

Research Article

Levels in Baryiamonds

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Abstract

This study examines a statistic called levels in baryiamonds. Generating functions are found that count the total number of four different kinds of levels in baryiamonds according to the number of cells and the perimeter. As a consequence, by singularity analysis, the average number of each type of levels over all baryiamonds with either n cells or having perimeter n , as $n \rightarrow \infty$, is studied.

Keywords: baryiamond; generating function; level; triangular lattice.

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1. Introduction

A *lattice animal* is a set of polygonal cells connected by shared edges on a two-dimensional lattice. An important class of lattice animals is *polyominoes*, which consist of edge-connected squares on the square lattice and are widely studied (for instance, see [7, 8, 10, 12, 17–19]). A *bargraph* is a polyomino formed by adjacent vertical columns of unit squares, all resting on the x -axis. Bargraphs have received considerable attention in the last few decades (for instance, see [2–6, 16] and references therein). In particular, Blecher, Brennan, and Knopfmacher [2] studied the number of levels according to the perimeter. A *level* in a bargraph B is a pair of two adjacent columns in B with the same height (number of cells).

A *polyiamond* is a polyomino on the triangular lattice. They have been studied in several recent works. Yang and Wilson [20] proved that a polyiamond with n cells has minimum perimeter $\lceil \sqrt{6n} \rceil$ or $\lceil \sqrt{6n} \rceil + 1$, depending on whether n is even or odd (for more recent works, see [1, 13, 14]).

Our main result in this paper is related to the enumeration of a class of polyiamonds according to the number of levels. First, we recall the definition of triangular lattice polyiamonds. Let \mathcal{T} be the two-dimensional plane in which the angle between the x -axis and the y -axis is $\pi/3$, measured counterclockwise, and both axes are marked with integer points. We partition \mathcal{T} into equilateral triangles of side length one, each having a horizontal edge parallel to the x -axis. This plane is called the *triangular lattice*, and the triangles are referred to as *triangular cells*, or simply *cells*. For $j \in \mathbb{Z}$, the line $x = j$ (respectively, $y = j$) denotes the line parallel to the y -axis (respectively, the x -axis) that intersects the x -axis (respectively, the y -axis) at j . A *triangular bargraph* B , or simple *baryiamond*, is an animal in \mathcal{T} with m columns, where the j -th column of B has a_j cells between the lines $x = j - 1$ and $x = j$ and between the lines $y = 0$ and $y = \lceil aj/2 \rceil$. Depending on the orientation of the top cell of each column of B , we identify two types of columns as seen in Figure 1.1.

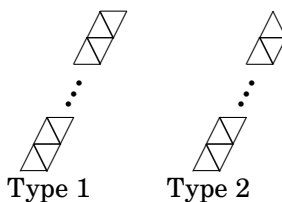


Figure 1.1: Two types of columns of baryiamonds.

In this paper, we study the statistic levels on *baryiamonds* formed by adjacent vertical columns of unit squares, all resting on the x -axis, see Figure 1.2. A *height* of a column in a baryiamond is the highest of the leftmost coordinate of the top cell of the column. For example, the heights of the columns of the baryiamond in Figure 1.2 are 5, 2, 5, 5, 1, 4, 4, and 1,

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Figure 1.2: A baryiamond.

from left to right. A *level* in a baryiamond is a pair of adjacent columns with the same height. We distinguish between different types of levels by considering the top cells of the columns. More precisely, a *ij*-level in a baryiamond is a pair of two adjacent columns AB with the same height k such that the line $y = k$ intersects the A by i cells and B by j cells. For instance, Figure 1.2 presents a baryiamond that has two levels, namely, 11-level (see the third and fourth columns) and 21-level (see the sixth and seventh columns). The *perimeter* of a polyiamond is the total number of its boundary edges. For instance, Figure 1.2 presents a baryiamond with perimeter 36.

Moreover, baryiamonds can be considered as a graphical presentation of a special class of compositions. A *composition* of n is an ordered collection $\pi_1 \pi_2 \cdots \pi_m$ of one or more positive integers whose sum is n (see [11]). Note that any baryiamond represents a composition of n such that $\pi_1, \dots, \pi_{m-1} \geq 2$ ($\pi_m \geq 1$). More precisely, we map π_i to a column with π_i cells. For example, Figure 1.2 presents the composition (10)4992871.

This paper investigates the statistic levels (11-levels, 12-levels, 21-level, and 22-levels) on *baryiamonds*. In the following sections, we present an enumeration of the baryiamonds according to the number of cells, the perimeter, and the number of *ij*-levels for $i, j \in \{1, 2\}$. In particular, have the following:

Statistic f	Average of f over all baryiamonds with perimeter n is asymptotic to
11-level	$\frac{(1 - \xi)\xi^6 n}{(1 + \xi)(\xi^4 + \xi^3 - 2\xi + 1)} \approx 0.0127956526n$
12-level	$\frac{-\sqrt{1 - \xi}(5\xi^4 - 2\xi^2 - 2\xi + 1)\xi^2 n}{(1 + \xi)\sqrt{\xi^4 + \xi^3 - 2\xi + 1}\sqrt{\xi^4 - 4\xi^3 - 6\xi^2 - 4\xi + 5}} \approx 0.0149076360n$
21-level	$\frac{(1 - \xi)\xi^5 n}{(1 + \xi)(\xi^4 + \xi^3 - 2\xi + 1)} \approx 0.0286787759n$
22-level	$\frac{(1 - \xi)(\xi^4 - 2\xi^3 - 2\xi^2 + 1)\xi^3 n}{2(1 + \xi)(\xi^4 + \xi^3 - 2\xi + 1)} \approx 0.0334123443n$
level	$\frac{(\xi - 1)(2\xi^3 + \xi^2 - 1)\xi^3 n}{(1 + \xi)(\xi^4 + \xi^3 - 2\xi + 1)} \approx 0.0897944089n$

During our computations, we observed an error in Theorem 1.1 of [15]. We show that the correct statement of Theorem 1.1 is as follows. The generating function for the number of baryiamonds counted by perimeter is

$$\frac{1 - 2p^2 - 2p^3 + p^4 - (1 + p)\sqrt{(1 - p)(1 - p - 2p^2 - 2p^3 + p^4 - p^5)}}{2p}.$$

Moreover, the number of baryiamonds with perimeter n is asymptotic to

$$\frac{(1 + \xi)\sqrt{\xi^4 + \xi^3 - 2\xi + 1}}{2\sqrt{\pi n^3}} \xi^{-n-1},$$

where $\xi = 0.44617150675\dots$ is the smallest positive root of the polynomial $p^5 - p^4 + 2p^3 + 2p^2 + p - 1$.

2. Main Results

In order to study the statistics ij -levels on baryiamonds, we define the generating function $F(p) = F(p; q_{11}, q_{12}, q_{21}, q_{22})$ for the number of baryiamonds according to the perimeter (marked by p), and the number of ij -levels (marked by q_{ij}) for all $i, j \in \{1, 2\}$. In order to find a formula for $F(x, p)$, we define $A_a(p) = A_a(p; q_{11}, q_{12}, q_{21}, q_{22})$ (respectively, $B_a(p) = B_a(p; q_{11}, q_{12}, q_{21}, q_{22})$) for the number of baryiamonds with first column of type 2 (respectively, type 1) and height a according to the number of cells (marked by x), the perimeter (marked by p), and the number of ij -levels (marked by q_{ij}) for all $i, j \in \{1, 2\}$.

To obtain the decomposition of $A_a(p)$ ($B_a(p)$), we classify the baryiamonds according to the type of their first column. In particular, we consider all baryiamonds whose first column is of type 2 (type 1) and has height a . Once the first column is fixed, by analyzing the possible configurations of the second column, when it exists, by considering both its height and its type, we obtain

$$\begin{aligned}
 A_1(p) &= p^3 \\
 A_a(p) &= p^{2a+1} + \sum_{j=1}^{a-1} p^{2a+1-2j} A_j(p) + q_{11}p^3 A_a(p) + \sum_{j \geq a+1} p^3 A_j(p) \\
 &\quad + \sum_{j=1}^{a-1} p^{2a+1-2j} B_j(p) + q_{12}p^3 B_a(p) + \sum_{j \geq a+1} p^3 B_j(p)
 \end{aligned}$$

and

$$\begin{aligned}
 B_a(p) &= p^{2a+2} + \sum_{j=1}^{a-1} p^{2a+2-2j} A_j(p) + q_{21}p^2 A_a(p) + \sum_{j \geq a+1} p^2 A_j(p) \\
 &\quad + \sum_{j=1}^{a-1} p^{2a+2-2j} B_j(p) + q_{22}p^2 B_a(p) + \sum_{j \geq a+1} p^2 B_j(p).
 \end{aligned}$$

Note that

$$(1 + p^2 - q_{22}p^2)B_1(p) = p^4 + (q_{21} - 1)p^5 + p^2 \sum_{j \geq 1} (A_j(p) + B_j(p)).$$

In order to solve these recurrence relations, we define $A(p; t) = \sum_{a \geq 1} A_a(p)t^a$ and $B(p; t) = \sum_{a \geq 1} B_a(p)t^a$. Then, by multiplying these recurrence relations by t^a and summing over all possible values of a , we obtain

$$\begin{aligned}
 A(p; t) &= \frac{p^3 t}{1 - p^2 t} + \frac{p^3 t}{1 - p^2 t} A(p; t) + q_{11}p^3 (A(p; t) - p^3 t) \\
 &\quad + \frac{p^3}{1 - t} (t^2 (A(p; 1) - p^3) - (A(p; t) - p^3 t)) \\
 &\quad + \frac{p^3 t}{1 - p^2 t} B(p; t) + q_{12}p^3 (B(p; t) - B_1(p)t) \\
 &\quad + \frac{p^3}{1 - t} (t^2 (B(p; 1) - B_1(p)) - (B(p; t) - B_1(p)t))
 \end{aligned} \tag{1}$$

$$\begin{aligned}
 B(p; t) &= \frac{p^4 t}{1 - p^2 t} + \frac{p^4 t}{1 - p^2 t} A(p; t) + q_{21}p^2 A(p; t) + \frac{p^2}{1 - t} (tA(p; 1) - A(p; t)) \\
 &\quad + \frac{p^4 t}{1 - p^2 t} B(p; t) + q_{12}p^2 B(p; t) + \frac{p^2}{1 - t} (tB(p; 1) - B(p; t)),
 \end{aligned} \tag{2}$$

and

$$B_1(p) = \frac{p^4 + (q_{21} - 1)p^5 + p^2 (A(p, 1) + B(p, 1))}{1 + p^2 - q_{22}p^2}. \tag{3}$$

By finding $A(p; t)$ from (1) and substituting the expressions of $A(p, t)$ and $B_1(p)$ (see (3)) into (2), we obtain

$$(1 + p^2 - q_{22}p^2)K(p, t)B(p; t) + a(p, t)(A(p; 1) + B(p; 1)) + b(p, t), \tag{4}$$

where

$$\begin{aligned} K(p, t) &= 1 + (1 - q_{22})p^2 + (1 - q_{11})p^3 + ((q_{11} - 1)(q_{22} - 1) - (q_{12} - 1)(q_{21} - 1) - q_{11})p^5 \\ &\quad + \left(-1 + (q_{22} - 1)p^2 + (q_{11} - 1)p^3 + (q_{22} - 2)p^4 - ((q_{11} - 1)(q_{22} - 1) - q_{12}(q_{12} - 1))p^5\right. \\ &\quad \left. - ((q_{11} - 1)(q_{22} - 2) - (q_{12} - 1)(q_{21} - 2))p^7\right)t \\ &\quad + \left(p^2 + p^3 + (1 - q_{22})p^4 - (q_{11} + q_{22} - q_{21})p^5 + (q_{11}(q_{22} - 1) - q_{21}(q_{12} - 1))p^7\right)t^2, \\ a(p, t) &= \left(-p^2 + (q_{22} - 1)p^4 + (q_{11} - 1)p^5 - ((q_{11} - 1)(q_{22} - 1) - (q_{12} - 1)(q_{21} - 1))p^7\right)t \\ &\quad + \left(p^4 + (1 - q_{21})p^5 + (1 - q_{22})p^6 + (q_{21}(q_{22} - q_{12}) - q_{11} - q_{22} + 2)p^7\right. \\ &\quad \left. + ((q_{11} - 1)(q_{22} - 1) - (q_{12} - 1)(q_{21} - 2))p^9\right)t^2, \\ b(p, t) &= \left(-p^4 + (1 - q_{21})p^5 + (q_{22} - 1)p^6 + (q_{21}q_{22} + q_{11} - q_{21} - q_{22})p^7\right. \\ &\quad \left. + (q_{21} - 1)(q_{11} - 1)p^8 - ((q_{11} - 1)(q_{22} - 1) - (q_{12} - 1)(q_{21} - 1))p^9\right. \\ &\quad \left. - (q_{21} - 1)((q_{11} - 1)(q_{22} - 1) - (q_{12} - 1)(q_{21} - 1))p^{10}\right)t \\ &\quad + \left(p^4 + q_{21}p^5 + (1 - q_{22})p^6 + (-q_{21}q_{22} - q_{11} + q_{21})p^7\right. \\ &\quad \left. - q_{21}(q_{11} - 1)p^8 + (q_{11}q_{22} - q_{12}q_{21} - q_{11} + q_{21})p^9\right. \\ &\quad \left. + ((q_{11} - 1)(q_{21}q_{22} + 2) - q_{21}(q_{12}q_{21} - q_{12} - q_{21} - 1 + 2q_{11}))p^{10}\right. \\ &\quad \left. - (q_{21} - 2)(q_{12} - 1)p^{11} + (q_{21} - 2)((q_{11} - 1)(q_{22} - 1) - (q_{12} - 1)(q_{21} - 1))p^{12}\right)t^2. \end{aligned}$$

Here, $K(p, t)$ plays the role of the kernel of the equation, that is, the coefficient multiplying the unknown generating function $B(p; t)$. The importance of this representation is that the equation becomes particularly useful when the kernel vanishes. Indeed, by choosing t such that $K(p, t) = 0$, the term involving $B(p; t)$ disappears, which allows us to eliminate the bulk contribution and obtain an explicit formula for the generating function $A(p; 1) + B(p; 1)$. Now, we choose $t = t_0$ as a power series in p , given by

$$\begin{aligned} t_0 = t_0(p) &= \frac{\alpha - \sqrt{\alpha^2 - 4p^2\beta}}{2p^2\gamma}, \\ \alpha &= ((q_{22} - 2)(q_{11} - 1) - (q_{21} - 2)(q_{12} - 1))p^7 + ((q_{22} - 1)(q_{11} - 1) - q_{21}(q_{12} - 1))p^5 \\ &\quad + (2 - q_{22})p^4 + (1 - q_{11})p^3 + (1 - q_{22})p^2, \\ \beta &= (((q_{22} - 1)(q_{11} - 1) - (q_{21} - 1)(q_{12} - 1))p^5 + (1 - q_{11})p^3 + (1 - q_{22})p^2 + 1) \\ &\quad \cdot ((q_{11}(q_{22} - 1) - q_{12}(q_{21} - 1))p^5 + (q_{21} - q_{22} - q_{11})p^3 + (1 - q_{22})p^2 + p + 1), \\ \gamma &= ((q_{11}(q_{22} - 1) - q_{12}(q_{21} - 1))p^5 - (q_{22} + q_{11} - q_{21})p^3 + (1 - q_{22})p^2 + p + 1), \end{aligned}$$

where t_0 is the root of the kernel equation satisfying $K(p, t_0) = 0$ and is analytic at $p = 0$. So, by taking $t = t_0$ and using the fact that $F(p) = A(p; 1) + B(p; 1)$, (4) gives the following result.

Theorem 2.1. *We have*

$$F(p) = F(p; q_{11}, q_{12}, q_{21}, q_{22}) = -\frac{b(p, t_0)}{a(p, t_0)}.$$

Theorem 2.1 with $q_{22} = q_{21} = q_{12} = q_{11} = 1$ gives

$$\begin{aligned} F(p; 1, 1, 1, 1) &= \frac{1 - 2p^2 - 2p^3 + p^4 - (1 + p)\sqrt{(1 - p)(1 - p - 2p^2 - 2p^3 + p^4 - p^5)}}{2p} \\ &= p^3 + p^4 + 2p^5 + 3p^6 + 6p^7 + 11p^8 + 21p^9 + 41p^{10} + 80p^{11} + 159p^{12} + 318p^{13} + \dots \end{aligned}$$

Note that this formula is a correction of the formula in [15, Theorem 1.1].

Let $\xi = 0.44617150675\dots$ be the smallest positive root of the polynomial $1 - p - 2p^2 - 2p^3 + p^4 - p^5$. The generating function for baryiamonds has a square-root singularity at $p = \xi$, which is the dominant singularity (closest to the origin). By the standard transfer theorem for algebraic singularities (see, for example, [9, Theorem VI.4]), a singularity of the form $(1 - p/\xi)^{1/2}$ leads to coefficients with asymptotic behavior of the form $cn^{-3/2}\xi^{-n}$. Applying this, we obtain that the number of baryiamonds with perimeter n is asymptotic to

$$\frac{(1 + \xi)\sqrt{\xi^4 + \xi^3 - 2\xi + 1}}{2\sqrt{\pi n^3}} \xi^{-n-1},$$

which again corrects the formula in [15, Theorem 1.1].

Now, we are ready to deal with all types of levels.

2.1. 11-Levels

By Theorem 2.1 with $q_{11} = q$ and $q_{12} = q_{21} = q_{22} = 1$, we obtain that the generating function $F(p; q, 1, 1, 1)$ for the number of baryiamonds according to the perimeter and 11-levels is given by

$$\frac{p^6(p^2 - p - 2)(q - 1) + (1 - 2q)p^5 + qp^4 + (q + 3)p^3 + 2p^2 - p - 1 + (p + 1)\sqrt{(p - 1)(p + 1)(p^3(q - 1) - 1)(p^7(p^2 + 1)(q - 1) - p^6 + 3(q - 1)p^5 - p^4 - (q + 3)p^3 - 3p^2 + 1)}}{2p(p^4q - p^4 - p - 1)}.$$

By differentiating the generating function $F(p; q, 1, 1, 1)$ at $q = 1$, and then by singularity analysis, we obtain that the total number of 11-levels over baryiamonds with perimeter n is asymptotic to

$$\frac{(1 - \xi)\xi^{5-n}}{2\sqrt{\xi^4 + \xi^3 - 2\xi + 1}\sqrt{\pi n}}.$$

Hence, the average of the statistic 11-levels over baryiamonds with perimeter n is asymptotic to

$$\frac{(1 - \xi)\xi^6 n}{(1 + \xi)(\xi^4 + \xi^3 - 2\xi + 1)}.$$

2.2. 12-Levels

By Theorem 2.1 with $q_{12} = q$ and $q_{11} = q_{21} = q_{22} = 1$, we obtain that the generating function $F(p; 1, q, 1, 1)$ for the number of baryiamonds according to the perimeter and 12-levels is given by

$$\frac{-p^6(2p^3 + p^2 - p - 1)(q - 1) - (q - 2)p^5 - p^4 - 4p^3 - 2p^2 + p + 1 - (p + 1)\sqrt{(p - 1)(p + 1)(p^{10}(p^2 - 1)(q - 1)^2 + 2p^9(q - 1) + p^6 + 2p^5(q - 1) + p^4 + 4p^3 + 3p^2 - 1)}}{2p(p^2(p^4 - p^2 + 1)(q - 1) + p + 1)}.$$

By differentiating the generating function $F(p; 1, q, 1, 1)$ at $q = 1$, and then by singularity analysis, we obtain that the total number of 12-levels over baryiamonds with perimeter n is asymptotic to

$$\frac{-\sqrt{1 - \xi}(5\xi^4 - 2\xi^2 - 2\xi + 1)\xi^{1-n}}{2\sqrt{\xi^4 - 4\xi^3 - 6\xi^2 - 4\xi + 5}\sqrt{\pi n}}.$$

Hence, the average of the statistic 12-levels over baryiamonds with perimeter n is asymptotic to

$$\frac{-\sqrt{1 - \xi}(5\xi^4 - 2\xi^2 - 2\xi + 1)\xi^2 n}{(1 + \xi)\sqrt{\xi^4 + \xi^3 - 2\xi + 1}\sqrt{\xi^4 - 4\xi^3 - 6\xi^2 - 4\xi + 5}}.$$

2.3. 21-Levels

By Theorem 2.1 with $q_{21} = q$ and $q_{11} = q_{12} = q_{22} = 1$, we obtain that the generating function $F(p; 1, 1, q, 1)$ for the number of baryiamonds according to the perimeter and 21-levels is given by

$$\frac{p^7(q - 1) - (4q - 5)p^5 - p^4 - (5 - q)p^3 - 2p^2 + p + 1 - (p^3(q - 1) + p + 1)\sqrt{(p - 1)(p + 1)(p^6 + 4p^5(q - 1) + p^4 + 4p^3 + 3p^2 - 1)}}{2p(p^2(q - 1) + p + 1)}.$$

By differentiating the generating function $F(p; 1, 1, q, 1)$ at $q = 1$, and then by singularity analysis, we obtain that the total number of 21-levels over baryiamonds with perimeter n is asymptotic to

$$\frac{(1 - \xi)\sqrt{\xi^4 + \xi^3 - 2\xi + 1}\xi^{4-n}}{2\sqrt{\pi n}}.$$

Hence, the average of the statistic 21-levels over baryiamonds with perimeter n is asymptotic to

$$\frac{(1 - \xi)\xi^5 n}{(1 + \xi)(\xi^4 + \xi^3 - 2\xi + 1)}.$$

2.4. 22-Levels

By Theorem 2.1 with $q_{22} = q$ and $q_{11} = q_{12} = q_{21} = 1$, we obtain that the generating function $F(p; 1, 1, 1, q)$ for the number of baryiamonds according to the perimeter and 22-levels is given by

$$\frac{(2 - 3q)p^5 + (2 - q)p^4 + (3 + q)p^3 + (1 + q)p^2 - p - 1 + (1 + p)\sqrt{(p^2 - 1)(4p^7(q - 1)^2 + p^6(q - 2)^2 - 8p^5(q - 1) - p^4(q^2 + 2q - 4) + 4p^3 + (2q + 1)p^2 - 1)}}{2p(p^2(q - 1) - p - 1)}.$$

By differentiating the generating function $F(p; 1, 1, 1, q)$ at $q = 1$, and then by singularity analysis, we obtain that the total number of 22-levels over baryiamonds with perimeter n is asymptotic to

$$\frac{(1 - \xi)(\xi^4 - 2\xi^3 - 2\xi^2 + 1)\xi^{2-n}}{4\sqrt{\xi^4 + \xi^3 - 2\xi + 1}\sqrt{\pi n}}.$$

Hence, the average of the statistic 22-levels over baryiamonds with perimeter n is asymptotic to

$$\frac{(1 - \xi)(\xi^4 - 2\xi^3 - 2\xi^2 + 1)\xi^3 n}{2(1 + \xi)(\xi^4 + \xi^3 - 2\xi + 1)}.$$

We end this paper by noting that the generating function $F(p; q, q, q, q)$ for the number of baryiamonds with perimeter n according to all the four kinds of levels is given

$$\frac{1 + p - (3q - 1)p^2 - (3q + 1)p^3 + (3q^2 - 2q - 2)p^4 + (3q^2 + 2q - 4)p^5 - (q - 1)(q^2 + q - 7)p^6 - (q - 1)(q^2 + 3q - 5)p^7 + (q - 1)(q^2 - 4q + 1)p^8 + (q - 1)(q^2 - 2q - 1)p^9 - (p^2q - p^2 - 1)(p^2q - p^2 - p - 1)\Delta}{2p(p + 1)(p^2(q - 1)(p^3q - p^2 - p - 1) + 1)},$$

where

$$\Delta = \sqrt{(p - 1)(p + 1)^2(p^7(q - 1)^2 + p^6(q - 3)(q - 1) - p^5(q^2 + 2q - 4) - p^4(q^2 + 2q - 2) + 2p^3 + 2p^2q + p - 1)}.$$

By differentiating the generating function $F(p; 1, 1, 1, q)$ at $q = 1$, and then by singularity analysis, we obtain that the total number of 22-levels over baryiamonds with perimeter n is asymptotic to

$$\frac{(\xi - 1)(2\xi^3 + \xi^2 - 1)\xi^{2-n}}{2\sqrt{\xi^4 + \xi^3 - 2\xi + 1}\sqrt{\pi n}}.$$

Hence, the average of the statistic levels (all different kinds of levels) over baryiamonds with perimeter n is asymptotic to

$$\frac{(\xi - 1)(2\xi^3 + \xi^2 - 1)\xi^3 n}{(1 + \xi)(\xi^4 + \xi^3 - 2\xi + 1)}.$$

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