

Research Article

## A Note on Orderenergetic Graphs and Unrecognized Direct Products

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### Abstract

A graph is orderenergetic if its energy is equal to its order. The direct product of graphs offers a simple way of creating new orderenergetic graphs from existing orderenergetic graphs. We show here that two unary graph operations introduced by Joseph in [MATCH Commun. Math. Comput. Chem. 87 (2022) 703–715] and [MATCH Commun. Math. Comput. Chem. 89 (2023) 665–686] for the construction of orderenergetic graphs are in fact unrecognized direct products with two specific graphs. We also briefly discuss an overlooked possibility for constructing new orderenergetic graphs as direct products of non-orderenergetic graphs.

**Keywords:** energy of graphs; orderenergetic graphs; direct product of graphs.

**2020 Mathematics Subject Classification:** 05C50.

## 1. Introduction

The *non-complete extended p-sum* (NEPS) of graphs was defined by Cvetković [3] already in his PhD thesis back in the 1970s. The following definition is taken from [4] with a minor modification.

**Definition 1.1.** Let  $\mathcal{B}$  be a set of binary  $n$ -tuples, i.e.,  $\mathcal{B} \subseteq \{0, 1\}^n - \{(0, \dots, 0)\}$  such that for every  $i = 1, \dots, n$  there exists  $\beta \in \mathcal{B}$  with  $\beta_i = 1$ . The NEPS of graphs  $G_1, \dots, G_n$  with the basis  $\mathcal{B}$ , denoted by  $\text{NEPS}(G_1, \dots, G_n; \mathcal{B})$ , is the graph with the vertex set  $V(G_1) \times \dots \times V(G_n)$ , in which two vertices  $(u_1, \dots, u_n)$  and  $(v_1, \dots, v_n)$  are adjacent if and only if there exists  $(\beta_1, \dots, \beta_n) \in \mathcal{B}$  such that  $u_i$  is adjacent to  $v_i$  in  $G_i$  whenever  $\beta_i = 1$ , and  $u_i = v_i$  whenever  $\beta_i = 0$ .

Most prominent special cases of NEPS are the direct product

$$G \times H = \text{NEPS}(G, H; \{(1, 1)\})$$

and the Cartesian product (also called the sum of graphs)

$$G + H = \text{NEPS}(G, H; \{(1, 0), (0, 1)\}).$$

One of the most useful properties of NEPS is that its adjacency eigenvalues are explicitly related to those of its factors.

**Theorem 1.1** ([4, Theorem 2.21]).  $\text{NEPS}(G_1, \dots, G_n; \mathcal{B})$  has adjacency matrix

$$A = \sum_{\beta \in \mathcal{B}} A_1^{\beta_1} \otimes \dots \otimes A_n^{\beta_n},$$

where  $A_i$  is the adjacency matrix of  $G_i$  and  $\otimes$  denotes Kronecker product of matrices.

**Theorem 1.2** ([4, Theorem 2.23]). The spectrum of  $\text{NEPS}(G_1, \dots, G_n; \mathcal{B})$  consists of all possible values  $\Lambda$  given by

$$\Lambda = \sum_{\beta \in \mathcal{B}} \lambda_1^{\beta_1} \dots \lambda_n^{\beta_n},$$

where  $\lambda_i$  is an arbitrary adjacency eigenvalue of  $G_i$ ,  $i = 1, \dots, n$ .

**Remark 1.1.** While it is customary in literature to consider simple graphs only, note that the above results continue to hold when the factors of NEPS are also weighted graphs or directed graphs or contain loops (with a single loop at a vertex indicated by 1 at the corresponding diagonal entry of adjacency matrix).

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From Theorem 1.2 we have that each eigenvalue of  $G \times H$  is equal to the product  $\lambda\mu$ , while each eigenvalue of  $G + H$  is equal to the sum  $\lambda + \mu$  of an eigenvalue  $\lambda$  of  $G$  and an eigenvalue  $\mu$  of  $H$ . If  $\lambda_1, \dots, \lambda_n$  denote the eigenvalues of simple graph  $G$ , then its energy is defined [6] (see also [11]) as

$$\mathcal{E}(G) = \sum_{i=1}^n |\lambda_i|. \tag{1}$$

It is now straightforward to see that, since the eigenvalues of  $G \times H$  are the products  $\lambda_i\mu_j$ , we have

$$\sum_{i,j} |\lambda_i\mu_j| = \left( \sum_i |\lambda_i| \right) \left( \sum_j |\mu_j| \right),$$

which yields

$$\mathcal{E}(G \times H) = \mathcal{E}(G)\mathcal{E}(H). \tag{2}$$

It is a bit less straightforward to see that the direct product is actually the only instance of NEPS for which its energy can be represented as a function of energy of its factors [19].

Graphs whose energy is simply related to their order received a lot of attention in literature. Here we are interested in *orderenergetic graphs*, defined by Akbari et al. [1] as simple graphs whose energy is equal to their order. They have already received considerable interest in literature [9, 10, 12–16, 18], with a main focus on identification of examples of such graphs obtained with various graph operations. Eq. (2) implies that the direct product  $G \times H$  of two orderenergetic graphs  $G$  and  $H$  is also orderenergetic. Akbari et al. [1, Problem 19] therefore posed the problem to find a method for constructing connected orderenergetic graphs without using the direct product. Here we show that some of the constructions of orderenergetic graphs, recently proposed in response to this problem, still represent special cases of the direct product of graphs only.

**Remark 1.2.** *It is important to note that the definition (1), representing the energy as the trace norm of its adjacency eigenvalues, refers to simple graphs only. In cases when matrix  $M_G$  associated to graph  $G$  may have non-zero trace, the energy is still defined as the trace norm, but only after the eigenvalues of  $M_G$  are shifted by their average. For example, Laplacian energy is defined in [8] as*

$$LE(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|,$$

where  $\mu_1, \dots, \mu_n$  are Laplacian eigenvalues of  $G$ , while the energy of a graph with  $\sigma$  loops attached at its vertices is defined in [7] as

$$\mathcal{E}(G) = \sum_{i=1}^n \left| \lambda_i - \frac{\sigma}{n} \right|.$$

Hence the relation  $\mathcal{E}(G \times H) = \mathcal{E}(G)\mathcal{E}(H)$  no longer holds if, say, graph  $G$  contains loops. Nevertheless, the direct product of a graph  $G$  with loops and a simple graph  $H$  is still a simple graph (it has no loops). The energy of  $G \times H$  is therefore the standard energy of a simple graph, and the multiplicative formula  $\mathcal{E}(G \times H) = \sum_{i=1}^n |\lambda_i|\mathcal{E}(H)$  still holds, where  $\lambda_i$  are the adjacency eigenvalues of  $G$ . To avoid confusion, for a graph  $G$  with loops, we will use  $\mathcal{E}^*(G) = \sum_{i=1}^n |\lambda_i|$  to denote the sum of absolute values of its adjacency eigenvalues.

## 2. Two Unrecognized Direct Products

Joseph [9, 10] introduced two “new” unary graph operations for construction of orderenergetic graphs.

**Definition 2.1** ([9]). *Let  $G$  be a graph with order  $n$  and number of edges  $m$ . Take  $p$  copies of  $G$ , denoted by  $G_i$ , for  $1 \leq i \leq p$ , and take  $nq$  number of isolated vertices for some positive integer  $q$ . Name the  $n$  vertices of  $G_i$  as  $V_i = \{v_{i1}, v_{i2}, \dots, v_{in}\}$ , for  $1 \leq i \leq p$ . Let the isolated vertices be named as  $\{u_{jk}\}$ , where  $1 \leq j \leq q$  and  $1 \leq k \leq n$ . Then construct a new graph  $H$  from the graphs  $\{G_i\}$  and the isolated vertices  $\{u_{jk}\}$  as follows.*

1. *The vertex set of  $H$  is the union of the vertices in  $G_i$  and the isolated vertices  $\{u_{jk}\}$ .*
2. *The edge set of  $H$  contains all the edges in  $G_i$ 's. In addition to these edges, the following two types edges will be added in the edge set of  $H$ . If  $v_{ij}$  is adjacent to  $v_{ik}$  in  $G_i$  for  $1 \leq i \leq p$ , then*
  - *add the edges  $\{v_{ij}, v_{lk}\}$  and  $\{v_{ik}, v_{lj}\}$  for all  $1 \leq l \leq n$  and  $l \neq i$ , i.e., add edges from  $v_{ij}$  to every  $k^{\text{th}}$  vertex in  $G_l$  for  $l \neq i$  and from  $v_{ik}$  to every  $j^{\text{th}}$  vertex in  $G_l$  for  $l \neq i$ .*
  - *add the edges  $\{v_{ij}, u_{lk}\}$  and  $\{v_{ik}, u_{lj}\}$  for all  $1 \leq l \leq q$ .*

A careful interpretation of this definition reveals a particular direct product hidden in it. Let  $CS_{p,q}^+$  be a complete split graph that consists of a clique on vertices  $c_1, \dots, c_p$ , with a loop attached at each vertex  $c_i$ , and an independent set on vertices  $s_1, \dots, s_q$ , so that  $c_i$  and  $s_j$  are adjacent for each  $1 \leq i \leq p$  and each  $1 \leq j \leq q$ . Vertex  $v_{ik}$  then corresponds to the pair  $(c_i, k)$ , while vertex  $u_{jk}$  corresponds to the pair  $(s_j, k)$ , where  $k$  is a vertex of  $G$ . The vertex set of  $H$  consist of all vertices  $v_{ik}$  and  $u_{jk}$ , which corresponds to  $V(CS_{p,q}^+) \times V(G)$ . The edge set of  $H$  consist of:

- all edges of  $G_i$ 's, which corresponds to the edges between pairs  $(c_i, j)$  and  $(c_i, k)$ , where  $c_i$  is adjacent to  $c_i$  in  $CS_{p,q}^+$  due to the attached loop, and  $j$  is adjacent to  $k$  in  $G$ ;
- edges  $\{v_{ij}, v_{lk}\}$  and  $\{v_{lj}, v_{ik}\}$  for  $i \neq l$  where  $v_{ij}$  is adjacent to  $v_{ik}$  in  $G_i$ , which corresponds to the edges between pairs  $(c_i, j)$  and  $(c_l, k)$  for each pair (of adjacent vertices)  $c_i, c_l$  in  $CS_{p,q}^+$  and each pair  $j, k$  of adjacent vertices in  $G$ ;
- edges  $\{v_{ij}, u_{lk}\}$  and  $\{v_{ik}, u_{lj}\}$  for all  $1 \leq l \leq q$ , which corresponds to the edges between pairs  $(c_i, j)$  and  $(s_l, k)$  for each pair (of adjacent vertices)  $c_i, s_l$  in  $CS_{p,q}^+$  and each pair  $j, k$  of adjacent vertices in  $G$ .

Hence  $H$  is nothing else but the direct product of  $CS_{p,q}^+$  and  $G$ . Graph  $CS_{p,q}^+$  itself is a join of the complete graph  $K_p^+$ , with a loop attached at each vertex, and the empty graph  $\overline{K_q}$ . Since both of these graphs are regular, the following theorem of Finck and Grohmann [5] enables us to quickly determine the spectrum of  $CS_{p,q}^+$ .

**Theorem 2.1.** *If  $G_i$  is a regular graph of degree  $r_i$  and order  $n_i$  for  $i = 1, 2$ , then the characteristic polynomial of the join  $G_1 \nabla G_2$  is equal to*

$$P_{G_1 \nabla G_2}(\lambda) = P_{G_1}(\lambda)P_{G_2}(\lambda) \frac{(\lambda - r_1)(\lambda - r_2) - n_1 n_2}{(\lambda - r_1)(\lambda - r_2)}.$$

The complete graph  $K_p^+$  has the all-one matrix as its adjacency matrix, hence  $P_{K_p^+}(\lambda) = (\lambda - p)\lambda^{p-1}$ . The empty graph  $\overline{K_q}$  has the zero matrix as its adjacency matrix, hence  $P_{\overline{K_q}}(\lambda) = \lambda^q$ . Therefore

$$P_{CS_{p,q}^+}(\lambda) = \lambda^{p+q-2}(\lambda^2 - p\lambda - pq),$$

so  $CS_{p,q}^+$  has simple eigenvalues  $\frac{1}{2} \left( p \pm \sqrt{p(p+4q)} \right)$  and eigenvalue 0 of multiplicity  $p+q-2$ , as well as the sum of absolute eigenvalues

$$\mathcal{E}^*(CS_{p,q}^+) = \sqrt{p(p+4q)}. \tag{3}$$

These observations directly imply Theorems 1 and 3 from Joseph's paper [9].

Joseph introduced another unary operation in [10].

**Definition 2.2 ([10]).** *Let  $G$  be a graph of order  $n$  with the vertex set  $V(G) = \{1, \dots, n\}$ . Given any two positive integers  $p = p_1 + p_2 + p_3$  and  $q = q_1 + q_2 + q_3$ , where  $p_i$ 's and  $q_i$ 's are non-negative integers, a new graph  $H$  is constructed from  $G$  as follows.*

1. *The vertex set of  $H$  is*

$$V(H) = \{u_{ij} : 1 \leq i \leq p, 1 \leq j \leq n\} \cup \{v_{kl} : 1 \leq k \leq q, 1 \leq l \leq n\}.$$

2. *The edges in  $H$  are obtained as follows. Let  $(j, k)$  be an edge of  $G$ , then*

- *the edges  $(u_{ij}, u_{ik}) \in E(H)$  for all  $1 \leq i \leq p_1 + p_2$ ,*
- *the edges  $(v_{ij}, v_{ik}) \in E(H)$  for all  $1 \leq i \leq q_1 + q_2$ ,*
- *the edges  $(u_{ij}, u_{lk}) \in E(H)$  for all  $1 \leq i \leq p_1, 1 \leq l \leq p_1, l \neq i$ ,*
- *the edges  $(v_{ij}, v_{lk}) \in E(H)$  for all  $1 \leq i \leq q_1, 1 \leq l \leq q_1, l \neq i$ ,*
- *the edges  $(u_{ij}, v_{lk}) \in E(H)$  for all  $1 \leq l \leq q, 1 \leq i \leq p$ ,*
- *the edges  $(u_{ik}, v_{lj}) \in E(H)$  for all  $1 \leq l \leq q, 1 \leq i \leq p$ .*

Another careful interpretation shows that this definition also represents a direct product of a particular graph and  $G$ , except that this particular graph is now a bit more complicated to define. Let  $A = \{a_1, \dots, a_p\}$  and  $B = \{b_1, \dots, b_q\}$  be two disjoint sets of vertices. Form the subsets:

$$\begin{aligned} A_1 &= \{a_1, \dots, a_{p_1}\}, & A_2 &= \{a_{p_1+1}, \dots, a_{p_1+p_2}\}, \\ B_1 &= \{b_1, \dots, b_{q_1}\}, & B_2 &= \{b_{q_1+1}, \dots, b_{q_1+q_2}\}. \end{aligned}$$

Let  $C_{p_1, p_2, p_3}^{q_1, q_2, q_3}$  be the complete bipartite graph on  $A \cup B$ , with  $A$  as one part and  $B$  as another part. Add new edges between the vertices of  $A_1$  to make  $A_1$  a clique. Similarly, add new edges between the vertices of  $B_1$  to make  $B_1$  a clique as well. Finally, attach a loop at each vertex of  $A_1 \cup A_2 \cup B_1 \cup B_2$ . The vertex  $u_{ij}$  of  $H$  then corresponds to the pair  $(a_i, j)$ , while  $v_{kl}$  corresponds to the pair  $(b_k, l)$ , where  $j$  and  $l$  are vertices of  $G$ . Since in all cases  $\{j, k\} \in E(G)$ :

- the first two edge types correspond to edges between  $(c, j)$  and  $(c, k)$  in  $C_{p_1, p_2, p_3}^{q_1, q_2, q_3} \times G$ , thanks to the loop attached at each vertex  $c \in A_1 \cup A_2 \cup B_1 \cup B_2$ ,
- the third edge type corresponds to edges between  $(a_i, j)$  and  $(a_l, k)$  in  $C_{p_1, p_2, p_3}^{q_1, q_2, q_3} \times G$ , thanks to a clique on  $A_1$ , while
- the fourth edge type corresponds to edges between  $(b_i, j)$  and  $(b_l, k)$  in  $C_{p_1, p_2, p_3}^{q_1, q_2, q_3} \times G$ , due to a clique on  $B_1$  and  $\{j, k\} \in E(G)$ .
- the last two edge types correspond to edges between  $(a, j)$  and  $(b, k)$ , as well as between  $(a, k)$  and  $(b, j)$ , in  $C_{p_1, p_2, p_3}^{q_1, q_2, q_3} \times G$  for each  $a \in A$  and  $b \in B$ , due to a complete bipartite subgraph between  $A$  and  $B$ .

Hence  $H$  is again nothing else but the direct product  $C_{p_1, p_2, p_3}^{q_1, q_2, q_3} \times G$ . Joseph determined the spectrum of  $C_{p_1, p_2, p_3}^{q_1, q_2, q_3}$  in [10, Theorem 1], although it could have been done in a more straightforward way by partitioning its full eigenspace into the eigenvectors whose entries sum to zero within each of the blocks  $A_1, A_2, A_3, B_1, B_2, B_3$ , and the eigenvectors which are constant on each of these blocks. The first type of eigenvectors lead to eigenvalues 1 of multiplicity  $(p_2 - 1) + (q_2 - 1)$  and 0 of multiplicity  $(p_1 - 1) + (p_3 - 1) + (q_1 - 1) + (q_3 - 1)$ . The second type of eigenvectors lead to the divisor matrix

$$\begin{bmatrix} p_1 & 0 & 0 & p_1 & p_1 & p_1 \\ 0 & 1 & 0 & p_2 & p_2 & p_2 \\ 0 & 0 & 0 & p_3 & p_3 & p_3 \\ q_1 & q_1 & q_1 & q_1 & 0 & 0 \\ q_2 & q_2 & q_2 & 0 & 1 & 0 \\ q_3 & q_3 & q_3 & 0 & 0 & 0 \end{bmatrix}$$

whose characteristic polynomial then yields the remaining six eigenvalues. To conclude this section, both unary graph operations introduced by Joseph [9, 10] turn out to actually be unrecognized direct products with graphs from two very specific families. It is fair to point out, as highlighted by an anonymous reviewer, that since these specific families contain loops, they are not simple graphs. Thus, if one restricts strictly to the domain of simple graphs, Joseph’s operations do indeed avoid the direct product of two *simple* graphs, answering the open problem from [1] in a strict sense. However, their fundamental nature is undeniably that of a direct product in the broader context of graphs with loops. We leave as an exercise to the reader to show that yet another unary operation for constructing orderenergetic graphs, recently introduced in [13, Section 3], also represents a special case of the direct product—answer to this exercise is visible from Theorem 1.1 and the formula for adjacency matrix of this operation given in [13, p. 4].

### 3. Products of Non-Orderenergetic Graphs May be Orderenergetic

While it is well known, thanks to Eq. (2), that the direct product of two orderenergetic graphs is again orderenergetic (see, e.g., [1, Lemma 5]), it appears that a simple observation that the direct product of non-orderenergetic graphs  $G$  and  $H$  can also be orderenergetic, has not been examined in literature so far. All one needs for  $G \times H$  to be orderenergetic is that

$$\frac{\mathcal{E}(G)}{|V(G)|} \cdot \frac{\mathcal{E}(H)}{|V(H)|} = 1. \tag{4}$$

To correct for this previously overlooked observation, here we will shortly point out a few new families of orderenergetic graphs that can be obtained in this way. We will rely on three simple graph families for which energy may be easily expressed in terms of their parameters:

- complete bipartite graphs  $K_{m,n}$  for which

$$\frac{\mathcal{E}(K_{m,n})}{|V(K_{m,n})|} = \frac{2\sqrt{mn}}{m+n}, \tag{5}$$

- balanced complete multipartite graphs  $\underbrace{K_{t, \dots, t}}_{n \text{ times}}$  for which [21]

$$\frac{\mathcal{E}(K_{t, \dots, t})}{|V(K_{t, \dots, t})|} = \frac{2(n-1)t}{nt} = 2 - \frac{2}{n}, \tag{6}$$

- complete split graphs  $CS_{p,q}^+$  with loops at clique vertices for which from Eq. (3)

$$\frac{\mathcal{E}^*(CS_{p,q}^+)}{|V(CS_{p,q}^+)|} = \frac{\sqrt{p(p+4q)}}{p+q}.$$

It is now straightforward to see that the following direct products of non-orderenergetic graphs yield orderenergetic graphs.

**Proposition 3.1.**  $K_{9s,s} \times K_{\underbrace{t,t,t,t,t,t}_{n \text{ times}}}$  is orderenergetic graph for all  $s, t \geq 1$ .

Note that the Diophantine equation

$$\frac{2\sqrt{pq}}{p+q} \left(2 - \frac{2}{n}\right) = 1$$

that stems from Eq. (4) for  $K_{p,q}$  and  $K_{\underbrace{t, \dots, t}_{n \text{ times}}}$  has infinitely many solutions. It can be easily seen by setting  $x = \sqrt{p/q}$ , which reduces it to the quadratic  $nx^2 - 4(n-1)x + n = 0$ . A rational solution  $x$  then leads to integer choices for  $p$  and  $q$ . In order for this quadratic to have rational solutions, its discriminant must be a perfect square, which leads to a Pell-type equation  $(3n-4)^2 - 3m^2 = 4$  that has infinitely many solutions. For example, the next feasible choice of parameters that yields orderenergetic direct product is

$$K_{121s,9s} \times K_{\underbrace{t, \dots, t}_{66 \text{ times}}}.$$

**Proposition 3.2.**  $K_{s,3s} \times CS_{2t,t}^+$  is orderenergetic graph for all  $s, t \geq 1$ .

Computational search—whether by exhaustive enumeration over a suitably restricted domain or by optimization-based exploration—has become a ubiquitous tool for discovering illustrative examples and counterexamples in literature [2, 17, 18, 20, 22–28]. Interestingly, however, when exhaustive enumeration was applied to the Diophantine equation

$$\frac{2\sqrt{mn}}{m+n} \cdot \frac{\sqrt{p(p+4q)}}{p+q} = 1,$$

that stems from Eq. (4) for  $K_{m,n}$  and  $CS_{p,q}^+$ , additionally requiring primitive solutions with  $\gcd(m, n) = \gcd(p, q) = 1$ , it yielded only the following solutions with  $1 \leq m, n, p, q \leq 10^6$ :

- $(m, n, p, q) = (1, 1, 1, 2)$ , leading to orderenergetic factors  $K_{s,s}$  and  $CS_{t,2t}^+$ ,
- $(m, n, p, q) = (1, 3, 2, 1)$ , leading to the family in the above proposition, and
- $(m, n, p, q) = (3, 1, 2, 1)$ , which is isomorphic to the previous family.

Although no theoretical proof has been found, this is a strong indicator that the above quadruplets are most probably the only primitive solutions.

**Proposition 3.3.**  $K_{\underbrace{t, \dots, t}_{14 \text{ times}}} \times CS_{s,12s}^+$  is orderenergetic graph for all  $t, s \geq 1$ .

The Diophantine equation

$$\left(2 - \frac{2}{n}\right) \frac{\sqrt{p(p+4q)}}{p+q} = 1$$

that stems from Eq. (4) for  $K_{\underbrace{t, \dots, t}_{n \text{ times}}}$  and  $CS_{p,q}^+$  also has infinitely many solutions. Setting  $x = p/q$  and  $r = \frac{n}{2n-2}$  reduces it to

the quadratic  $(1-r^2)x^2 + (4-2r^2)x - r^2 = 0$ , whose discriminant is  $\Delta(n) = \frac{13n^2-32n+16}{(n-1)^2}$ . Setting further  $a = 13n - 16$  and  $b^2 = 13n^2 - 32n + 16$  yields Pell-type equation  $a^2 - 13b^2 = 48$  that has infinitely many solutions. The above solution  $n = 14$ ,  $q = 12p$  stems from  $(a, b) = (166, 46)$ . A few other small solutions that yield orderenergetic graphs  $K_{\underbrace{t, \dots, t}_{n \text{ times}}} \times CS_{p,q}^+$  are:

- $a = 1024, b = 284 \Rightarrow n = 80, p = 16s, q = 221s$  for  $s \in \mathbb{Z}_+$ ,
- $a = 19744, b = 5476 \Rightarrow n = 1520, p = 100s, q = 1419s$  for  $s \in \mathbb{Z}_+$ ,
- $a = 121846, b = 33794 \Rightarrow n = 9374, p = 1849s, q = 26270s$  for  $s \in \mathbb{Z}_+$ .

As a final remark, let us mention that the fact that  $G \times H$  does not contain loops as long as not both  $G$  and  $H$  contain loops, does not seem to be exploited in literature either. Allowing for loops in one factor, the simplest example is probably obtained by taking the complete graph of order  $n$  with a loop attached at each vertex, denoted  $K_n^+$ . Its adjacency matrix is the all-one matrix with a simple eigenvalue  $n$  and eigenvalue  $0$  of multiplicity  $n - 1$ . While  $K_n^+$  itself is not an orderenergetic graph under the proper energy definition for graphs with loops [7], the sum of absolute values of its adjacency eigenvalues equals its order,  $\mathcal{E}^*(K_n^+) = n = |V(K_n^+)|$ . Consequently,  $K_n^+ \times G$  is an orderenergetic simple graph whenever  $G$  is an orderenergetic simple graph. Note only that  $K_n^+ \times G$  allows very simple reinterpretation as that of taking  $n$  copies of  $G$ , denoting their vertices as  $u_i = (u, i)$  with  $u \in V(G)$  and  $i \in \{1, \dots, n\}$ , with two vertices  $u_i$  and  $v_j$  adjacent whenever  $u$  and  $v$  are adjacent in  $G$ , possibly also hiding its direct product structure from a less attentive reader.

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