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Research Article

Maximum Non-Self-Centrality Number of Trees With a Fixed Number of Pendant Vertices

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Abstract

The non-self-centrality number of an n-vertex, simple, and connected graph G is defined as $\mathcal{N}(G) = \sum_{u \neq v} |e(u) - e(v)|$, where $e(u)$ (respectively, $e(v)$) represents the eccentricity of the vertex u (respectively, v) in G, and the summation is taken over all unordered pairs of vertices of G. Let $\mathcal{T}(n,p)$ denote the class of n-vertex trees with p pendant vertices, where $4 \leq p \leq n-3$. Farooq and Mudusar in [*Discrete Appl. Math.* **311** (2022) 26–34] proposed the problem of finding trees with the maximum non-self-centrality number in the class $\mathcal{T}(n,p)$. In this paper, we solve this problem.

Keywords: non-self-centrality; eccentricity; pendant vertices.

2020 Mathematics Subject Classification: 05C12, 05C35.

1. Introduction

We consider finite, simple, and connected graphs throughout this paper. Let G be a graph with vertex set V_G and edge set E_G. The open neighborhood of a vertex $v \in V_G$, denoted by $N_G(v)$, is the set of all vertices adjacent to v. The elements of $N_G(v)$ are called the neighbors of v. The degree of $v \in V_G$, denoted by $d_G(v)$, is the number $|N_G(v)|$. If $d_G(v) = 1$, then v is called a pendant vertex. The distance between two vertices $u, v \in V_G$, denoted as $d_G(u,v)$, is defined as the length of a shortest path connecting u and v. The eccentricity of a vertex $v \in V_G$, denoted by $e_G(v)$, is defined by $e_G(v) = \max\{d_G(v,u) \mid u \in V_G\}$. A vertex $u \in V_G$ is called an eccentric vertex of a vertex $v \in V_G$ if $e_G(v) = d_G(v,u)$. The diameter of G, denoted by $d(G)$, is the maximum eccentricity among all vertices of G. The radius of G, denoted by $r(G)$, is the minimum eccentricity among all vertices of G. If $e_G(v) = r(G)$ then v is a central vertex in G. A diametric path in a graph is a shortest path between two vertices whose length is equal to the diameter of the graph. If there is no ambiguity, we write $d(v,u)$ and $e(v)$ instead of $d_G(v,u)$ and $e_G(v)$.

If the condition $e(v) = r(G)$ holds for every vertex $v \in V_G$, then the graph G is called self-centered; otherwise, it is called non-self-centered. We refer to [10] for the study of results on self-centered graphs. A tree is a connected acyclic graph. A caterpillar graph is a tree with the property that removing all pendant vertices results in a path. Throughout, let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The total irregularity $\mathrm{irr}_t(G)$ of G, introduced in [4], is defined as

$$\mathrm{irr}_t(G) = \sum_{v_i \neq v_j} |d_G(v_i) - d_G(v_j)|,$$

where the summation is taken across all unordered pairs of vertices in the graph G. Readers are referred to [2–5, 7, 11] for more results.

Recently, Xu et al. [13] introduced a new graph invariant to efficiently indicate the non-self-centrality of a graph. This invariant is called the non-self-centrality number (hereafter, abbreviated as NSC number) and is defined by:

$$\mathcal{N}(G) = \sum_{v_i \neq v_j} |e(v_i) - e(v_j)|,$$

where the summation is taken across all unordered pairs of vertices in the graph G. We cite [12, 14] for a study of relationship between the total irregularity of graphs and the non-self-centrality number. See [1, 6, 8, 9] for a detailed study of recent and past results on the non-self-centrality number of graphs.

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Let $e_1 > e_2 > \cdots > e_k$ be the distinct eccentricities of G and $l_1(G), l_2(G), ..., l_k(G)$ be their respective multiplicities. If there is no ambiguity in the context, we can denote $l_i(G)$ by l_i. The eccentricity sequence $\zeta(G) = \{e_1^{l_1}, e_2^{l_2}, \ldots, e_k^{l_k}\}$ of G is the set of eccentricities of G along with their multiplicities. Using the eccentricity sequence, we now redefine the NSC number as follows:

$$\mathcal{N}(G) \quad = \sum_{1\le i<j\le k} l_i l_j (e_i - e_j). \tag{1}$$

Xu et al. [13] characterized certain graphs with the smallest and largest NSC numbers. Specifically, they showed that the n-vertex star and n-vertex path have the smallest and largest NSC numbers, respectively, among all n-vertex trees, where $n \ge 3$. The authors proposed several interesting directions for future research, one of which is to find a tree with the largest NSC number among n-vertex trees with a fixed maximum degree Δ, where $3 < \Delta \le n-1$. Farooq and Mudusar [9] solved this problem and found the tree with the largest NSC number in the family of n-vertex trees with a fixed maximum degree Δ, where $3 < \Delta \le n-1$.

The motivation for this work comes from the paper by Farooq and Mudusar [9]. In their paper, the authors posed the problem of finding trees with the maximum non-self-centrality number in the class $\mathcal{T}(n,p)$, consisting of n-vertex trees with p pendant vertices. We solve this problem by finding trees with the largest NSC number in the class $\mathcal{T}(n,p)$ when $n \ge 7$, $4 \le p \le n-3$ and $|\zeta(T)| = k \ge 3$ for each $T \in \mathcal{T}(n,p)$. It is important to note that, for a tree $T \in \mathcal{T}(n,p)$, if $|\zeta(T)| = 2$ or $p = n-1$, then T is a star. In the case where $p = n-2$, each tree in $\mathcal{T}(n,p)$ is a double star. For $p = 3$, each tree has maximum degree 3, and the result for this case was obtained by Xu et al. [13]. Finally, when $p = 2$, the only tree in $\mathcal{T}(n,p)$ is an n-vertex path, for which the required result is trivial.

2. Trees With the Largest NSC Number

In this section, we find trees with the largest NSC number in $\mathcal{T}(n,p)$. We define a subclass $\mathcal{D}(n,p)$ of $\mathcal{T}(n,p)$ in which every $T \in \mathcal{D}(n,p)$ has an eccentricity sequence given below:

$$\zeta(T) = \left\{ e_1^{\lceil \frac{p+2}{2} \rceil}, e_2^2, \ldots, e_{k-2}^2, e_{k-1}^{\lfloor \frac{p+2}{2} \rfloor}, e_k^{l_k} \right\}. \tag{2}$$

Obviously, $d(T) = n-p+1$ for each $T \in \mathcal{D}(n,p)$. Also, because T is a tree, the graph induced by its central vertices is either K_1 or K_2. Thus, $l_k = 1$ when $d(T)$ is even and $l_k = 2$ when $d(T)$ is odd.

Let $\mathcal{P}(n,p)$ denote the class of n-vertex trees with p pendant vertices, such that the eccentricity sequence of any $B \in \mathcal{P}(n,p)$ is given as follows:

$$\zeta(B) = \left\{ e_1^{\lceil \frac{p+2}{2} \rceil}, e_2^2, \ldots, e_{s-1}^2, e_s^3, e_{s+1}^2, \ldots, e_{k-2}^2, e_{k-1}^{\lfloor \frac{p+2}{2} \rfloor}, e_k^{l_k} \right\}, \tag{3}$$

where $1 < s < k-1$ and $k \ge 4$. It is easy to see that $d(B) = n-p$ for each $B \in \mathcal{P}(n,p)$. As special cases, different trees in $\mathcal{D}(18,10)$ and $\mathcal{P}(18,10)$ are depicted in Figures 2.1 and 2.2, respectively, to help readers understand these classes.

The next lemma gives a general expression for the NSC number of a tree in $\mathcal{D}(n,p)$.

Lemma 2.1. *Let $T \in \mathcal{D}(n,p)$. Then the* NSC *number of T is derived as follows:*

$$\mathcal{N}(T) = \begin{cases} kl_k \left\lceil \dfrac{p+2}{2} \right\rceil + l_k(k^2-3k-1) + (p+2)(k-3)(k-2) \\ \quad + \dfrac{2(k-4)(k-3)(k-2)}{3} + \left\lceil \dfrac{p+2}{2} \right\rceil \left\lfloor \dfrac{p+2}{2} \right\rfloor (k-2) & \text{if } p \text{ is odd} \\ \\ (p+2)(k-3)(k-2) + \dfrac{2(k-4)(k-3)(k-2)}{3} \\ \quad + \dfrac{(p+2)^2(k-2)}{4} + \dfrac{kl_k(p+2k-4)}{2} & \text{if } p \text{ is even.} \end{cases} \tag{4}$$

Proof. We consider two cases, based on whether p is even or odd.

Case 1. p is odd.
Using equation (2) and formula (1), we obtain

$$\mathcal{N}(T) = kl_k \left\lceil \frac{p+2}{2} \right\rceil + l_k(k^2-3k-1) + (p+2)(k-3)(k-2) + \frac{2(k-4)(k-3)(k-2)}{3} + \left\lceil \frac{p+2}{2} \right\rceil \left\lfloor \frac{p+2}{2} \right\rfloor (k-2).$$

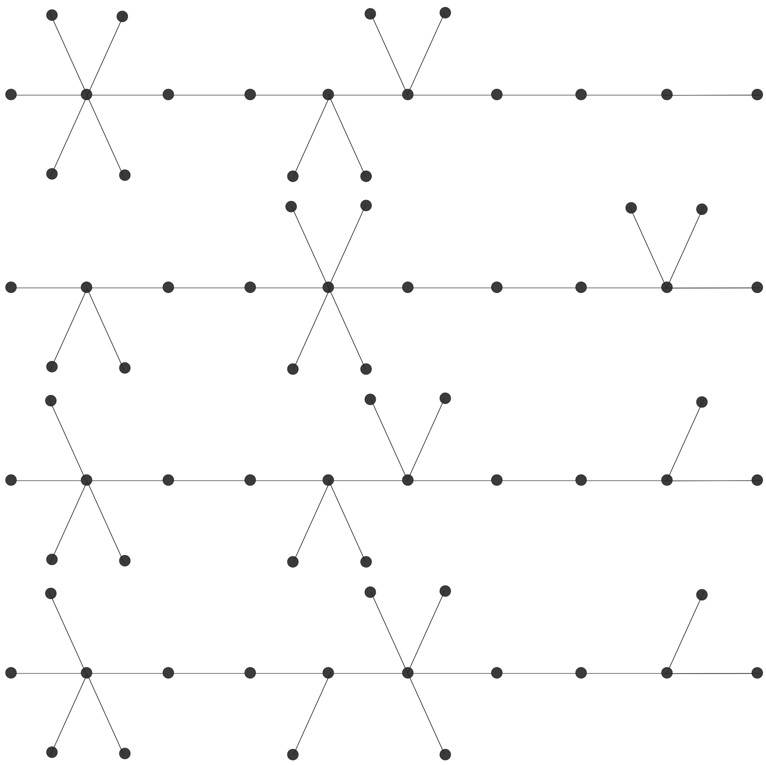

Figure 2.1: Different trees in $\mathcal{D}(18, 10)$.

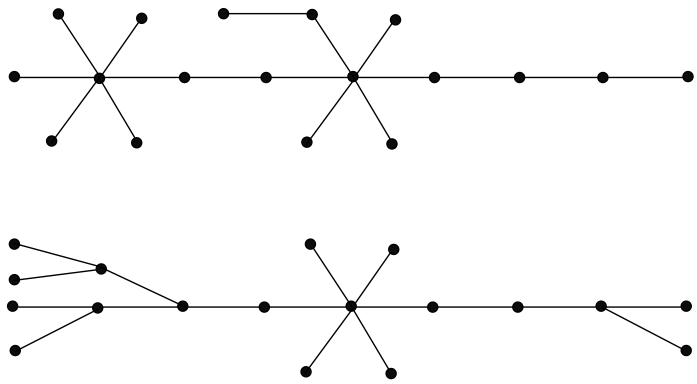

Figure 2.2: Different trees in $\mathcal{P}(18, 10)$.

Case 2. p is even.
In this case, $\lceil \frac{p+2}{2} \rceil = \lfloor \frac{p+2}{2} \rfloor = \frac{p+2}{2}$. From equation (2) and formula (1), we derive

$$\mathcal{N}(T) = (p+2)(k-3)(k-2) + \frac{2(k-4)(k-3)(k-2)}{3} + \frac{(p+2)^2(k-2)}{4} + \frac{kl_k(p+2k-4)}{2}.$$

The proof is complete. □

The following theorem shows that the NSC number of a tree in $\mathcal{D}(n, p)$ is greater than the NSC number of any tree in $\mathcal{P}(n, p)$.

Theorem 2.1. *For $T \in \mathcal{D}(n, p)$ and $B \in \mathcal{P}(n, p)$, it holds that $\mathcal{N}(T) > \mathcal{N}(B)$.*

Proof. We divide the proof into two cases based on the parity of $d(B)$. We will address the case when $d(B)$ is odd, and the case in which $d(B)$ is even can be shown similarly.

Case 1. $d(B)$ is odd.
We have $d(B) = n - p$ and $d(T) = n - p + 1$. If $d(B)$ is odd, then $d(T)$ is even. Equation (2) is rewritten as follows:

$$\zeta(T) \quad = \left\{ e_1^{\lceil \frac{p+2}{2} \rceil}, e_2^2, \ldots, e_s^2, \ldots, e_{k-2}^2, e_{k-1}^{\lfloor \frac{p+2}{2} \rfloor}, e_k^1 \right\}. \tag{5}$$

Also, if $|\zeta(T)| = k$, then $|\zeta(B)| = k - 1$. Equation (3) gives the following expression:

$$\zeta(B) \quad = \left\{ e_2^{\lceil \frac{p+2}{2} \rceil}, e_3^2, \ldots, e_{s-1}^2, e_s^3, e_{s+1}^2, \ldots, e_{k-2}^2, e_{k-1}^{\lfloor \frac{p+2}{2} \rfloor}, e_k^2 \right\}. \tag{6}$$

Here, e_i represents the eccentricity in T for each i.

Now, we consider two cases depending on whether p is even or odd.

Case 1.1. p is odd.
Equations (5) and (6), along with (1), give

$$\begin{aligned} \mathcal{N}(T) - \mathcal{N}(B) \quad &= \quad ((k-4)(k-3) - (k-s-1)(k-s-2)) + \left\lceil \frac{p+2}{2} \right\rceil \left\lfloor \frac{p+2}{2} \right\rfloor \\ &+ \quad 3 + ((k-4)(k-3) - (s-3)(s-2)) + k\left(\left\lceil \frac{p+2}{2} \right\rceil - \left\lfloor \frac{p+2}{2} \right\rfloor \right) \\ &+ \quad k(k-3) - 2(k-s) + (s-1)\left\lfloor \frac{p+2}{2} \right\rfloor - s\left\lceil \frac{p+2}{2} \right\rceil \\ &+ \quad 2(k-3)\left((p+2) - \left\lceil \frac{p+2}{2} \right\rceil \right) > 0. \end{aligned}$$

Case 1.2. p is even.
In this case, $\lceil \frac{p+2}{2} \rceil = \lfloor \frac{p+2}{2} \rfloor = \frac{p+2}{2}$. Equations (5) and (6), together with formula (1), give

$$\mathcal{N}(T) - \mathcal{N}(B) \quad = \quad p(k-3) + \frac{p(p+2)}{4} + 2(s-1) + ((k-4)(k-3) - (k-s-1)(k-s-2)) > 0.$$

Therefore, $\mathcal{N}(T) > \mathcal{N}(B)$.

Case 2. $d(B)$ is even.
In this case, the inequality $\mathcal{N}(T) > \mathcal{N}(B)$ can be proved using the same approach used in Case 1. □

It is evident that for any tree $T \in \mathcal{T}(n,p)$, there exists an n-vertex caterpillar H such that $\zeta(T) = \zeta(H)$ and vice versa. Clearly, the number of pendant vertices in a tree $T \in \mathcal{T}(n,p)$ is less than or equal to the number of pendant vertices in H, where H is the corresponding caterpillar of T satisfying $\zeta(T) = \zeta(H)$. Our objective is to construct a tree $T_1 \in \mathcal{T}(n,p)$ from a given tree $T \in \mathcal{T}(n,p)$ such that $\mathcal{N}(T) < \mathcal{N}(T_1)$.

Let $\mathcal{G}(n,\Delta)$ with $\Delta \geq 4$ be the class of n-vertex trees with fixed maximum degree Δ. A broom $B_{n,\Delta} \in \mathcal{G}(n,\Delta)$ is a tree obtained by attaching $\Delta - 1$ pendant vertices to an end-vertex of a path $P_{n-\Delta+1}$. An n-vertex 1-broom B in $\mathcal{G}(n,\Delta)$ is a tree whose $\Delta - 1$ pendant vertices are adjacent to one end-vertex of a path with $n - \Delta$ vertices and has a unique pendant vertex with eccentricity less than $d(B)$. The eccentricity sequence of 1-broom B is given by

$$\zeta(B) \quad = \quad \{e_1^\Delta, e_2^2, \ldots, e_{p-1}^2, e_p^3, e_{p+1}^2, \ldots, e_{k-1}^2, e_k^{l_k}\}$$

where $2 \leq p \leq k-1$. Also $d(B) = n - \Delta$. The class of n-vertex 1-brooms with maximum degree Δ is denoted by $\tilde{\mathcal{B}}_n^\Delta$. For further details on $\mathcal{G}(n,\Delta)$, $B_{n,\Delta}$, and $\tilde{\mathcal{B}}_n^\Delta$, we refer the reader to [9].

Lemma 2.2 (see [9]). *Let $T \in \mathcal{G}(n,\Delta)$ be such that $T \ncong B_{n,\Delta}$, $T \notin \tilde{\mathcal{B}}_n^\Delta$ and*

$$\zeta(T) \quad = \quad \{e_1^{l_1}, e_2^{l_2}, \ldots, e_p^{l_p}, \ldots, e_k^{l_k}\}, \tag{7}$$

where $2 \leq p \leq k-1$ and $l_1, l_p > 2$. Then there exists $T_1 \in \mathcal{G}(n,\Delta)$ with $d(T_1) = d(T) + 2$ and $\mathcal{N}(T_1) > \mathcal{N}(T)$.

Let us now establish some results that will help in proving the main result.

Lemma 2.3. *Let $T \in \mathcal{T}(n,p) \setminus \mathcal{D}(n,p)$ be such that $d(T) < n - p$ and the eccentricity sequence is given by*

$$\zeta(T) = \left\{ e_1^{l_1}, e_2^{l_2}, \ldots, e_s^{l_s}, \ldots, e_k^{l_k} \right\},$$

where $1 \leq s \leq k-1$ and $l_s \geq 3$. Then there exists $T_1 \in \mathcal{T}(n,p)$ such that $d(T_1) = d(T) + 2$ and $\mathcal{N}(T_1) > \mathcal{N}(T)$.

Proof. *We prove this by replacing* Δ *with* $\lceil \frac{p+2}{2} \rceil$ *and making simple modifications to the proof of Lemma 2.2.* □

Lemma 2.4. *Let* $T \in \mathcal{T}(n,p) \setminus (\mathcal{D}(n,p) \cup \mathcal{P}(n,p))$ *be such that* $d(T) \in \{n-p, n-p+1\}$ *and the eccentricity sequence is given by*

$$\zeta(T) = \left\{ e_1^{l_1}, e_2^{l_2}, \dots, e_s^{l_s}, \dots, e_k^{l_k} \right\},$$

where $1 \le s \le k-1$ *and* $l_s \ge 3$. *Then there exists a tree* $T_1 \in \mathcal{T}(n,p)$ *with* $\mathcal{N}(T_1) > \mathcal{N}(T)$ *such that*

(a) if $l_1(T) < \lceil \frac{p+2}{2} \rceil$ *then* $l_1(T_1) = l_1(T) + 1$*;*

(b) if $l_1(T) \ge \lceil \frac{p+2}{2} \rceil$ *then* $l_{k-1}(T_1) = l_{k-1}(T) + 1$*.*

Proof. Let H be an n-vertex caterpillar such that $\zeta(T) = \zeta(H)$. Consider a diametric u,v-path P in H. Let x be a pendant vertex distinct from both u and v, with $e(x) = e_s$ where $1 \le s \le k-1$. Additionally, let y be the neighbor of x on path P. Introduce another vertex w on P that is distinct from u, v, and y. Now construct a caterpillar H_1 such that

$$H_1 \cong (H - xy) + wx. \tag{8}$$

Selecting the vertex w with an appropriate eccentricity and determining the value of s, where $1 \le s \le k-1$, relies on the relationship between $l_1(T)$ and $\lceil \frac{p+2}{2} \rceil$. We consider two cases accordingly.

(a) $l_1(T) = l_1(H) < \lceil \frac{p+2}{2} \rceil$. In this case, we select the smallest value of s such that $1 < s \le k-1$ and $l_i(T) = l_i(H) = 2$ for $1 < i < s$. Therefore, the eccentricity sequence for H_1 is given by:

$$\zeta(H_1) \quad = \left\{ e_1^{l_1+1}, e_2^{2}, \dots, e_{s-1}^{2}, e_s^{l_s-1} \dots, e_k^{l_k} \right\}.$$

Using the formula (1), we get the following:

$$\mathcal{N}(H_1) = \mathcal{N}(H) + \sum_{i=s}^{k} l_i(s-1) - (l_1+1)(s-1). \tag{9}$$

We denote by p_i the number of pendant vertices in H with eccentricity e_i. Note that $p_1 = l_1$ and $p_k = 0$. Additionally, for $1 < i < k$, the following relationship holds:

$$l_i = p_i + 2. \tag{10}$$

Therefore by (10), we can write as follows:

$$\begin{aligned}\sum_{i=s}^{k} l_i(s-1) - (l_1+1)(s-1) &= \sum_{i=s}^{k-1}(p_i+2)(s-1) + l_k(s-1) - (p_1+1)(s-1)\\ &= (s-1)\left(\left(\sum_{i=s}^{k-1} p_i - p_1\right) + 2(k-s) + (l_k - 1)\right).\end{aligned}$$

As $l_1(H) < \lceil \frac{p+2}{2} \rceil$, it follows that

$$\sum_{i=s}^{k} p_i \ge \left\lceil \frac{p+2}{2} \right\rceil.$$

Consequently,

$$(s-1)\left(\sum_{i=s}^{k-1} p_i - p_1 + 2(k-s) + (l_k-1)\right) > 0.$$

Thus, $\mathcal{N}(H_1) > \mathcal{N}(H)$.

(b) $l_1(T) \ge \lceil \frac{p+2}{2} \rceil$. If $l_1(T) > \lceil \frac{p+2}{2} \rceil$, we set $s = 1$. In this situation, $l_{k-1} < \lfloor \frac{p+2}{2} \rfloor$, and consequently, we take w as the central vertex of H, that is, $e(w) = e_k$. Then the eccentricity sequence for H_1 is given as:

$$\zeta(H_1) \quad = \left\{ e_1^{l_1-1}, e_2^{l_2}, \dots, e_{k-1}^{l_{k-1}+1}, e_k^{l_k} \right\}.$$

Therefore, by applying formula (1), we obtain

$$\mathcal{N}(H_1) = \mathcal{N}(H) + k\left(\sum_{i=2}^{k-2} l_i - 1\right) + (k-2)\left(l_1 - l_{k-1} - l_k\right),$$

where

$$k\left(\sum_{i=2}^{k-2} l_i - 1\right) + (k-2)\left(l_1 - l_{k-1} - l_k\right) > 0.$$

Otherwise, if $l_1(T) = \lceil \frac{p+2}{2} \rceil$, then, as $T \notin \mathcal{D}(n,p)$, we must have $k \geq 4$ and $l_{k-1} < \lfloor \frac{p+2}{2} \rfloor$. Therefore, we choose the smallest s in $1 < s \leq k-2$ such that $l_s(T) \geq 3$. Subsequently, if $s \neq 2$, then $l_i(T) = 2$ for $2 \leq i < s$. In this situation, we take w as the central vertex of H, that is, $e(w) = e_k$. Then the eccentricity sequence for H_1 is given by:

$$\zeta(H_1) = \left\{ e_1^{l_1}, e_2^{l_2}, \ldots, e_s^{l_s - 1}, \ldots, e_{k-1}^{l_{k-1}+1}, e_k^{l_k} \right\}.$$

Therefore, by using formula (1), we obtain

$$\mathcal{N}(H_1) = \mathcal{N}(H) + (l_1 - l_{k-1} - 1)(k-s-1) + \sum_{i=2}^{s-1} l_i(k-s-1) + (l_s - l_k)(k-s-1).$$

Note that $l_1 = \lceil \frac{p+2}{2} \rceil$ and $l_{k-1} < \lfloor \frac{p+2}{2} \rfloor$. Consequently, $l_1 - l_{k-1} > 1$. In addition, given that $l_s \geq 3$ and $l_k \in \{1, 2\}$, we obtain $l_s - l_k > 0$. Moreover, as $1 < s \leq k-2$, we have $k - s - 1 > 0$. This gives

$$(l_1 - l_{k-1} - 1)(k-s-1) + \sum_{i=2}^{s-1} l_i(k-s-1) + (l_s - l_k)(k-s-1) > 0.$$

Hence $\mathcal{N}(H_1) > \mathcal{N}(H)$. □

Theorem 2.2. *Among the trees in $\mathcal{T}(n,p)$, the trees in the class $\mathcal{D}(n,p)$ have the maximum NSC number.*

Proof. Let $T \in \mathcal{T}(n,p)$ with the eccentricity sequence given by $\zeta(T) = \{e_1^{l_1}, e_2^{l_2}, \ldots, e_k^{l_k}\}$. Then $d(T) \leq n-p+1$. Based on the diameter of T, we divide the proof into two cases.

Case 1. $d(T) < n-p$.
We iteratively apply Lemma 2.3, starting from T, to obtain a tree $T_1 \in \mathcal{T}(n,p)$ such that $d(T_1) = d(T) + 2$ and $\mathcal{N}(T_1) > \mathcal{N}(T)$. We repeat this process for the trees with diameter less than $n-p$, eventually obtaining a tree in $\mathcal{T}(n,p)$ with diameter in $\{n-p, n-p+1\}$, after which we proceed to Case 2.

Case 2. $T \notin (\mathcal{D}(n,p) \cup \mathcal{P}(n,p))$ and $d(T) \in \{n-p, n-p+1\}$.
We consider two subcases.

Case 2.1. $l_1(T) < \lceil \frac{p+2}{2} \rceil$.
We iteratively apply Lemma 2.4(a), starting from T, to obtain a tree $T_1 \in \mathcal{T}(n,p)$ such that $l_1(T_1) = l_1(T)+1$ and $\mathcal{N}(T_1) > \mathcal{N}(T)$. We repeat this process until we reach a tree that either belongs to $\mathcal{D}(n,p)$, in which case we stop, or for which Lemma 2.4(b) applies, in which case we proceed to the second subcase.

Case 2.2. $l_1(T) \geq \lceil \frac{p+2}{2} \rceil$ and $l_{k-1}(T) < \lfloor \frac{p+2}{2} \rfloor$.
We iteratively apply Lemma 2.4(b) to obtain a tree $T_1 \in \mathcal{T}(n,p)$ such that $l_{k-1}(T_1) = l_{k-1}(T) + 1$ and $\mathcal{N}(T_1) > \mathcal{N}(T)$. We repeat this process until we reach a tree that belongs to $\mathcal{D}(n,p)$, in which case we stop.

The above procedure will give a finite sequence of trees $T_i \in \mathcal{T}(n,p)$, where $i = 1, \ldots, m$, that satisfies

$$\mathcal{N}(T) < \mathcal{N}(T_1) < \cdots < \mathcal{N}(T_m),$$

where $T_m \in \mathcal{D}(n,p) \cup \mathcal{P}(n,p)$. If $T_m \in \mathcal{D}(n,p)$ then we observe that $\mathcal{N}(T) < \mathcal{N}(T_m)$. If $T_m \in \mathcal{P}(n,p)$, we apply Theorem 2.1 to prove that $\mathcal{N}(T_m) < \mathcal{N}(D)$ for each $D \in \mathcal{D}(n,p)$, which gives $\mathcal{N}(T) < \mathcal{N}(D)$. This completes the proof. □

3. Conclusion

In this paper, we proved that trees in the class $\mathcal{D}(n,p)$ have the maximum NSC number among all trees in the class $\mathcal{T}(n,p)$, where $4 \leq p \leq n-3$. For future research, one can find trees in the class $\mathcal{T}(n,p)$ with minimum NSC number. In addition, one can consider the problem of finding extremal trees in the class of n-vertex trees with a given number of vertices having a fixed maximum degree.

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