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Research Article

Realizability in Proper Total Domination in Graphs

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Abstract

A vertex u in a graph G totally dominates a vertex v if u is adjacent to v. A set S of vertices of G is a total dominating set if every vertex of G is totally dominated by some vertex of S. The minimum cardinality of a total dominating set of G is its total domination number $\gamma_t(G)$ of G. A proper total dominating set (or pt-dominating set) of G is a total dominating set S where no two adjacent vertices of G are totally dominated by the same number of vertices of S. The minimum cardinality of a pt-dominating set of G is the pt-domination number $\gamma_{pt}(G)$, while the maximum cardinality of such a set is the upper pt-domination number $\Gamma_{pt}(G)$. While every graph without isolated vertices has a total dominating set, this need not be the case for pt-dominating sets. For every graph G having a pt-dominating set, $\gamma_t(G) \le \gamma_{pt}(G) \le \Gamma_{pt}(G)$. It is shown that for every pair a, b of integers with $3 \le a \le b$, there exists a connected graph G with $\gamma_{pt}(G) = a$ and $\Gamma_{pt}(G) = b$. Furthermore, those triples a, b, c of integers with $2 \le a \le b \le c$ are studied for which there is a connected graph G with $\gamma_t(G) = a$, $\gamma_{pt}(G) = b$, and $\Gamma_{pt}(G) = c$.

Keywords: proper total dominating set; proper total domination number.

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1. Introduction

A vertex u in a graph G is said to *totally dominate* a vertex v if u is adjacent to v. A set S of vertices in a graph G is a *total dominating set* of G if every vertex of G is totally dominated by at least one vertex of S. Consequently, a graph G has a total dominating set if and only if G has no isolated vertices. The minimum cardinality of a total dominating set in G is its *total domination number* $\gamma_t(G)$. A total dominating set S of G with $|S| = \gamma_t(G)$ is called a γ_t*-set* of G. These concepts were introduced by Cockayne, Dawes, and Hedetniemi in [3]. The total domination numbers of all paths and cycles have been determined (see [1], for example).

Theorem 1.1. *For a path P_n of order $n \ge 2$ and a cycle C_n of order $n \ge 3$,*

$$\gamma_t(C_n) = \gamma_t(P_n) = \begin{cases} \left\lceil \dfrac{n}{2} \right\rceil & \text{if } n \not\equiv 2 \pmod 4 \\[2ex] \dfrac{n+2}{2} & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

A graph G has a total dominating set if and only if each component of G has a total dominating set. Because the total domination number of G is the sum of the total domination numbers of the components of G, only connected graphs are considered here.

For a total dominating set S of a graph G, the number of vertices of S that totally dominate a vertex v of G is denoted by $\sigma_S(v)$. While there exists no graph G with a total dominating set S such that $\sigma_S(u) \ne \sigma_S(v)$ for every two distinct vertices u and v of G, such sets do exist for every two adjacent vertices of G. A total dominating set S in a graph G is called a *proper total dominating set* (or a pt*-dominating set*) if $\sigma_S(u) \ne \sigma_S(v)$ for every two adjacent vertices u and v of G.

A graph G is often called *locally irregular* if $\deg u \ne \deg v$ for every two adjacent vertices u and v of G. If G is a locally irregular graph, then the vertex set $S = V(G)$ of G is a total dominating set of G since $\sigma_S(v) = \deg v$ for every vertex v of G. In fact, locally irregular graphs are the only graphs G for which $V(G)$ is a pt-dominating set.

Observation 1.1. [10] *If S is a pt-dominating set of a connected graph G, then the subgraph $G[S]$ induced by S is a locally irregular subgraph without isolated vertices in G. Then (i) $|S| \ge 3$ while $|S| = 3$ if and only if $G[S] = P_3$ and (ii) $S = V(G)$ if and only if G is locally irregular.*

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The *neighborhood* $N(v)$ of a vertex v in a graph G is the set of vertices adjacent to v and the set $N[v] = \{v\} \cup N(v)$ is the *closed neighborhood* of v. Two vertices u and v in a graph G are *twins* if $N(u) = N(v)$ and two adjacent vertices u and v in G are *adjacent twins* if $N[u] = N[v]$.

Observation 1.2. [10] *Let G be a connected graph with pt-dominating set S. If u and v are adjacent twins of G, then exactly one of u and v belongs to S. Consequently, if G contains the complete subgraph K_t of order $t \geq 3$ such that every two vertices of K_t are twins of G, then G has no pt-dominating set. In particular, every complete graph of order 3 or more has no pt-dominating set.*

The minimum cardinality of a pt-dominating set in G is its *proper total domination number* (or *pt-domination number* $\gamma_{pt}(G)$). By Observation 1.1, $\gamma_{pt}(G) \geq 3$ for every graph G having a pt-dominating set. A pt-dominating set S with $|S| = \gamma_{pt}(G)$ is called a *γ_{pt}-set* of G. All paths and cycles possessing a total proper dominating set were determined in [2].

Theorem 1.2. [2] *For a path P_n of order $n \geq 2$ and a cycle C_n of order $n \geq 3$,*

(*i*) *P_n has a pt-dominating set if and only if $n \equiv 3 \pmod 4$ and*

(*ii*) *C_n has a pt-dominating set if and only if $n \equiv 0 \pmod 4$.*

Furthermore,

$$\begin{aligned} \gamma_{pt}(P_n) &= \frac{3(n+1)}{4} \quad \text{if } n \geq 3 \text{ and } n \equiv 3 \pmod 4 \\ \gamma_{pt}(C_n) &= \frac{3n}{4} \quad \text{if } n \geq 4 \text{ and } n \equiv 0 \pmod 4. \end{aligned}$$

Since proper total domination is more restrictive than total domination, it follows that $2 \leq \gamma_t(G) \leq \gamma_{pt}(G)$ for every connected graph G possessing a pt-dominating set. In [2], all pairs a, b of integers with $2 \leq a \leq b$ have been determined for which there exists a graph G with $\gamma_t(G) = a$ and $\gamma_{pt}(G) = b$.

Theorem 1.3. [2] *For each pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G such that $\gamma_t(G) = a$ and $\gamma_{pt}(G) = b$ if and only if (1) $a \in \{2, 3, 4\}$ and $b \geq a + 1$ or (2) $5 \leq a \leq b$.*

The set $S = \{u_1, u_2, u_3, u_5, u_6, u_7\}$ of vertices in the graph G of Figure 1.1 is a γ_{pt}-set of G and $\gamma_{pt}(G) = 6$. This set S cannot be extended to a larger pt-dominating set of G. That is, there is no pt-dominating set of G containing S as a proper subset. For the graph H of Figure 1.1, the set $S' = \{v, v_1, v_3, v_5, v_7\}$ is a γ_{pt}-set of H and so $\gamma_{pt}(H) = 5$. However, S' can be extended to a larger pt-dominating set of H. In fact, since H is locally irregular, $V(H)$ is a pt-dominating set of H.

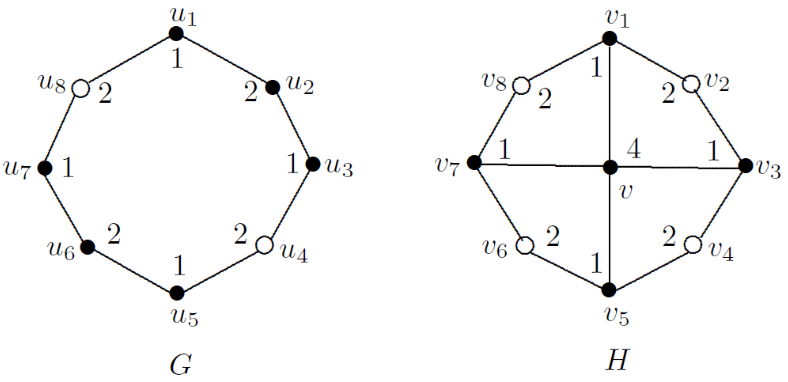

Figure 1.1: Two graphs G and H.

These observations lead to the following question.

> *For a connected graph G having a pt-dominating set, what is the maximum cardinality of a pt-dominating set in G?*

The maximum cardinality of a pt-dominating set in a graph G is referred to as the *upper proper total domination number* (or *upper pt-domination number*) $\Gamma_{pt}(G)$ of the graph G. A pt-dominating set S with $|S| = \Gamma_t(G)$ is called a *Γ_{pt}-set* of G. Therefore, for every graph G having a pt-dominating set,

$$\max\{\gamma_t(G), 3\} \leq \gamma_{pt}(G) \leq \Gamma_{pt}(G).$$

For example, for the graph G of Figure 1.1, the set S is a Γ_{pt}-set and $\Gamma_{pt}(G) = 6$, while for the graph H of Figure 1.1, the vertex set $V(H)$ of H is a Γ_{pt}-set of H and $\Gamma_{pt}(H) = 9$.

The following observation is a consequence of Observation 1.1.

Observation 1.3. *A graph G of order $n \geq 3$ has $\Gamma_{pt}(G) = n$ if and only if G is locally irregular.*

Consequently, since the graph H of order 9 in Figure 1.1 is locally irregular, $\Gamma_{pt}(H) = 9$, as we already saw. For positive integers s and t with $s \neq t$, the complete bipartite graph $K_{s,t}$ is a locally irregular graph of order $s + t$ and so we have the following.

Proposition 1.1. *For positive integers s and t with $s \neq t$,*

$$\gamma_{pt}(K_{s,t}) = 3 \text{ and } \Gamma_{pt}(K_{s,t}) = s + t.$$

In particular, $\gamma_{pt}(K_{1,t}) = 3$ and $\Gamma_{pt}(K_{1,t}) = 1 + t$ for every star $K_{1,t}$ of order $1 + t \geq 3$.

Since there is a unique pt-dominating set for the path P_n of order n when $n \geq 3$ and $n \equiv 3 \pmod 4$ and for the cycle C_n of order n when $n \geq 4$ and $n \equiv 0 \pmod 4$, we have the following.

Proposition 1.2. *For a path P_n of order $n \geq 2$ and a cycle C_n of order $n \geq 3$,*

$$\begin{aligned} \Gamma_{pt}(P_n) &= \gamma_{pt}(P_n) = \frac{3(n+1)}{4} \quad \text{if } n \geq 3 \text{ and } n \equiv 3 \pmod 4 \\ \Gamma_{pt}(C_n) &= \gamma_{pt}(C_n) = \frac{3n}{4} \quad \text{if } n \geq 4 \text{ and } n \equiv 0 \pmod 4. \end{aligned}$$

The primary goal here is to investigate the relationship among the total domination number, the pt-domination number and the upper pt-domination number for graphs having a pt-dominating set. We refer to the books [1, 7] for notation and terminology not defined here. Further details and perspectives on the concepts related to the present paper can be found in [4–6, 8, 9, 11].

2. Realizable Pairs of Integers

If G is a connected graph with a pt-dominating set such that $\gamma_{pt}(G) = a$ and $\Gamma_{pt}(G) = b$, then $3 \leq a \leq b$. A pair a, b of integers with $3 \leq a \leq b$ is said to be *realizable* if there exists a connected graph G with $\gamma_{pt}(G) = a$ and $\Gamma_{pt}(G) = b$. This gives rise to the following question:

> *Which pairs a, b of integers with $3 \leq a \leq b$ are realizable?*

We show that every pair a, b of integers with $3 \leq a \leq b$ is realizable. In order to establish this fact, we first present some preliminary information.

For two vertex-disjoint graphs F and H, the graph $F \vee H$ is the *join* of F and H. That is, $V(F \vee H) = V(F) \cup V(H)$ and $E(F \vee H) = E(F) \cup E(H) \cup \{uv : u \in V(F), v \in V(H)\}$. A graph G is called a *friendship graph* if for every two vertices u and v of G, there is a unique vertex of G that is adjacent to both u and v. For each positive integer k, the graph $F_k = kK_2 \vee K_1$ is the unique friendship graph of order $2k + 1$. Thus, F_k has one vertex of degree $2k$ (which is the *center* of F_k) and all other vertices have degree 2. Every two adjacent vertices of degree 2 in F_k are therefore adjacent twins. The friendship graphs $F_1 = K_3$ and $F_2 = 2K_1 \vee K_1$ have no pt-dominating sets by Observation 1.2. In fact, F_1 and F_2 are exceptional friendship graphs.

Theorem 2.1. *For a positive integer k, the friendship graph F_k has a pt-dominating set if and only if $k \geq 3$. Furthermore, if $k \geq 3$, then F_k has a unique pt-dominating set and $\gamma_{pt}(F_k) = \Gamma_{pt}(F_k) = k + 1$.*

Proof. As mentioned earlier, $F_1 = K_3$ and $F_2 = 2K_1 \vee K_1$ have no pt-dominating set. For $k \geq 3$, let

$$V(F_k) = \{u\} \cup \{v_i, w_i : 1 \leq i \leq k\},$$

where u is the center of F_k and $v_i w_i \in E(F_k)$ for $1 \leq i \leq k$. We show that F_k has a unique (up to isomorphism) pt-dominating set. Let $S = \{u\} \cup \{v_i : 1 \leq i \leq k\}$. Since $\sigma_S(u) = k \geq 3$, $\sigma_S(v_i) = 1$, and $\sigma_S(w_i) = 2$ for $1 \leq i \leq k$, it follows that S is a pt-dominating set of F_k.

Next, let S' be a pt-dominating set of F_k. By Observation 1.2, exactly one of v_i and w_i belongs to S' for each integer i with $1 \leq i \leq k$. Thus, we may assume that $v_i \in S'$ and $w_i \notin S'$. Since $\sigma_{S'}(v_i) \geq 1$ and $w_i \notin S'$ $(1 \leq i \leq k)$, it follows that $u \in S'$. Hence, $S' = S$ and so S is the unique pt-dominating set of F_k. Therefore, $\gamma_{pt}(G) = \Gamma_{pt}(G) = |S| = k + 1$. □

As a consequence of Theorem 2.1, for every integer b with $b \geq 3$, the pair $3, b$ is realizable. This fact can be extended. For two vertex-disjoint graphs F and H, the graph $F + H$ is the *union* of F and H, that is, $V(F + H) = V(F) \cup V(H)$ and $E(F + H) = E(F) \cup E(H)$.

Theorem 2.2. *Every pair a, b of integers with $3 \leq a \leq b$ is realizable. Furthermore, for each integer k with $a \leq k \leq b$, there is a pt-dominating set in the graph with k vertices.*

Proof. First, suppose that $a = 3$ and $b \geq 3$. Let $G = K_{1,b-1}$ with $V(G) = \{v, v_1, v_2, \ldots, v_{b-1}\}$ where v is the center of G. Then $\gamma_{pt}(G) = 3$. Since G is a locally irregular graph of order b, it follows by Observation 1.3 that $\Gamma_{pt}(G) = b$. For an integer k with $3 \leq k \leq b$, let $S_k = \{v, v_1, v_2, \ldots, v_{k-1}\}$. Since $\sigma_{S_k}(v) = k - 1 \geq 2$ and $\sigma_{S_k}(v_i) = 1$ for $1 \leq i \leq b - 1$, it follows that S_k is a pt-dominating set of G with $|S_k| = k$.

Next, suppose that $a \geq 4$. If $b = a \geq 4$, then for the friendship graph $G = (a - 1)K_2 \vee K_1$ of order $2a - 1$, it follows by Theorem 2.1 that $\gamma_{pt}(G) = \Gamma_{pt}(G) = a$. Thus, we may assume that $4 \leq a < b$. Let $G = [(a - 1)K_2 + (b - a)K_1] \vee K_1$ be the graph of order $a + b - 1$ with

$$V(G) = \{u\} \cup \{v_i, w_i : 1 \leq i \leq a\} \cup \{x_j : 1 \leq j \leq b - a\},$$

where $\deg u = a + b - 2$, $\deg v_i = \deg w_i = 2$, $v_i w_i \in E(G)$ for $1 \leq i \leq a - 2$, and $\deg x_j = 1$ for $1 \leq j \leq b - a$. For each integer k with $a \leq k \leq b$, let

$$S_k = \begin{cases} \{u\} \cup \{v_i : 1 \leq i \leq a - 1\} & \text{if } k = a \\ \{u\} \cup \{v_i : 1 \leq i \leq a - 1\} \cup \{x_1, x_2, \ldots, x_{k-a}\} & \text{if } k > a. \end{cases}$$

Then $|S_k| = k$ for $a \leq k \leq b$. Since $\sigma_{S_k}(u) = (a - 1) + (k - a) = k - 1 \geq 3$, $\sigma_{S_k}(v_i) = 1$ and $\sigma_{S_k}(w_i) = 2$ for $1 \leq i \leq k$, and $\sigma_{S_k}(x_j) = 1$ for $1 \leq j \leq b - a$, it follows that S_k is a pt-dominating set of G. It remains to show that $\gamma_{pt}(G) = a$ and $\Gamma_{pt}(G) = b$.

First, we show that $\gamma_{pt}(G) = a$. Since S_a is a pt-dominating set of G, it follows that $\gamma_{pt}(G) \leq |S_a| = a$. To show that $\gamma_{pt}(G) \geq a$, let S' be a pt-dominating set of G with $|S'| = \gamma_{pt}(G)$. By Observation 1.2, exactly one of v_i and w_i belongs to S', say $v_i \in S'$ and $w_i \notin S'$ for $1 \leq i \leq a - 1$. Since each vertex v_i $(1 \leq i \leq a - 1)$ is dominated only by u, it follows that $u \in S'$. Thus, $\gamma_{pt}(G) = |S'| \geq a$ and so $\gamma_{pt}(G) = a$.

Next, we show that $\Gamma_{pt}(G) = b$. Since S_b is a pt-dominating set of G, it follows that $\Gamma_{pt}(G) \geq |S_b| = b$. To show that $\Gamma_{pt}(G) \leq b$, let S'' be a pt-dominating set of G with $|S''| = \Gamma_{pt}(G)$. By Observation 1.2, exactly one of v_i and w_i belongs to S'' for $1 \leq i \leq a - 1$. Thus, $\Gamma_{pt}(G) = |S''| \leq |V(G)| - (a - 1) = (a + b - 1) - (a - 1) = b$. Therefore, $\Gamma_{pt}(G) = b$. ☐

3. Realizable Triples of Integers

We have mentioned that $2 \leq \gamma_t(G) \leq \gamma_{pt}(G) \leq \Gamma_{pt}(G)$ for every graph G having a pt-dominating set. A triple a, b, c of integers with $2 \leq a \leq b \leq c$ is said to be *realizable* if there exists a connected graph G with $\gamma_t(G) = a$, $\gamma_{pt}(G) = b$, and $\Gamma_{pt}(G) = c$. Theorem 2.2 therefore gives rise to a natural question:

> *Which triples a, b, c of integers with $2 \leq a \leq b \leq c$ are realizable?*

By Theorem 1.3, it suffices to consider only those triples a, b, c such that

$$(i)\ a \in \{2, 3, 4\} \text{ and } b \geq a + 1 \text{ or } (ii)\ 5 \leq a \leq b. \tag{1}$$

We begin with those triples a, b, c of integers where $a = 2$ and $3 \leq b \leq c$.

Proposition 3.1. *Every triple $2, b, c$ of integers with $3 \leq b \leq c$ is realizable.*

Proof. For $b = 3$ and $c \geq 3$, let G be the star $K_{1,c-1}$. Then $\gamma_t(G) = 2$, $\gamma_{pt}(G) = 3$, and $\Gamma_{pt}(G) = c$. For $b = c \geq 4$, let G be the friendship graph $F_{b-1} = (b - 1)K_2 \vee K_1$. Then $\gamma_t(G) = 2$ and $\gamma_{pt}(G) = \Gamma_{pt}(G) = b$ by Theorem 2.1. For $c > b \geq 4$, let $G = [(b - 1)K_2 + (c - b)K_1] \vee K_1$. Then $\gamma_t(G) = 2$, $\gamma_{pt}(G) = b$, and $\Gamma_{pt}(G) = c$ from the proof of Theorem 2.2. ☐

By Proposition 3.1, we may now assume that $a \geq 3$. Next, we consider those triples a, b, c of integers with $3 \leq a \leq b \leq c$ where b is sufficiently large.

Theorem 3.1. *Let a, b, c be three integers with $3 \leq a \leq b \leq c$. If*

$$(i)\ a = 3 \text{ and } b \geq 7,\ (ii)\ a = 4 \text{ and } b \geq 8, \text{ or } (iii)\ a \geq 5 \text{ and } b \geq 2a,$$

then a, b, c is realizable.

Proof. First, we verify the following claim.

> **Claim.** For each pair a, b of integers with $a \geq 3$ and $b \geq \max\{a+4, 2a\}$, let $H_{a,b}$ be the graph of order $2b-a$ constructed from the friendship graph $F_{b-2a+1} = (b-2a+1)K_2 \vee K_1$ and the graph $(a-1)K_3$ by joining exactly two vertices of each copy of K_3 to the center of F_{b-2a+1}. Then $H_{a,b}$ has a unique γ_t-set (up to isomorphism) consisting of a vertices and a unique pt-dominating set (up to isomorphism) consisting of b vertices. Consequently,
>
> $$\gamma_t(H_{a,b}) = a \text{ and } \gamma_{pt}(H_{a,b}) = \Gamma_{pt}(H_{a,b}) = b.$$

To verify this claim, let $V(F_{b-2a+1}) = \{u\} \cup \{v_i, w_i : 1 \leq i \leq b-2a+1\}$, where u is the center and u_i is adjacent to w_i for $1 \leq i \leq b-2a+1$. For $1 \leq j \leq a-1$, let $K_3 = (x_j, y_j, z_j, x_j)$ in the graph $(a-1)K_3$ where x_j and y_j are adjacent to u. The vertex u of degree $2(b-a)$ is referred to as the *center* of $H_{a,b}$.

First, we show that $H_{a,b}$ has a unique γ_t-set. Let

$$A = \{u\} \cup \{x_j : 1 \leq i \leq a-1\}. \tag{2}$$

Then A is a total dominating set of $H_{a,b}$ and so $\gamma_t(H_{a,b}) \leq |A| = a$. Next, we show that $\gamma_t(H_{a,b}) \geq a$. Assume, to the contrary, that $\gamma_t(H_{a,b}) < a$ and let A' be a γ_t-set of $H_{a,b}$. Then $|A'| = \gamma_t(H_{a,b})$. Since z_j is dominated only by x_j or y_j for each integer j with $1 \leq j \leq a-1$, it follows that $x_j \in A'$ or $y_j \in A'$, say the former. This implies that $A' = \{x_j : 1 \leq j \leq a-1\}$. Since A' is an independent set of vertices in $H_{a,b}$, it follows that A' is not a total dominating set of $H_{a,b}$, a contradiction. Therefore, $\gamma_t(H_{a,b}) \geq a$ and so $\gamma_t(H_{a,b}) = a$. Since every γ_t-set of $H_{a,b}$ contains u and exactly one of x_j and y_j for $1 \leq j \leq a-1$, it follows that A is the unique γ_t-set of $H_{a,b}$.

Next, we show that $H_{a,b}$ has a unique pt-dominating set. Let

$$B = \{u\} \cup \{v_i : 1 \leq i \leq b-2a+1\} \cup \{x_j, z_j : 1 \leq j \leq a-1\}. \tag{3}$$

Then $|B| = b$. Since $\sigma_B(u) = b-a \geq 4$, $\sigma_B(v_i) = \sigma_B(z_j) = 1$, $\sigma_B(w_i) = \sigma_B(x_j) = 2$, and $\sigma_B(y_j) = 3$ where $1 \leq i \leq b-2a+1$ and $1 \leq j \leq a-1$, it follows that B is a pt-dominating set of $H_{a,b}$. Next, we show that B is the unique pt-dominating set of G. Let B' be a pt-dominating set of $H_{a,b}$. Since v_i and w_i are adjacent twins for $1 \leq i \leq b-2a+1$ while x_j and y_j are adjacent twins for $1 \leq j \leq a-1$, exactly one of v_i and w_i belongs to B' and exactly one of x_j and y_j belongs to B'. Thus, we may assume that $v_i \in B'$ and $w_i \notin B'$ for $1 \leq i \leq b-2a+1$ while $x_j \in B'$ and $y_j \notin B'$ $1 \leq j \leq a-1$. Since each vertex v_i $(1 \leq i \leq b-2a+1)$ is dominated only by u, it follows that $u \in B'$. Furthermore, if $z_j \notin B'$ for some integer j with $1 \leq j \leq a-1$, then $\sigma_{B'}(z_j) = \sigma_{B'}(x_j) = 1$, which is impossible. Thus, $z_j \in B'$ for $1 \leq j \leq a-1$. Therefore, $B = B'$ and so B is the unique pt-dominating set of G. Consequently, $\gamma_{pt}(H_{a,b}) = \Gamma_{pt}(H_{a,b}) = b$. This verifies the claim.

Let a, b, c be integers with $3 \leq a \leq b \leq c$. If a and b satisfy one of the conditions (i), (ii), (iii), then $b - a \geq 4$. For $b = c$, let $G = H_{a,b}$. It follows by the claim that $\gamma_t(G) = a$ and $\gamma_{pt}(G) = \Gamma_{pt}(G) = b$. For $b < c$, let G be the graph obtained from $H_{a,b}$ by adding $c-b$ pendant edges at the center of $H_{a,b}$. We apply the same notation for the vertices of $H_{a,b}$ in the claim, namely, let $V(F_{b-2a+1}) = \{u\} \cup \{v_i, w_i : 1 \leq i \leq b-2a+1\}$, where u is the center and u_i is adjacent to w_i for $1 \leq i \leq b-2a+1$. For $1 \leq j \leq a-1$, let $K_3 = (x_j, y_j, z_j, x_j)$ in the graph $(a-1)K_3$ where x_j and y_j are adjacent to u. Let

$$P = \{p_1, p_2, \ldots, p_{c-b}\} \tag{4}$$

be the set of $c-b$ end-vertices of G that are adjacent to u. Then the order of G is

$$n = |V(H_{a,b})| + (c-b) = (2b-a) + (c-b) = c+b-a.$$

We now show that $\gamma_t(G) = a$, $\gamma_{pt}(G) = b$, and $\Gamma_{pt}(G) = c$. Since the set A in (2) is a total dominating set of G, it follows that $\gamma_t(G) \leq |A| = a$. Since $H_{a,b} \subseteq G$, it follows that $\gamma_t(G) \geq \gamma(H_{a,b}) = a$ and so $\gamma_t(G) = a$.

Next, we show that $\gamma_{pt}(G) = b$. Since the set B in (3) is a pt-dominating set of G, it follows that $\gamma_{pt}(G) \leq |B| = b$. To show that $\gamma_{pt}(G) \geq b$, let B' be a γ_{pt}-set of G. Since v_i and w_i are adjacent twins for $1 \leq i \leq b-2a+1$ while x_j and y_j are adjacent twins for $1 \leq j \leq a-1$, exactly one of v_i and w_i belongs to B' and exactly one of x_j and y_j belongs to B'. Thus, we may assume that $v_i \in B'$ and $w_i \notin B'$ for $1 \leq i \leq b-2a+1$ while $x_j \in B'$ and $y_j \notin B'$ $1 \leq j \leq a-1$. Since each vertex v_i $(1 \leq i \leq b-2a+1)$ is dominated only by u, it follows that $u \in B'$. Furthermore, if $z_j \notin B'$ for some integer j with $1 \leq j \leq a-1$, then $\sigma_{B'}(z_j) = \sigma_{B'}(x_j) = 1$, which is impossible. Thus, $z_j \in B'$ for $1 \leq j \leq a-1$. Thus, $|B'| \geq (b-2a+1) + (a-1) + 1 + (a-1) = b$. Therefore, $\gamma_{pt}(G) = b$.

Finally, we show that $\Gamma_{pt}(G) = c$. Since $T = B \cup P$ is a pt-dominating set of G, where B and P are the sets described in (3) and (4), respectively, it follows that $\Gamma_{pt}(G) \geq |B| + |P| = b + (c - b) = c$. Next, let T be a Γ_{pt}-set of G. Then exactly one of v_i and w_i $(1 \leq i \leq b - 2a + 1)$ belongs to T and exactly one of x_j and y_j $(1 \leq j \leq a - 1)$ belongs to T. Then $\Gamma_{pt}(G) = |T| \leq |V(G)| - [(b - 2a + 1) + (a - 1)] = (c + b - a) - (b - a) = c$. Therefore, $\Gamma_{pt}(G) = c$. □

A path $P_k = (v_1, v_2, \ldots, v_k)$ of order $k \geq 3$ in a graph G is a *pendant k-path at the vertex v_k* in G if v_1 is an end-vertex in G and v_i has degree 2 in G for $2 \leq i \leq k - 1$. In this case, v_k is referred to as the *terminal vertex* of P_k.

Lemma 3.1. [10] *Let $P_k = (v_1, v_2, \ldots, v_k)$ be a pendant k-path of order $k \geq 3$ in a graph G. If S is a pt-dominating set of G, then $v_i \in S$ if $i \not\equiv 0 \pmod 4$ and $v_i \notin S$ if $i \equiv 0 \pmod 4$ for $1 \leq i \leq k$. In particular, every vertex of a pendant 3-path belongs to S and the terminal vertex of a pendant 4-path is the only vertex in the path that does not belong to S.*

The *subdivision graph* $S(G)$ of a graph G is obtained by subdividing each edge of G exactly once.

Theorem 3.2. *Let a, b, c be three integers with $3 \leq a \leq b \leq c$. If*

$$(i)\ (a, b) \in \{(3, 6), (4, 6)\} \text{ or } (ii)\ a \geq 4 \text{ and } b = 2a - 1,$$

then a, b, c is realizable.

Proof. First, suppose that $a = 3$ and $b = 6$. Then $c \geq 6$. Let G be the locally irregular graph of order c obtained from the star $K_{1,c-3}$ by subdividing each of two edges of $K_{1,c-3}$ exactly once. Then $\gamma_t(G) = 3$. By Observation 1.3 and Lemma 3.1, it follows that $\gamma_{pt}(G) = 6$ and $\Gamma_{pt}(G) = c$.

Next, suppose that $a = 4$ and $b = 6$. For $c = 6$, let $G = P_7 = (v_1, v_2, \ldots, v_7)$. Then $\gamma_t(G) = 4$ and $\gamma_{pt}(G) = \Gamma_{pt}(G) = 6$ by Theorems 1.1, 1.2, and Proposition 1.2. For $b \geq 7$, let H be the graph of order $c + 1$ obtained from P_7 by adding $b - 6$ pendant edges at the vertex v_6. That is, H is the broom of order $c + 1$ and diameter 6. Then $\gamma_t(G) = 4$. First, we show that $\gamma_{pt}(G) = 6$. Since $V(P_7) - \{v_4\}$ is a pt-dominating set of G, it follows that $\gamma_{pt}(G) \leq 6$. Next, let S be a pt-dominating set of G. Then $v_i \in S$ for $i = 1, 2, 3, 5, 6$ by Lemma 3.1. Since $\sigma_S(v_6) \in \{2, 3\}$, it follows that S must contain one leaf adjacent to v_6 and so $\gamma_{pt}(G) \geq 6$. Hence, $\gamma_{pt}(G) = 6$. Next, we show that $\Gamma_{pt}(G) = c$. Since $V(H) - \{v_4\}$ is a pt-dominating set of H, it follows that $\Gamma_{pt}(H) \geq c$. On the other hand, H is not locally irregular and so $\Gamma_{pt}(H) \leq c$ by Observation 1.3. Hence, $\Gamma_{pt}(H) = c$.

Finally, suppose that $a \geq 4$ and $b = 2a - 1 \geq 7$. Then $c \geq 7$ and $c - a \geq a - 1$. Let G be the locally irregular graph of order c obtained from the star $K_{1,c-a}$ by subdividing each of $a - 1$ edges of $K_{1,c-a}$ exactly once. Then $\gamma_t(G) = a$. By Observation 1.3 and Lemma 3.1, we have $\gamma_{pt}(G) = 2a - 1$ and $\Gamma_{pt}(G) = c$. □

By Proposition 3.1 and Theorems 3.1 and 3.2, it remains to consider those integers a, b, c of integers with $3 \leq a \leq b \leq c$ that satisfy one of the following two conditions:

$$\text{(i) } (a, b) \in \{(3, 4), (3, 5), (4, 5)\} \text{ and (ii) } 5 \leq a \leq b \leq 2a - 2.$$

First, we consider those triples a, b, c with $a \leq b \leq c$ such that $(a, b) \in \{(3, 4), (3, 5), (4, 5)\}$. We show that except for the four triples $(3, 4, 4), (3, 4, 5), (4, 5, 5), (4, 5, 6)$, all triples in (i) are realizable. We begin with those triples where $(a, b) \in \{(3, 4), (3, 5)\}$.

Theorem 3.3. *Let $3, b, c$ be three integers where $b \in \{4, 5\}$ and $b \leq c$. If $(b, c) \notin \{(4, 4), (4, 5)\}$, then $3, b, c$ is realizable.*

Proof. We consider two cases, according to whether $b = 4$ or $b = 5$.

Case 1. $b = 4$. Since $(b, c) \notin \{(4, 4), (4, 5)\}$, it follows that $c \geq 6$. We consider two subcases, depending on whether $c = 6$ or $c \geq 7$.

Subcase 1.1. $c = 6$. Let G be the graph of order 7 obtained from the 6-cycle $C_6 = (v_1, v_2, \ldots, v_6, v_1)$ by adding a vertex v and joining v to v_1, v_3, and v_5. We show that $\gamma_t(G) = 3$, $\gamma_{pt}(G) = 4$, and $\Gamma_{pt}(G) = 6$. Since $\{v, v_1, v_3\}$ is a total dominating set of G and there is no total dominating set of G consisting of two vertices, it follows that $\gamma_t(G) = 3$. By Theorem 1.3, $\gamma_{pt}(G) \geq 4$. Since $\{v, v_1, v_3, v_5\}$ is a pt-dominating set of G, it follows that $\gamma_{pt}(G) = 4$. It remains to show that $\Gamma_{pt}(G) = 6$. Since G is not locally irregular, it follows that $\Gamma_{pt}(G) \leq 6$ by Observation 1.3. Furthermore, $V(G) - \{v_1\}$ is a pt-dominating set and so $\Gamma_{pt}(G) \geq 6$. Hence, $\Gamma_{pt}(G) = 6$.

Subcase 1.2. $c \geq 7$. Let H be the graph of order $c + 1$ constructed from the graph G of order 7 in Subcase 1.1 by adding $c - 6$ new vertices $u_1, u_2, \ldots, u_{c-6}$ and joining each vertex u_i $(1 \leq i \leq c - 6)$ to v_1 and v_3. Since $\deg v = \deg v_5 = 3$ and v is adjacent to v_5, it follows that H is not locally irregular. The same argument used when $c = 6$ shows that $\gamma_t(H) = 3$ and $\gamma_{pt}(H) = 4$. Since $V(H) - \{v_1\}$ is a pt-dominating set, it follows that $\Gamma_{pt}(H) \geq |V(H) - \{v_1\}| = c$. On the other hand, H is not locally irregular and so $\Gamma_{pt}(G) \leq c$. Therefore, $\Gamma_{pt}(G) = c$.

Case 2. $b = 5$. Then $c \geq 5$. We consider three subcases, depending on whether $c = 5$, $c = 6$, or $c \geq 7$.

Subcase 2.1. $c = 5$. Let G be the graph of order 7 obtained from the graph $K_4 - e$ with vertex set $\{v_1, v_2, v_3, v_4\}$ and $e = v_2v_4$ by adding three new vertices w_1, w_2, w_3 and joining w_i to v_i for $1 \leq i \leq 3$. We show that $\gamma_t(G) = 3$ and $\gamma_{pt}(G) = \Gamma_{pt}(G) = 5$. Since $\{v_1, v_2, v_3\}$ is a total dominating set of G and there is no total dominating set of G consisting of two vertices, it follows that $\gamma_t(G) = 3$. It remains to show that $\gamma_{pt}(G) = \Gamma_{pt}(G) = 5$.

First, we show that $\gamma_{pt}(G) = 5$. Since $\{v_1, v_2, v_3, v_4, w_3\}$ is a pt-dominating set of G, it follows that $\gamma_{pt}(G) \leq 5$. By Theorem 1.3, $\gamma_{pt}(G) \geq 4$. Thus, $\gamma_{pt}(G) \in \{4, 5\}$. We show that $\gamma_{pt}(G) \neq 4$. Assume, to the contrary, that $\gamma_{pt}(G) = 4$. Let S be a γ_{pt}-set of G. Since w_i is an end-vertex of G, it follows that $v_i \in S$ for $i = 1, 2, 3$. Since $G[S]$ is a locally irregular graph of order 4 by Lemma 1.1, it follows that $G[S] \cong K_{1,3}$. Thus, $S = \{v_1, v_2, v_3, w_2\}$. However then, $\sigma_S(v_1) = \sigma_S(v_3) = 2$, which is impossible. Thus, $\gamma_{pt}(G) = 5$. Next, we show that $\Gamma_{pt}(G) = 5$. Assume, to the contrary, that $\Gamma_{pt}(G) \geq 6$. Since G is not locally irregular, $\Gamma_{pt}(G) = 6$. Let T be a Γ_{pt}-set of G with $|T| = 6$. Then $T = V(G) - \{v\}$ for some vertex $v \neq v_i$ where $i = 1, 2, 3$. Since $G[T] = G - v$ is not locally irregular, this is impossible by Lemma 1.1. Therefore, $\Gamma_{pt}(G) = 5$.

Subcase 2.2. $c = 6$. Let G be the graph of order 7 in Figure 3.1. We show that $\gamma_t(G) = 3$, $\gamma_{pt}(G) = 5$, and $\Gamma_{pt}(G) = 6$. First, every total dominating set of G contains v_1, v_2 and at least one of v_4 and v_5. Hence, $\gamma_t(G) \geq 3$. Since $\{v_1, v_2, v_4\}$ is a total dominating set of G, it follows that $\gamma_t(G) = 3$. Next, since $\{v_1, v_2, v_3, v_4, v_5\}$ is a pt-dominating set of G, it follows that $\gamma_{pt}(G) \leq 5$. Thus, $\gamma_{pt}(G) = 4$ or $\gamma_{pt}(G) = 5$. We claim that $\gamma_{pt}(G) \neq 4$, for otherwise, let S be a γ_{pt}-set of G. Then $G[S] = K_{1,3}$. Since $\{v_1, v_2, v_4\} \subseteq S$ or $\{v_1, v_2, v_5\} \subseteq S$, say the former, it follows that $S = \{v_1, v_2, v_3, v_4\}$. However, $\sigma_S(w_1) = \sigma_S(v_1)$, which is impossible. Thus, $\gamma_{pt}(G) = 5$. Finally, $VG) - \{v_5\}$ is a pt-dominating set of G and so $\Gamma_{pt}(G) \geq 6$. Since G is not locally irregular, it follows that $\Gamma_{pt}(G) \leq 6$ by Observation 1.3. Hence, $\Gamma_{pt}(G) = 6$.

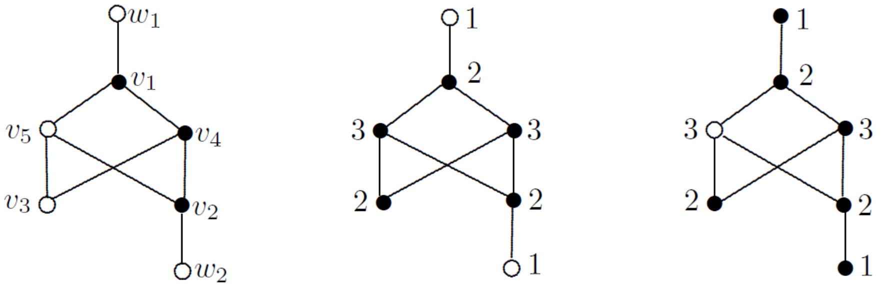

Figure 3.1: A graph G with $\gamma_t(G) = 3$, $\gamma_{pt}(G) = 5$, and $\Gamma_{pt}(G) = 6$.

Subcase 2.3. $c \geq 7$. Let H be the graph of order $c + 1$ obtained from the graph G of order 7 in Figure 3.1 by adding $c - 6$ pendant edges at the vertex v_4. The same argument used in Subcase 2.2 shows that $\gamma_t(H) = 3$ and $\gamma_{pt}(H) = 4$. Since H is not locally irregular and $VH) - \{v_5\}$ is a pt-dominating set of H, it follows that $\Gamma_{pt}(H) = c$. ☐

We now consider those triples where $a = 4$, $b = 5$, and $c \geq 7$.

Theorem 3.4. *Every triple* $4, 5, c$ *of integers where* $c \geq 7$ *is realizable.*

Proof. First, suppose that $c = 7$. Let G be the graph of order 8 obtained from the graph of order 7 in Figure 3.1 by adding a new vertex w_3 and joining w_3 to v_3. We show that $\gamma_t(G) = 4$, $\gamma_{pt}(G) = 5$, and $\Gamma_{pt}(G) = 7$. First, every total dominating set of G contains v_1, v_2, v_3 and at least one of v_4 and v_5; so $\gamma_t(G) \geq 4$. Since $\{v_1, v_2, v_3, v_4\}$ is a total dominating set of G, it follows that $\gamma_t(G) = 4$. Next, since $\{v_1, v_2, v_3, v_4, v_5\}$ is a pt-dominating set of G, it follows that $\gamma_{pt}(G) \leq 5$. Thus, $\gamma_{pt}(G) = 4$ or $\gamma_{pt}(G) = 5$. We claim that $\gamma_{pt}(G) \neq 4$, for otherwise, let S be a γ_{pt}-set of G with $|S| = 4$. By Lemma 1.1, $G[S] = K_{1,3}$. Because every total dominating set of G contains v_1, v_2, v_3 and at least one of v_4 and v_5, we may assume that $S = \{v_1, v_2, v_3, v_4\}$ is a pt-dominating set of G. However then, $\sigma_S(w_1) = \sigma_S(v_1)$, which is impossible. Thus, $\gamma_{pt}(G) = 5$. Finally, $VG) - \{v_5\}$ is a pt-dominating set of G and so $\Gamma_{pt}(G) \geq 7$. Since G is not locally irregular, it follows that $\Gamma_{pt}(G) \leq 7$. Hence, $\Gamma_{pt}(G) = 7$.

Next, suppose that $c \geq 8$. Let H be the graph of order $c + 1$ obtained from the graph G of order 8 (where $c = 7$) by adding $c - 7$ pendant edges at the vertex v_4. Then $\gamma_t(H) = 4$ and $\gamma_{pt}(H) = 5$. Since H is not locally irregular and $VH) - \{v_5\}$ is a pt-dominating set of H, it follows that $\Gamma_{pt}(H) = c$. ☐

By Theorem 1.3, every realizable triple a, b, c must satisfy either (i) $a \in \{2, 3, 4\}$ and $b \geq a + 1$ or (ii) $5 \leq a \leq b$. All known realizable triples a, b, c are stated next.

- ⋆ $a = 2$ and $3 \leq b \leq c$ by Proposition 3.1,
- ⋆ $a = 3$ and $(b, c) \notin \{(4, 4), (4, 5)\}$ by Theorems 3.1, 3.2, and 3.3,
- ⋆ $a = 4$ and $(b, c) \neq (5, 6)$ by Theorems 3.1, 3.2, and 3.4,
- ⋆ $a \geq 5$ and $2a - 1 \leq b \leq c$ by Theorems 3.1 and 3.2.

Furthermore, it can be shown that there are infinitely many realizable triples a, b, c for which $5 \leq a \leq b \leq 2a - 2$. The results obtained suggest the following problem.

Problem 3.1. *Does there exist a triple* a, b, c *with*

$$(a, b, c) \in \{(3, 4, 4), (3, 4, 5), (4, 5, 5), (4, 5, 6)\} \text{ or } 5 \leq a \leq b \leq 2a - 2$$

that is not realizable?

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