frac{1}{\sqrt{\prod\limits_{v\in e} d_{\mathcal{H}}(v)}} + \sqrt{\prod_{v\in e} d_{\mathcal{H}}(v)}.$$

Then Equation (2) can be rewritten as

$$(R+RR)(\mathcal{H}) = \sum_{e\in E(\mathcal{H})} (R+RR)_{\mathcal{H}}(e). \tag{3}$$

Lemma 2.4. *Let $\mathcal{H}$ be a k-uniform hypergraph with $m \geq 3$ hyperedges, $e_1, e_2 \in E(\mathcal{H})$ with $e_1 \cap e_2 \neq \phi$ and $u \in e_1 \cap e_2$. Let $\mathcal{H}'$ be a hypergraph obtained from $\mathcal{H}$ by replacing the hyperedge e_1 with a new hyperedge $e_1' = (e_1 \setminus \{u\}) \cup \{u'\}$, where u' is a new vertex. Then $\mathcal{H}'$ is also a k-uniform hypergraph with m hyperedges, and*

$$(R+RR)(\mathcal{H}) > (R+RR)(\mathcal{H}').$$

Proof. By assumptions, $d_{\mathcal{H}}(u) \geq 2$, $d_{\mathcal{H}'}(u) = d_{\mathcal{H}}(u) - 1$, and $d_{\mathcal{H}'}(u') = 1$. Let $d_{\mathcal{H}}(u) = s$, and the hyperedges that contain the vertex u, except for e_1 and e_2, be $e_3, \ldots, e_s$. Denote

$$A_i = \sqrt{\prod_{v\in e_i\setminus\{u\}} d_{\mathcal{H}}(v)}, \quad i = 1, 2, \ldots, s.$$

Then

$$(R+RR)_{\mathcal{H}}(e_i) = \frac{1}{\sqrt{s}A_i} + \sqrt{s}A_i, \quad i = 1, 2, \ldots, s,$$
$$(R+RR)_{\mathcal{H}'}(e_1') = \frac{1}{A_1} + A_1,$$
$$(R+RR)_{\mathcal{H}'}(e_i) = \frac{1}{\sqrt{s-1}A_i} + \sqrt{s-1}A_i, \quad i = 2, \ldots, s.$$

By Lemmas 2.1 and 2.2,

$$(R+RR)_{\mathcal{H}}(e_1) - (R+RR)_{\mathcal{H}'}(e_1') = \frac{1}{\sqrt{s}A_1} + \sqrt{s}A_1 - \left(\frac{1}{A_1} + A_1\right)$$
$$\geq \left(\frac{1}{\sqrt{s}A_1} + \sqrt{s}A_1 - \left(\frac{1}{A_1} + A_1\right)\right)\Big|A_1 = 1$$
$$= \frac{1}{\sqrt{s}} + \sqrt{s} - 2 > 0,$$

and for $i = 2, 3, \ldots, s$,

$$(R+RR)_{\mathcal{H}}(e_i) - (R+RR)_{\mathcal{H}'}(e_i) = \frac{1}{\sqrt{s}A_i} + \sqrt{s}A_i - \left(\frac{1}{\sqrt{s-1}A_i} + \sqrt{s-1}A_i\right)$$
$$\geq \left(\frac{1}{\sqrt{s}A_i} + \sqrt{s}A_i - \left(\frac{1}{\sqrt{s-1}A_i} + \sqrt{s-1}A_i\right)\right)\Big|A_i = 1$$
$$= \frac{1}{\sqrt{s}} + \sqrt{s} - \left(\frac{1}{\sqrt{s-1}} + \sqrt{s-1}\right) > 0.$$

Note that for any hyperedge $e \in E(\mathcal{H}) \setminus \{e_1, e_2, \ldots, e_s\}$,

$$(R+RR)_{\mathcal{H}}(e) = (R+RR)_{\mathcal{H}'}(e).$$

Then, by Equation (3), we have $(R+RR)(\mathcal{H}) > (R+RR)(\mathcal{H}')$. □

Let s, t be two positive integers, and $\mathcal{H}$ be a k-uniform hypergraph with $u, v \in V(\mathcal{H})$. We use $\mathcal{H}_u(s)$ to denote the k-uniform hypergraph obtained from $\mathcal{H}$ by attaching a pendant path of length s at u, and $\mathcal{H}_{u,v}(s,t)$ the k-uniform hypergraph obtained from $\mathcal{H}$ by attaching a pendant path of length s at u, and a pendant path of length t at v. In particular, if $u = v$, then $\mathcal{H}_{u,v}(s,t)$ can be simply denoted as $\mathcal{H}_u(s,t)$.

Lemma 2.5. *Let $\mathcal{H}$ be a connected k-uniform hypergraph and $u \in V(\mathcal{H})$. Then for $s \geq t \geq 1$,*

$$(R+RR)(\mathcal{H}_u(s,t)) > (R+RR)(\mathcal{H}_u(s+t)).$$

Proof. Let $d_{\mathcal{H}}(u) = p$, and the hyperedges that contain the vertex u be $e_1, e_2, \ldots, e_p$. Denote

$$A_i = \sqrt{\prod_{v \in e_i \setminus \{u\}} d_{\mathcal{H}}(v)}, \quad i = 1, 2, \ldots, p.$$

Note that $d_{\mathcal{H}_u(s,t)}(u) = p+2$ and $d_{\mathcal{H}_u(s+t)}(u) = p+1$. Then for $i = 1, 2, \ldots, p$,

$$\begin{aligned}
(R+RR)_{\mathcal{H}_u(s,t)}(e_i) &= \frac{1}{\sqrt{p+2}A_i} + \sqrt{p+2}A_i, \\
(R+RR)_{\mathcal{H}_u(s+t)}(e_i) &= \frac{1}{\sqrt{p+1}A_i} + \sqrt{p+1}A_i.
\end{aligned}$$

By Lemma 2.2 and Lemma 2.3(2),

$$\begin{aligned}
\sum_{i=1}^{p} \left((R+RR)_{\mathcal{H}_u(s,t)}(e_i) - (R+RR)_{\mathcal{H}_u(s+t)}(e_i)\right) &\geq \sum_{i=1}^{p} \left(\frac{1}{\sqrt{p+2}A_i} + \sqrt{p+2}A_i - \frac{1}{\sqrt{p+1}A_i} - \sqrt{p+1}A_i\right)\Big|_{A_i=1} \\
&= p\left(\frac{1}{\sqrt{p+2}} + \sqrt{p+2} - \frac{1}{\sqrt{p+1}} - \sqrt{p+1}\right) \\
&\geq p\left(\frac{1}{\sqrt{p+2}} + \sqrt{p+2} - \frac{1}{\sqrt{p+1}} - \sqrt{p+1}\right)\Big|_{p=1} = \frac{1}{6}\left(8\sqrt{3} - 9\sqrt{2}\right).
\end{aligned}$$

Note that for any hyperedge $e \in E(\mathcal{H}) \setminus \{e_1, e_2, \ldots, e_p\}$,

$$(R+RR)_{\mathcal{H}_u(s,t)}(e) = (R+RR)_{\mathcal{H}_u(s+t)}(e).$$

By Equation (3), we have

$$\begin{aligned}
&(R+RR)(\mathcal{H}_u(s,t)) - (R+RR)(\mathcal{H}_u(s+t)) \\
=& \sum_{i=1}^{p} \left((R+RR)_{\mathcal{H}_u(s,t)}(e_i) - (R+RR)_{\mathcal{H}_u(s+t)}(e_i)\right) \\
&+ \sum_{e \in E(\mathcal{H}_u(s,t)) \setminus E(\mathcal{H})} (R+RR)_{\mathcal{H}_u(s,t)}(e) - \sum_{e \in E(\mathcal{H}_u(s+t)) \setminus E(\mathcal{H})} (R+RR)_{\mathcal{H}_u(s+t)}(e) \\
\geq& \frac{1}{6}\left(8\sqrt{3} - 9\sqrt{2}\right) + \sum_{e \in E(\mathcal{H}_u(s,t)) \setminus E(\mathcal{H})} (R+RR)_{\mathcal{H}_u(s,t)}(e) - \sum_{e \in E(\mathcal{H}_u(s+t)) \setminus E(\mathcal{H})} (R+RR)_{\mathcal{H}_u(s+t)}(e). \qquad (4)
\end{aligned}$$

Case 1. $s = t = 1$.
By (4) and Lemma 2.3(3), we have

$$\begin{aligned}
&(R+RR)(\mathcal{H}_u(s,t)) - (R+RR)(\mathcal{H}_u(s+t)) \\
\geq& \frac{1}{6}\left(8\sqrt{3} - 9\sqrt{2}\right) + 2\left(\sqrt{p+2} + \frac{1}{\sqrt{p+2}}\right) - \left(\sqrt{2(p+1)} + \frac{1}{\sqrt{2(p+1)}} + \sqrt{2} + \frac{1}{\sqrt{2}}\right) \\
\geq& \frac{1}{6}\left(8\sqrt{3} - 9\sqrt{2}\right) - \left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) + \left(2\sqrt{p+2} + \frac{2}{\sqrt{p+2}} - \sqrt{2(p+1)} - \frac{1}{\sqrt{2(p+1)}}\right)\Big|_{p=1} \\
=& \frac{1}{2}\left(8\sqrt{3} - 6\sqrt{2} - 5\right) \approx 0.1856 > 0.
\end{aligned}$$

Case 2. $s \geq 2$ and $t = 1$.
By (4) and Lemma 2.3(4), we have

$$\begin{aligned}
&(R+RR)(\mathcal{H}_u(s,t)) - (R+RR)(\mathcal{H}_u(s+t)) \\
\geq& \frac{1}{6}\left(8\sqrt{3}-9\sqrt{2}\right) + \left(\sqrt{p+2} + \frac{1}{\sqrt{p+2}} + \sqrt{2(p+2)} + \frac{1}{\sqrt{2(p+2)}}\right) - \left(\sqrt{2(p+1)} + \frac{1}{\sqrt{2(p+1)}} + 2 + \frac{1}{2}\right) \\
\geq& \frac{1}{6}\left(8\sqrt{3}-9\sqrt{2}\right) - \frac{5}{2} + \left(\sqrt{p+2} + \frac{1}{\sqrt{p+2}} + \sqrt{2(p+2)} + \frac{1}{\sqrt{2(p+2)}} - \sqrt{2(p+1)} - \frac{1}{\sqrt{2(p+1)}}\right)\Big|_{p=1} \\
=& \frac{1}{6}\left(7\sqrt{6}+16\sqrt{3}-9\sqrt{2}-30\right) \approx 0.3552 > 0.
\end{aligned}$$

Case 3. $s \geq t \geq 2$.
By (4) and Lemma 2.3(5),

$$\begin{aligned}
&(R+RR)(\mathcal{H}_u(s,t)) - (R+RR)(\mathcal{H}_u(s+t)) \\
\geq& \frac{1}{6}\left(8\sqrt{3}-9\sqrt{2}\right) + \left(2\sqrt{2(p+2)} + \frac{2}{\sqrt{2(p+2)}} + \sqrt{2} + \frac{1}{\sqrt{2}}\right) - \left(\sqrt{2(p+1)} + \frac{1}{\sqrt{2(p+1)}} + 2\left(2 + \frac{1}{2}\right)\right) \\
\geq& \frac{1}{6}\left(8\sqrt{3}-9\sqrt{2}\right) + \sqrt{2} + \frac{1}{\sqrt{2}} - 5 + \left(2\sqrt{2(p+2)} + \frac{2}{\sqrt{2(p+2)}} - \sqrt{2(p+1)} - \frac{1}{\sqrt{2(p+1)}}\right)\Big|_{p=1} \\
=& \frac{1}{6}\left(14\sqrt{6}+8\sqrt{3}-45\right) \approx 0.5249 > 0.
\end{aligned}$$

The lemma now follows. □

Lemma 2.6. *Let $\mathcal{H}$ be a connected k-uniform hypergraph, $u_1, u_2 \in V(\mathcal{H})$ be two pendent vertices, $u_i \in e_i \in E(\mathcal{H})$ for $i = 1, 2$, and $e_1 \neq e_2$. If e_2 is a non-pendent hyperedge, then for $s, t \geq 1$,*

$$(R+RR)(\mathcal{H}_{u_1,u_2}(s,t)) > (R+RR)(\mathcal{H}_{u_1}(s+t)).$$

Proof. Let e_2 be a non-pendent hyperedge, and denote

$$\sqrt{\prod_{v \in e_2} d_{\mathcal{H}}(v)} = A.$$

Then $A \geq 2$, and by Lemma 2.2,

$$(R+RR)_{\mathcal{H}_{u_1,u_2}(s,t)}(e_2) - (R+RR)_{\mathcal{H}_{u_1}(s+t)}(e_2) = \frac{1}{\sqrt{2}A} + \sqrt{2}A - \frac{1}{A} - A \geq \left(\frac{1}{\sqrt{2}A} + \sqrt{2}A - \frac{1}{A} - A\right)\Big|_{A=2} = \frac{1}{4}\left(9\sqrt{2}-10\right).$$

Note that for any hyperedge $e \in E(\mathcal{H}) \setminus \{e_2\}$,

$$(R+RR)_{\mathcal{H}_u(s,t)}(e) = (R+RR)_{\mathcal{H}_u(s+t)}(e).$$

Since both u_1 and u_2 are pendent vertices, by Equation (3), we have

$$\begin{aligned}
&(R+RR)(\mathcal{H}_{u_1,u_2}(s,t)) - (R+RR)(\mathcal{H}_{u_1}(s+t)) \\
=& (R+RR)_{\mathcal{H}_{u_1,u_2}(s,t)}(e_2) - (R+RR)_{\mathcal{H}_{u_1}(s+t)}(e_2) \\
&+ \sum_{e \in E(\mathcal{H}_{u_1,u_2}(s,t)) \setminus E(\mathcal{H})} (R+RR)_{\mathcal{H}_{u_1,u_2}(s,t)}(e) - \sum_{e \in E(\mathcal{H}_{u_1}(s+t)) \setminus E(\mathcal{H})} (R+RR)_{\mathcal{H}_{u_1}(s+t)}(e) \\
\geq& \frac{1}{4}\left(9\sqrt{2}-10\right) + \left(\sqrt{2} + \frac{1}{\sqrt{2}} - 2 - \frac{1}{2}\right) = \frac{5}{4}\left(3\sqrt{2}-4\right) > 0.
\end{aligned}$$

Hence, the lemma holds. □

Lemma 2.7. *Let $\mathcal{H}$ be a connected k-uniform hypergraph such that $|E(\mathcal{H})| \geq 2$. Let $u_1, u_2 \in V(\mathcal{H})$ be two pendent vertices and $u_i \in e_0 \in E(\mathcal{H})$ for $i = 1, 2$. Then, for $s, t \geq 1$,*

$$(R+RR)(\mathcal{H}_{u_1,u_2}(s,t)) > (R+RR)(\mathcal{H}_{u_1}(s+t)).$$

Proof. Denote

$$\sqrt{\prod_{v\in e_0} d_{\mathcal{H}}(v)} = A.$$

Since $|E(\mathcal{H})| \geq 2$ and $\mathcal{H}$ is connected, we have $A \geq \sqrt{2}$. Then by Lemma 2.2,

$$\begin{aligned}&(R+RR)_{\mathcal{H}_{u_1,u_2}(s,t)}(e_0) - (R+RR)_{\mathcal{H}_{u_1}(s+t)}(e_0)\\ =& \frac{1}{2A} + 2A - \frac{1}{\sqrt{2}A} - \sqrt{2}A \geq \left(\frac{1}{2A} + 2A - \frac{1}{\sqrt{2}A} - \sqrt{2}A\right)\Big|_{A=\sqrt{2}} = \frac{1}{4}\left(9\sqrt{2}-10\right).\end{aligned}$$

Note that for any hyperedge $e \in E(\mathcal{H}) \setminus \{e_0\}$,

$$(R+RR)_{\mathcal{H}_u(s,t)}(e) = (R+RR)_{\mathcal{H}_u(s+t)}(e).$$

Since both u_1 and u_2 are pendent vertices, by Equation (3), we have

$$\begin{aligned}&(R+RR)(\mathcal{H}_{u_1,u_2}(s,t)) - (R+RR)(\mathcal{H}_{u_1}(s+t))\\ =& (R+RR)_{\mathcal{H}_{u_1,u_2}(s,t)}(e_0) - (R+RR)_{\mathcal{H}_{u_1}(s+t)}(e_0)\\ &+ \sum_{e\in E(\mathcal{H}_{u_1,u_2}(s,t))\setminus E(\mathcal{H})} (R+RR)_{\mathcal{H}_{u_1,u_2}(s,t)}(e) - \sum_{e\in E(\mathcal{H}_{u_1}(s+t))\setminus E(\mathcal{H})} (R+RR)_{\mathcal{H}_{u_1}(s+t)}(e)\\ \geq& \frac{1}{4}\left(9\sqrt{2}-10\right) + \left(\sqrt{2} + \frac{1}{\sqrt{2}} - 2 - \frac{1}{2}\right) = \frac{5}{4}\left(3\sqrt{2}-4\right) > 0.\end{aligned}$$

□

3. Minimum and Maximum $(R+RR)-$Values of k-Uniform Hypertrees

In this section, the minimum and maximum values of the sum $R+RR$ among all k-uniform hypertrees with $m \geq 3$ hyperedges are obtained. The extremal hypertrees that reach the minimum and maximum values are characterized, respectively. The main result is Theorem 3.1.

We recall that a linear k-uniform hyperpath is a hyperpath in which any two hyperedges have at most one vertex in common, and each hyperedge contains k vertices. We use $\mathcal{P}_{m,k}$ to denote the linear k-uniform hyperpath with m hyperedges.

Let m, h, k be positive integers with $m > 0$ and $0 < h < k$. A *sunflower* hypergraph $\mathcal{S}(m,h,k)$ is a k-uniform hypergraph defined [1] as follows. Let A be a set of h vertices, and $\{B_i\}_{i=1}^m$ be m disjoint sets of $k-h$ vertices. The hyperedge set of $\mathcal{S}(m,h,k)$ consists of $A \cup B_i$, $1 \leq i \leq m$. It is clear that $\mathcal{S}(m,h,k)$ is a k-uniform hypertree with m hyperedges and $h+m(k-h)$ vertices.

The following two lemmas give the values of $R+RR$ for $\mathcal{P}_{m,k}$ and $\mathcal{S}(m,h,k)$, respectively.

Lemma 3.1. *Let $m \geq 3$. Then*

$$(R+RR)(\mathcal{P}_{m,k}) = 3\sqrt{2} + \frac{5}{2}(m-2).$$

Proof. Note that $\mathcal{P}_{m,k}$ has two pendent hyperedges in which each contains one 2-degree vertex, and $m-2$ non-pendant hyperedges in which each contains two 2-degree vertices and $k-2$ pendant vertices. Then

$$(R+RR)(\mathcal{P}_{m,k}) = 2\left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) + (m-2)\left(\frac{1}{2} + 2\right) = 3\sqrt{2} + \frac{5}{2}(m-2).$$

□

Lemma 3.2. *Let $\mathcal{S}(m,h,k)$ be a sunflower hypergraph with $m \geq 3$ and $0 < h < k$. Then*

$$(R+RR)(\mathcal{S}(m,h,k)) = m\left(\frac{1}{\sqrt{m^h}} + \sqrt{m^h}\right).$$

Proof. Note that each hyperedge of $\mathcal{S}(m,h,k)$ has h vertices of degree m and $k-h$ pendant vertices. Then for each hyperedge $e \in E(\mathcal{S}(m,h,k))$,

$$(R+RR)_{\mathcal{S}(m,h,k)}(e) = \frac{1}{\sqrt{m^h}} + \sqrt{m^h}.$$

So, by Equation (3),

$$(R+RR)(\mathcal{S}(m,h,k)) = m\left(\frac{1}{\sqrt{m^h}} + \sqrt{m^h}\right).$$

□

Corollary 3.1. *Let $m \geq 3$. Then*

$$(R + RR)(\mathcal{S}(m, k-1, k)) = m\left(\frac{1}{\sqrt{m^{k-1}}} + \sqrt{m^{k-1}}\right).$$

Theorem 3.1. *Let $\mathcal{H}$ be a k-uniform hypertree with $m \geq 3$ hyperedges. Then*

$$3\sqrt{2} + \frac{5}{2}(m-2) \leq (R + RR)(\mathcal{H}) \leq m\left(\frac{1}{\sqrt{m^{k-1}}} + \sqrt{m^{k-1}}\right),$$

the left equality holds if and only if $\mathcal{H} \cong \mathcal{P}_{m,k}$, and the right equality holds if and only if $\mathcal{H} \cong \mathcal{S}(m, k-1, k)$.

Proof. Let $\mathcal{H}$ be a k-uniform hypertree with m hyperedges. In order to minimize $(R + RR)(\mathcal{H})$, by Lemma 2.4, it is easy to see that $\mathcal{H}$ is a linear hypertree. By Lemmas 2.5, 2.6 and 2.7, $\mathcal{H}$ is a hyperpath. It implies that the linear k-uniform hyperpath with m hyperedges minimizes $(R + RR)(\mathcal{H})$. From Lemma 3.1, the lower bound holds.

In order to maximize $(R + RR)(\mathcal{H})$, the maximum number of vertices should attain the maximum degree and these vertices has to be included uniformly among all the hyperedges. It is key to observe that a vertex can have maximum degree of m. So the k-uniform hypertree that maximizes $(R + RR)(\mathcal{H})$ must be the sunflower hypergraph $\mathcal{S}(m, k-1, k)$. By Corollary 3.1, the upper bound holds. □

4. Minimum $(R + RR)-$Value of Connected k-Uniform Hypergraphs

In this section, the minimum value of the sum $R + RR$ among all connected k-uniform hypergraphs with $m \geq 3$ hyperedges is obtained, and the extremal hypergraph that reach the minimum value is characterized.

Theorem 4.1. *Let $\mathcal{H}$ be a connected k-uniform hypergraph with $m \geq 3$ hyperedges. Then*

$$(R + RR)(\mathcal{H}) \geq 3\sqrt{2} + \frac{5}{2}(m-2),$$

the equality holds if and only if $\mathcal{H} \cong \mathcal{P}_{m,k}$.

Proof. Among all connected k-uniform hypergraphs with $m \geq 3$ hyperedges, let $\mathcal{H}_0$ be a hypergraph having the minimum value of $R + RR$. Then by Lemma 2.4, $\mathcal{H}_0$ must be a hypertree. From Theorem 3.1, the result holds. □

5. Conclusions

In this paper, we study the sum $R + RR$ for k-uniform hypergraphs with $m \geq 3$ hyperedges. In Section 3, the minimum and maximum values of $R + RR$ for k-uniform hypertrees with $m \geq 3$ hyperedges are obtained, and the extremal hypertrees are characterized, respectively. In Section 4, the minimum value of $R + RR$ for connected k-uniform hypergraphs with $m \geq 3$ hyperedges is obtained, and the extremal hypergraph is characterized. However, the maximum value of $R + RR$ for connected k-uniform hypergraphs with $m \geq 3$ hyperedges remains open. Therefore, we have raised the following problem for further research.

Problem 5.1. *Let $\mathcal{H}$ be a connected k-uniform hypergraph with $m \geq 3$ hyperedges. Find an upper bound for $(R + RR)(\mathcal{H})$.*

If we consider the connected k-uniform hypergraphs with $n \geq 3$ vertices, the following theorem proves that the complete k-uniform hypergraph with $n \geq 3$ vertices has the maximum value of $R + RR$.

Theorem 5.1. *Let $\mathcal{H}$ be a connected k-uniform hypergraph with $n \geq 3$ vertices. Then*

$$(R + RR)(\mathcal{H}) \leq \binom{n}{k}\left(\frac{1}{\sqrt{\binom{n-1}{k-1}^k}} + \sqrt{\binom{n-1}{k-1}^k}\right),$$

the equality holds if and only if $\mathcal{H} \cong \mathcal{K}_n^{(k)}$.

Proof. Since each hyperedge in $\mathcal{H}$ contains k vertices, we have that each vertex $v \in V(\mathcal{H})$ can be contained in at most $\binom{n-1}{k-1}$ hyperedges, so $d_{\mathcal{H}}(v) \leq \binom{n-1}{k-1}$. Thus for each hyperedge $e \in E(\mathcal{H})$,

$$\prod_{v \in e} d_{\mathcal{H}}(v) \leq \binom{n-1}{k-1}^k,$$

and by Lemma 2.1,

$$(R + RR)_{\mathcal{H}}(e) = \frac{1}{\sqrt{\prod_{v \in e} d_{\mathcal{H}}(v)}} + \sqrt{\prod_{v \in e} d_{\mathcal{H}}(v)} \leq \frac{1}{\sqrt{\binom{n-1}{k-1}^k}} + \sqrt{\binom{n-1}{k-1}^k}.$$

Since a k-uniform hypergraph with n vertices contains at most $\binom{n}{k}$ hyperedges, by Equation (3), we obtain that

$$(R + RR)(\mathcal{H}) \leq \binom{n}{k} \left(\frac{1}{\sqrt{\binom{n-1}{k-1}^k}} + \sqrt{\binom{n-1}{k-1}^k} \right),$$

where the equality holds if and only if there are $\binom{n}{k}$ hyperedges in $\mathcal{H}$ and each vertex has degree $\binom{n-1}{k-1}$, which means that $\mathcal{H} \cong \mathcal{K}_n^{(k)}$. □

Based on the above theorem, we now propose the following conjecture on Problem 5.1.

Conjecture 5.1. *Let $\mathcal{H}$ be a connected k-uniform hypergraph with $m \geq 3$ hyperedges having the maximum value of the sum of the Randić index and the reciprocal Randić index among all k-uniform hypergraphs with m hyperedges.*

(1) *If there is a positive integer n such that $m = \binom{n}{k}$, then $\mathcal{H}$ is the complete k-uniform hypergraph $\mathcal{K}_n^{(k)}$.*

(2) *If there is a positive integer n such that $\binom{n}{k} < m < \binom{n+1}{k}$, then $\mathcal{H}$ has $n+1$ vertices, and for any two vertices $v_1, v_2 \in V(\mathcal{H})$,*

$$|d_{\mathcal{H}}(v_1) - d_{\mathcal{H}}(v_2)| \leq 1.$$

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