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Research Article

Note on the Eternal Domination Number of Planar Graphs and Vertex-Critical Graphs

Kenji Kimura[1,*], Naoki Matsumoto[2], Tsukasa Sato[3]

[1]*Department of Information Technology and Electronics, Ishinomaki Senshu University, 986-8580, 1, Shinmito, Minamisakai, Ishinomaki-shi, Miyagi, Japan*
[2]*Faculty of Education, University of the Ryukyus, 903-0213, 1, Senbaru, Nakagami-Gun Nishihara-Cho, Okinawa, Japan*
[3]*MOVE Co., Ltd., 986-0853, 64-28, Ichibanyachi, Kadonowaki, Ishinomaki-shi, Miyagi, Japan*

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Abstract

An eternal dominating set S of G is a dominating set with the following property: for any infinite sequence of selecting vertices, called attacks, moving one vertex in S along an edge from its current position to the attacked vertex results in another dominating set of G. The size of a minimum eternal dominating set of G is called the eternal domination number of G, denoted by $\gamma^\infty(G)$. Klostermeyer and Mynhardt [*Appl. Anal. Discrete Math.* **10** (2016) 1–29] asked whether every planar graph G satisfies $\gamma^\infty(G) = \theta(G)$, where $\theta(G)$ is the clique covering number of G. Further, MacGillivray, Mynhardt and Virgile [*Electron. J. Graph Theory Appl.* **10** (2022) 603–624] showed that a smallest planar graph G with $\gamma^\infty(G) < \theta(G)$ is 2-connected and vertex-critical with respect to $\theta(G)$ if such a graph exists. In this paper, we give some results for the eternal domination number of planar graphs and vertex-critical graphs. Moreover, we verify the existence of graphs G with $\gamma^\infty(G) < \theta(G)$ for every non-spherical surface.

Keywords: eternal domination number; clique covering number; vertex-critical graph.

2020 Mathematics Subject Classification: 05C69.

1. Introduction

In this paper, we deal with only finite simple graphs unless otherwise specified. The eternal domination game, introduced in [4], is defined as follows: Given a graph G, there are two players, a defender and an attacker, who play alternately, beginning with the defender. At the start of the game, the defender chooses a dominating set $D_0 \subseteq V(G)$ on which to place guards (at most one guard on each vertex of G), where $V(G)$ denotes the set of vertices of G. A vertex subset S of G is a *dominating set* of G if each vertex of G is in S or adjacent to a vertex in S. At each time $t = 1, 2, 3, \ldots$, the attacker selects a vertex v; we say the attacker attacks v. The defender responds by moving a guard on a neighbor of v to v; we say the defender defends v. The defender wins if he can respond to the sequence of attacks, that is, the set of guards always forms a dominating set of G throughout the game; otherwise, that is, if no guard dominates some vertex at some time t, the attacker wins. The eternal domination number of G, denoted by $\gamma^\infty(G)$, is the minimum number of guards required such that the defender can respond to any sequence of attacks on G. For a survey of the eternal domination number and related topics, see [6–8].

A *clique cover* of a graph G is a set $\{X_1, X_2, \ldots, X_k\}$ for which each $X_i \subseteq V(G)$ induces a clique, $X_i \cap X_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^k X_i = V(G)$. For the sake of simplicity, we refer to X_i as a clique. The cardinality of a minimum clique cover of G is called the *clique covering number* of G, denoted by $\theta(G)$. A graph G is *vertex-critical* if $\theta(G - v) < \theta(G)$ for every vertex $v \in V(G)$; in fact, $\theta(G - v) = \theta(G) - 1$ holds. It is easy to see that for every graph G, $\gamma^\infty(G) \leq \theta(G)$; for a minimum clique cover $\{X_1, X_2, \ldots, X_{\theta(G)}\}$ of G, the defender puts a guard to each clique X_i at first. For any attack at a vertex $v \in X_i$, the defender can respond to it by moving a guard in X_i to v.

It is a natural problem whether $\gamma^\infty(G) = \theta(G)$ holds for a given graph G. MacGillivray et al. [9] proved that $\gamma^\infty(G) < \theta(G)$ for almost all graphs G. Klostermeyer and Mynhardt [7] proposed the following problem.

Problem 1.1 (Klostermeyer and Mynhardt [7]). *Is it true that $\gamma^\infty(G) = \theta(G)$ if a graph G is planar?*

It is known that there is no planar graph of order 11 or less with $\gamma^\infty(G) < \theta(G)$ [9]. Moreover, MacGillivray et al. [9] showed that if there is a planar graph G with $\gamma^\infty(G) < \theta(G)$, then G is 2-connected and vertex-critical.

*Corresponding author (kimura@isenshu-u.ac.jp).

In this paper, we focus on planar graphs and vertex-critical ones and consider some conditions for those graphs to satisfy the equality of the eternal domination number and the clique covering number. The organization of the rest of the paper and our results are as follows: In the next section, we first investigate the basic properties and some results of vertex-critical graphs. It is known that every non-empty bipartite graph G has a vertex that is contained in every maximum matching of G [3, Exercises 16.1.15]. As a consequence, the following proposition follows (because deleting such a vertex does not decrease the clique covering number).

Proposition 1.1. *Let G be a connected vertex-critical graph with at least three vertices. Then $\chi(G) \geq 3$, that is, G is non-bipartite, where $\chi(G)$ is the chromatic number of G.*

In particular, we prove a sufficient condition for a triangle-free vertex-critical graph G to satisfy $\gamma^\infty(G) = \theta(G)$, as follows.

Theorem 1.1. *Let G be a vertex-critical triangle-free graph with at least three vertices. If there is a vertex v such that $G - v$ is bipartite, then $\gamma^\infty(G) = \theta(G)$.*

As detailed in the next section, for any vertex-critical graph G, the complement $\overline{G}$ is vertex color-critical. A graph G is *vertex k-color-critical* if $\chi(G) = k$ and $\chi(G - v) = k - 1$ for any $v \in V(G)$. As a consequence of Proposition 1.1, we have the following corollary.

Corollary 1.1. *For any vertex color-critical graph G with at least three vertices, $\theta(G) \geq 3$.*

In Section 3, we investigate the eternal domination number for planar graphs. The main theorem in that section is the following, which is a partial solution for Problem 1.1.

Theorem 1.2. *Let G be a plane graph each of whose faces has size at least 12. Then $\gamma^\infty(G) = \theta(G)$.*

Moreover, we also verify that the planarity condition for Problem 1.1 is necessary. That is, for any non-spherical surface $\mathbb{F}$, we find a graph G embedded on $\mathbb{F}$ with $\gamma^\infty(G) < \theta(G)$ (Proposition 3.1)[†].

2. Vertex-Critical Graphs

By the definition of clique cover and vertex k-color-critical, the following fact is trivial.

Fact 2.1. *For any graph G, $\theta(G) = \chi(\overline{G})$. In particular, G is vertex-critical if and only if the complement $\overline{G}$ is vertex $\theta(G)$-color-critical.*

From this fact, we obtain the complementary version of known results for vertex color-critical graphs. Here we introduce one of them, that is the complement of the result in [11], which helps us to investigate vertex-critical graphs.

Theorem 2.1 (Stehlík [11]). *Let G be a vertex-critical connected graph with n vertices. There is a minimum clique cover F of G such that there exists exactly one K_1 in F and every other clique in F has at least two vertices.*

From Theorem 2.1, we have the following.

Corollary 2.1. *Let G be a vertex-critical connected graph. If G is triangle-free, then any minimum clique cover of G contains exactly one K_1 and each other clique is K_2, that is, $|V(G)|$ is odd and $\theta(G) = \frac{|V(G)|+1}{2}$.*

Hereafter, we give several fundamental properties of vertex-critical graphs and prove our main result.

Proposition 2.1. *Let G be a vertex-critical graph. Then*

$$\gamma^\infty(G) = \theta(G) \Rightarrow \gamma^\infty(G - v) = \theta(G - v) \text{ for any vertex } v \in V(G).$$

Proof. Suppose there is a vertex $u \in V(G)$ with $\gamma^\infty(G - u) < \theta(G - u)$. Since G is vertex-critical, $\theta(G - u) = \theta(G) - 1$. Thus, $\gamma^\infty(G) \leq \gamma^\infty(G - u) + 1 < \theta(G - u) + 1 = \theta(G)$, which is a contradiction. □

The converse of Proposition 2.1 does not hold in general. The complement $\overline{M_4}$ of the Grötzsch graph M_4 satisfies $3 = \gamma^\infty(\overline{M_4}) < \theta(\overline{M_4}) = 4$ [5]. Since Grötzsch graph is vertex 4-color-critical, $\overline{M_4}$ is vertex-critical by Fact 2.1. Moreover, we can verify $\gamma^\infty(\overline{M_4} - v) = \theta(\overline{M_4} - v) = 3$ for any vertex $v \in V(\overline{M_4})$ (since all but two of the 10-vertex graphs H satisfy $\gamma^\infty(H) = \theta(H)$ [9]).

[†]In this paper, we verify some results using a computer. The program/data files used in this paper appear on the following website: https://sites.google.com/view/naokimatsumoto/data

Observe that the Grötzsch graph has 11 vertices. In [9], it is verified, using a computer, that every vertex-critical (respectively, triangle-free) graph G with at most 9 (respectively, 14) vertices satisfies $\gamma^\infty(G) = \theta(G)$ (respectively, $\alpha(G) = \gamma^\infty(G) = \theta(G)$), where $\alpha(G)$ denotes the independence number of G. Thus, such graphs satisfy the converse of Proposition 2.1. Moreover, they also found a 10-vertex vertex-critical graph H with $\gamma^\infty(H) < \theta(H)$ and a 13-vertex triangle-free graph H' with $\alpha(H') < \gamma^\infty(H') < \theta(H')$. Independent of their work, we have also verified, using a computer, that every vertex-critical graph (respectively, triangle-free vertex-critical graph) with at most 9 (respectively, 11) vertices satisfies the converse of Proposition 2.1, and also found a 13-vertex vertex-critical triangle-free graph (see Figure 2.1).

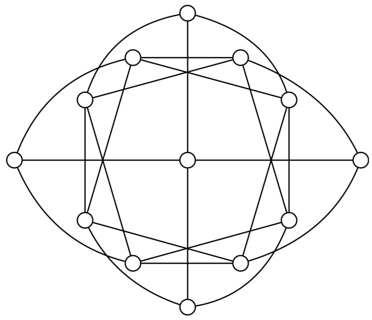

Figure 2.1: A vertex-critical triangle-free graph H with 13 vertices and $\gamma^\infty(H) < \theta(H)$.

Based on experimental results in our computer search, every triangle-free vertex-critical graph satisfies the converse of Proposition 2.1 when the number of edges is linear. Then, based on the computational results obtained so far, we believe that the following holds.

Conjecture 2.1. *Let G be a vertex-critical triangle-free graph of order n. If $|E(G)| \le 2n-3$, then*

$$\gamma^\infty(G) = \theta(G) \Leftrightarrow \gamma^\infty(G-v) = \theta(G-v) \text{ for any vertex } v \in V(G).$$

For a graph G and a vertex v of G, $N_G(v)$ denotes the set of neighbors of v in G. For a set $S \subseteq V(G)$,

$$N_G(S) = \bigcup_{u \in S} N_G(u) \setminus S.$$

Observation 2.1. *Let G be a vertex-critical graph with at least three vertices. For every vertex $v \in V(G)$, there is a minimum clique cover F of G such that $\{v\} \in F$, that is, it forms K_1 in F.*

Observation 2.2. *Let G be a vertex-critical graph with at least three vertices, and let Q be a clique of G of order at least 2. Then G has no vertex $v \in V(Q)$ with $N_G(v) \subseteq V(Q)$.*

Proof. Suppose to the contrary that G has a vertex $v \in V(Q)$ with $N_G(v) \subseteq V(Q)$. By Observation 2.1 and $|V(Q)| \ge 2$, there is a minimum clique cover F with $\{x\} \in F$ for a vertex $x \in V(Q) \setminus \{v\}$. Note that for a clique $R \in F$ with $v \in R$, $R \subseteq V(Q)$ since $N_G(v) \subseteq V(Q)$. Thus, $F' = F \setminus \{R, \{x\}\} \cup \{R \cup \{x\}\}$ is a smaller clique cover of G, which is a contradiction. □

Observation 2.3. *Let G be a vertex-critical graph with at least three vertices and suppose that the minimum degree of G, denoted by $\delta(G)$, is at least one. Then $\delta(G) \ge 2$.*

Proof. For the contradiction, let v be a vertex of degree 1 and let u be the unique neighbor of v. By Observation 2.1, G has a minimum clique cover F with $\{u\} \in F$. However, because now $\{v\} \in F$, $F' = F \setminus \{\{u\}, \{v\}\} \cup \{\{u, v\}\}$ is a smaller clique cover of G, which is a contradiction. □

Observation 2.4. *Let G be a vertex-critical graph with at least three vertices. For every vertex v and any minimum clique cover F with $\{v\} \in F$, each clique $X \in F$ has a vertex which is not adjacent to v.*

Observation 2.5. *Let G be a vertex-critical graph with at least three vertices. If G is not a complete graph, then the maximum degree is at most $|V(G)| - 2$.*

At the end of this section, we prove Theorem 1.1.

Proof of Theorem 1.1. Let G be a vertex-critical triangle-free graph with at least three vertices and v a vertex of G such that $G - v$ is bipartite. Recall that the number of vertices is odd and $\theta(G) = \frac{n+1}{2}$ by Corollary 2.1. Let D be a dominating set of G with size $\theta(G) - 1 = \frac{n-1}{2}$, which is chosen by the defender at the beginning of the game, and let Z be a clique cover of $G - v$ with size $\theta(G) - 1$; note that Z is a perfect matching of $G - v$. It suffices to show that the attacker can win the game in G.

Let B and W be the bipartition of $G - v$. Note that $|B| = |W| = |D|$ since $G - v$ has a perfect matching Z. We may suppose that $D = B$ (by attacking all vertices in B). First, the attacker attacks all vertices in $N_G(N_G(v)) \cap W$ in a certain order. Note that any two vertices in $N_G(v)$ are not adjacent and no vertex in $N_G(N_G(v))$ is adjacent to v since G is triangle-free.

Let D_1 be the resulting set obtained from D by the above attack. If $D_1 \cap N_G(v) = \emptyset$, then we are done. Thus, $D_1 \cap N_G(v) \neq \emptyset$; note that $D_1 \cap N_G(v) \subseteq B$. Note that for some vertex $u \in N_G(v)$,

$$D_1 \cap (N_G(u) \cup \{u\}) = (N_G(u) \cup \{u\}) \setminus \{v\}. \quad (*)$$

Otherwise, there is a vertex $y \in N_G(x) \cap W$ with $y \notin D_1$ where $x \in N_G(v) \cap D_1$. But this contradicts the fact that the attacker attacks all vertices in $N_G(N_G(v)) \cap W$ because any two vertices in $N_G(N_G(v)) \cap W$ are not adjacent.

Suppose that a vertex $u \in N_G(v)$ satisfies this last condition $(*)$. Let $u_1, u_2, \ldots, u_{\deg_G(u)-1}$ be vertices in $N_G(u) \setminus \{v\}$ and $uu_1 \in Z$. Let T_i be a connected component of $G - v - u$ containing u_i. Recall that $u \in B$ and then $u_i \in W$. Since $uu_1 \in Z$, $|T_1|$ is odd and $|T_i|$ is even for any $i \neq 1$; otherwise, there is a vertex in $T_1 - u_1$ not contained in Z, a contradiction. Moreover, we have

$$|V(T_1) \cap D_1| < \theta(T_1) = \frac{|T_1| + 1}{2} \quad (1)$$

or there exists an index $j \neq 1$ such that

$$|V(T_j) \cap D_1| < \theta(T_j) = \frac{|T_j|}{2} \quad (2)$$

because $u, u_1 \in D_1$ and $|D_1| = |Z|$.

If (1) (respectively, (2)) holds, then the attacker attacks all vertices in $(W \cap V(T_1)) \setminus \{u_1\}$ (respectively, $(W \cap V(T_j)) \setminus \{u_j\}$). In the resulting set obtained by this attack, some vertex in W cannot be protected. This completes the proof. □

The converse of Theorem 1.1 does not hold since there exists a vertex-critical triangle-free graph H with $\gamma^\infty(H) = \theta(H)$ which has no vertex v such that $H - v$ is bipartite.

Let H be the graph shown in Figure 2.2. Since H is triangle-free, $\theta(H) = 6$. It is easy to check that H is vertex-critical ($H - v$ has a perfect matching for any vertex $v \in V(H)$). Moreover, since H has two vertex-disjoint 5-cycles, H has no vertex v such that $H - v$ is bipartite. Then it suffices to prove $\gamma^\infty(H) = \theta(H) = 6$. Let D be a dominating set with $|D| = 5$. First, the attacker attacks x_1, x_2, x_3, x_4 in this order. Let D_1 be the obtained dominating set from D by this attack. Moreover, D_1 contains y_3, y_4, x_5 or y_5; otherwise, by attacking y_4, an element x_3 (respectively, x_4) in D_1 must be moved into y_4, and then y_3 (respectively, x_5) is not dominated as a result. If $y_3 \in D_1$ or $y_4 \in D_1$, then similarly to the above, the attacker wins by attacking y_1, by symmetry, because y_2 or y_5 is not protected as a result. Then we may suppose $x_5 \in D_1$ by symmetry. The attacker attacks y_1. Since $y_5 \notin D_1$, the defender has to move x_1 in D_1 to y_1; otherwise, if he moves x_2 in D_1 to y_1, y_2 is not protected as a result. Then the attacker wins the game by attacking y_5, y_4 and c (respectively, y_3); the final attack depends on the defender's move whether he moves x_4 (respectively, x_3). Therefore, $\gamma^\infty(H) = \theta(H) = 6$.

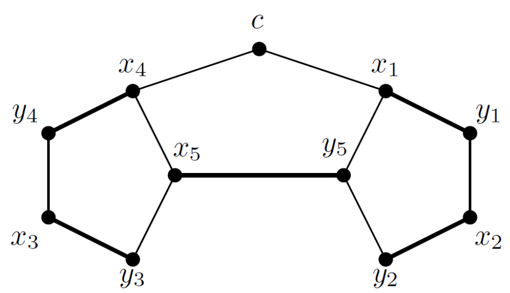

Figure 2.2: A vertex-critical triangle-free graph H with $\gamma^\infty(H) = \theta(H)$ which has no vertex v such that $H - v$ is bipartite.

3. Planar Graphs

Now we concentrate on planar graphs. We begin by presenting two lemmas that will be used in our proof of Theorem 1.2.

Lemma 3.1 (MacGillivray et al. [9]**).** *Let $\mathcal{F}$ be a family of graphs satisfying a hereditary property $\mathcal{P}$. Suppose $\mathcal{F}$ contains a graph G such that $\gamma^\infty(G) < \theta(G)$. Then the smallest such graph G is a 2-connected vertex-critical graph.*

Lemma 3.2 (Anderson et al. [1]**).** *Suppose G is obtained from a graph H by adding a path $P = v_1v_2\dots v_m$ with $m \geq 4$ between two distinct vertices v_1 and v_m of H. If H, $H - v_1$, $H - v_m$ and $H - v_1 - v_m$ satisfy $\gamma^\infty(K) = \theta(K)$ where K is one of these graphs, then so does G.*

By using these lemmas, we shall prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a smallest counterexample with respect to the number of vertices. Since the planarity and the minimum size of faces are a hereditary property, G is 2-connected by Lemma 3.1.

By the discharging method, we first prove that G has two adjacent vertices of degree 2. Suppose that no two vertices of degree 2 are adjacent. We give the initial charge $(\frac{12}{5} - k)$ to every vertex of degree k. Note that $2|E(G)| = \sum_{i=2}^{\Delta(G)} ip_i$ and $|V(G)| = \sum_{i=2}^{\Delta(G)} p_i$, where p_i is the number of vertices of degree i and $\Delta(G)$ denotes the maximum degree of G. By Euler's formula and $12|F(G)| \leq 2|E(G)|$, where $F(G)$ denotes the number of faces, we have

$$\sum_{i=2}^{\Delta(G)} ip_i \leq \frac{12}{5}\sum_{i=2}^{\Delta(G)} p_i - \frac{24}{5}.$$

Therefore, the total of the initial charge is

$$\sum_{i=2}^{\Delta(G)} p_i\left(\frac{12}{5} - i\right) \geq \frac{24}{5} > 0. \tag{3}$$

We discharge according to the following simple rule: for every vertex v of degree 2, half of the initial charge of v is discharged to each neighbor of v. After this discharging, the charge of every vertex of degree 2 is zero since any two vertices of degree 2 are not adjacent. For any other vertex of degree t with $t \geq 3$, the charge of this is at most

$$\left(\frac{12}{5} - t\right) + \frac{t}{2}\left(\frac{12}{5} - 2\right) = \frac{12 - 4t}{5}.$$

This means that the total charge after the discharging is at most zero, which contradicts to (3). Therefore, there are two adjacent vertices u, v of degree 2.

Let a and b be neighbors of u and v, respectively, where $a \neq v$ and $b \neq u$. Moreover, $a \neq b$ by the 2-connectedness of G and because the smallest face size of G is at least 12. Let $H = G - u - v$. By the minimality of G, the proposition holds for all of H, $H - a$, $H - b$ and $H - a - b$. Therefore, we have $\gamma^\infty(G) = \theta(G)$ by Lemma 3.2. □

Finally, we reveal the necessity of the planarity condition in Problem 1.1. In fact, we prove the following proposition.

Proposition 3.1. *For any integer $g \geq 1$ (respectively, $k \geq 1$), there are graphs with $\gamma^\infty < \theta$ and orientable genus g (respectively, non-orientable genus k).*

Proof. Let M_4 be the Grötzsch graph. It is known that $\gamma^\infty(\overline{M_4}) < \theta(\overline{M_4})$ [5]. We can easily check that the complement $\overline{M_4}$ can be embedded on the double torus S_2. Moreover, a 10-vertex graph G_1 with $\gamma^\infty(G_1) < \theta(G_1)$ shown in Figure 3.1 (which is constructed in [9, Figure 3]; `IEhbtn{ro` in Graph6 format [9, Appendix]) can be embedded on the torus; see Figure 3.2.

It is not difficult to check that the graph G_1 can also be embedded on the projective plane. Then it suffices to prove that for any integer $g \geq 2$ (respectively, $k \geq 2$), we can construct a graph H with $\gamma^\infty(H) < \theta(H)$ and orientable genus g (respectively, non-orientable genus k). Since a simple additivity of the orientable genus holds (cf. [2]), we can construct a desired graph by repeatedly adding a copy of K_5 and joining a vertex of the K_5 and a vertex of G_1.

Finally, we prove the non-orientable case. Let $g(H)$ and $\tilde{g}(H)$ be the orientable and non-orientable genus of a graph H, respectively. A graph G is *orientably simple* if $\tilde{g}(G) = 2g(G) + 1$. Stahl and Beineke [10] showed that a graph G is orientably simple if and only if each *block* of G, which is a maximal 2-connected subgraph of G, is orientably simple. Thus, the graph H_t constructed by adding t copies of K_5 and joining a fixed vertex of G_1 and a designated vertex of each copy of K_5 by an edge is not orientably simple. (Note that blocks of H are G_1, K_5s and K_2s.) Moreover, it is also proved in [10] that if G is not orientably simple, then $\tilde{g}(G) = 2n - \sum_{i=1}^{n} eg(B_i)$ where $B_1, \dots, B_n$ are blocks of G and $eg(B_i) = \max\{2 - 2g(B_i), 2 - \tilde{g}(B_i)\}$ is the Euler genus of B_i. Therefore, since $eg(G_1) = 1$, $eg(K_5) = 1$ and $eg(K_2) = 2$, we have

$$\tilde{g}(H_t) = 2(1 + 2t) - (1 + t + 2t) = t + 1.$$

This completes the proof of the proposition. □

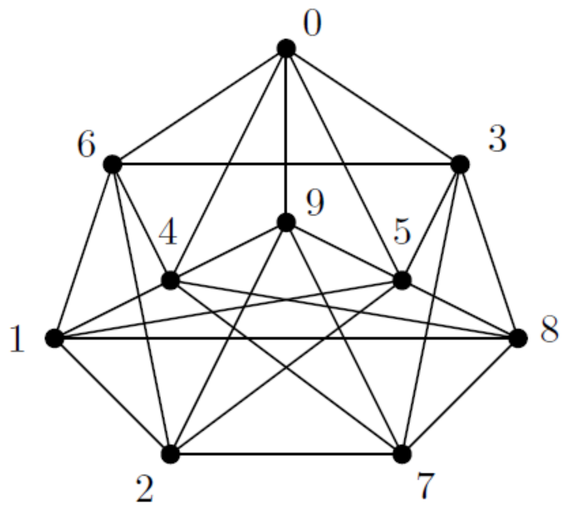

Figure 3.1: The graph G_1.

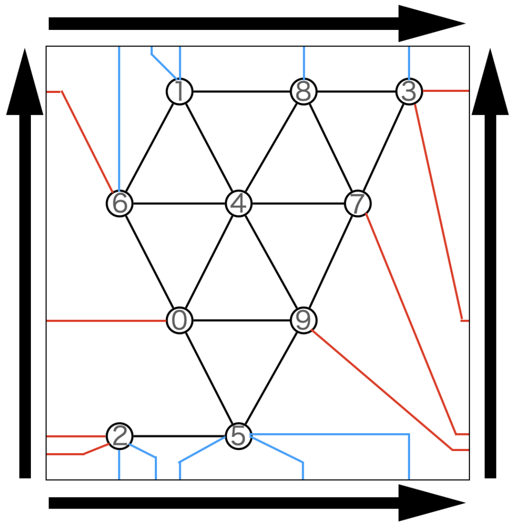

Figure 3.2: Embedding of G_1 on the torus.

References

[1] M. Anderson, C. Barrientos, R. C. Brigham, J. R. Carrington, R. P. Vitray, J. Yellen, Maximum-demand graphs for eternal security, *J. Combin. Math. Combin. Comput.* **61** (2007) 111–127.

[2] J. Battle, F. Harary, Y. Kodama, J.W.T. Youngs, Additivity of the genus of a graph, *Bull. Amer. Math. Soc.* **68** (1962) 565–568.

[3] A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, New York, 2008.

[4] A. P. Burger, E. J. Cockayne, W. R. Gründlingh, C. M. Mynhardt, J. H. van Vuuren, W. Winterbach, Infinite order domination in graphs, *J. Combin. Math. Combin. Comput.* **50** (2004) 179–194.

[5] W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, Eternal security in graphs, *J. Combin. Math. Combin. Comput.* **52** (2005) 169–180.

[6] W. F. Klostermeyer, C. M. Mynhardt, Domination, eternal domination and clique covering, *Discuss. Math. Graph Theory* **35** (2015) 283–300.

[7] W. F. Klostermeyer, C. M. Mynhardt, Protecting a graph with mobile guards, *Appl. Anal. Discrete Math.* **10** (2016) 1–29.

[8] W. F. Klostermeyer, C. M. Mynhardt, Eternal and secure domination in graphs, In: T. W. Haynes, S. T. Hedetniemi, M. A. Henning (Eds.), *Topics in Domination in Graphs*, Springer, Cham, 2020, 445–478.

[9] G. MacGillivray, C. M. Mynhardt, V. Virgile, Eternal domination and clique covering, *Electron. J. Graph Theory Appl.* **10** (2022) 603–624.

[10] S. Stahl, L. W. Beineke, Blocks and the nonorientable genus of graphs, *J. Graph Theory* **1** (1977) 75–78.

[11] M. Stehlík, Critical graphs with connected complements, *J. Combin. Theory Ser. B* **89** (2003) 189–194.