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Review Article

Survey on Extremal Results and Bounds for Elliptic Sombor and Euler–Sombor Indices

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Abstract

Degree-based topological indices play a central role in chemical graph theory due to their good correlations with physicochemical properties of molecular structures. Among the recently introduced indices, the Sombor index and its variants have attracted significant attention, particularly from the extremal graph theory perspective. Motivated by the geometric interpretation of degree-based indices, the elliptic Sombor index (ESO index, for short) and the Euler–Sombor index (EUSO index, for short) have emerged as novel extensions that enrich the family of Sombor-type indices. In this survey paper, we provide a comprehensive overview of existing results concerning the ESO index and the EUSO index. Particularly, we collect known extremal results and bounds for ESO and EUSO indices. The paper also outlines several open problems.

Keywords: topological index; elliptic Sombor index; Euler–Sombor index; extremal problem; bound.

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1. Introduction

Chemical graph theory is a well-established interdisciplinary field that applies graph-theoretical concepts to model molecular structures, where vertices represent atoms and edges correspond to chemical bonds [47,73,74]. Over the past several decades, topological indices have become indispensable tools in this area, owing to their effectiveness in quantitative structure-property and structure-activity relationships (QSPR/QSAR); for instance, see [21,28,47,72].

Among various classes of topological indices, the class of degree-based topological indices [10,11,15,27,33,57] occupies a prominent place in chemical graph theory. These indices depend solely on vertex degrees and often admit elegant mathematical properties. Notable early examples include the Zagreb indices [40,41], whose development was thoroughly discussed in the survey paper [22]. The first Zagreb index and the second Zagreb index of a graph G with edge set $E(G)$ are denoted by $\mathcal{Z}_1(G)$ and $\mathcal{Z}_2(G)$, and can be defined as

$$\mathcal{Z}_1(G) = \sum_{uv\in E(G)} (d_u + d_v) \quad \text{and} \quad \mathcal{Z}_2(G) = \sum_{uv\in E(G)} d_u\, d_v,$$

respectively, where d_u and d_v represent the degrees of vertices u and v of G. The so-called forgotten topological index, which first appeared in the same paper where $\mathcal{Z}_1$ appeared, is another topological index whose definition we need in this paper. For a graph G, the forgotten topological index is denoted as $\mathcal{F}(G)$ and can be defined [30] as

$$\mathcal{F}(G) = \sum_{uv\in E(G)} (d_u^2 + d_v^2).$$

A systematic overview of many well-known degree-based indices and their theoretical aspects was provided by one of the present authors in [33].

In 2021, one of the present authors introduced a new topological index, namely the Sombor index, motivated by a geometric interpretation of vertex degrees [34]. The Sombor index of a graph G is defined as

$$\mathcal{SO}(G) = \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}.$$

This index rapidly gained popularity, leading to a substantial body of literature devoted to its mathematical properties, particularly the bounds and extremal results. A comprehensive review of extremal results and inequalities related to the Sombor index was presented in [50].

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Motivated by the success of the Sombor index, numerous variants have been proposed, extending the original concept and providing additional flexibility for capturing structural information. In the present paper, we are concerned with the following two such variants: the elliptic Sombor index [38] (ESO index, for short) and the Euler–Sombor index [35, 71] (EUSO index, for short); for a graph G, these indices are denoted by $\mathcal{ESO}(G)$ and $\mathcal{EUSO}(G)$, respectively, and are defined as

$$\mathcal{ESO}(G) = \sum_{uv \in E(G)} (d_u + d_v)\sqrt{d_u^2 + d_v^2} \quad \text{and} \quad \mathcal{EUSO}(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2 + d_u d_v}.$$

In [34], the ordered pair (d_u, d_v) of the degrees of endvertices of an edge uv in a graph was interpreted geometrically by representing it with a point $A = (a, b)$ in the Euclidean plane so that each edge uv corresponds to a coordinate point called its degree point. Reversing the coordinates yields $B = (b, a) = (d_v, d_u)$, which represents the dual degree point of the same edge uv. A more recent geometric framework proposed in [38] treats these two points in a fully symmetric manner: they are chosen as the foci of an ellipse passing through the origin (see Figure 1.1).

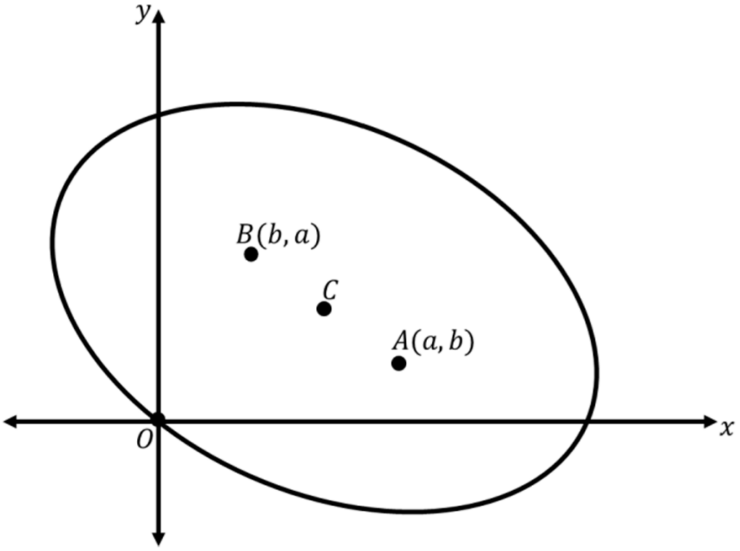

Figure 1.1: Ellipse whose foci are the degree-point A and the dual degree-point B of the edge uv of a graph. The point C is the center of the ellipse.

It was shown in [38] that the lengths of the semi-major and the semi-minor axes of the ellipse depicted in Figure 1.1 are

$$r_1 = \sqrt{d_u^2 + d_v^2} \quad \text{and} \quad r_2 = d_u + d_v.$$

The area of the ellipse under consideration is $\pi\sqrt{r_1 r_2}$, which motivates the introduction of the ESO index. In contrast, evaluating the perimeter of the ellipse is a substantially more difficult problem. Owing to its importance in astronomy, numerous approximation formulas have been developed; an overview is given in [38]. A classical approximation due to Leonhard Euler expresses the perimeter of the ellipse under discussion as $\pi\sqrt{r_1^2 + r_2^2}$, leading to the discovery of the EUSO index.

Chemical applicability of the ESO index and the EUSO index was examined in [17, 32, 38, 60] and [17, 67, 71, 75], respectively. Extremal problems related to these Sombor-type indices, where one seeks graphs maximizing or minimizing a given index under prescribed constraints over a certain class of graphs, have rapidly developed into a highly active research direction in chemical graph theory. However, the results concerning their extremal properties are scattered across the literature and have not yet been systematically collected. The primary aim of this survey is to fill this gap by collecting and organizing known extremal results related to the elliptic Sombor index and the Euler–Sombor index. We hope that this survey will serve as a useful reference for researchers working in chemical graph theory and will stimulate further investigations into these and related Sombor-type indices.

2. Elliptic Sombor Index

We divide this section into two subsections. The first subsection is concerned with extremal results on the elliptic Sombor index, while the second subsection is devoted to its bounds.

2.1. Extremal Results on Elliptic Sombor Index

In this subsection, we summarize the known extremal results concerning the ESO index. In particular, we collect results identifying graphs that maximize or minimize this index within specific graph classes or under prescribed structural constraints.

Theorem 2.1.1. *[38] The path graph P_n (respectively, complete graph K_n) uniquely minimizes (respectively, maximizes) the ESO index in the class of all n-order connected graphs for every $n \geq 3$. If the aforementioned class of graphs is restricted to trees, then the star graph S_n uniquely maximizes the considered index.*

We denote by $m_{i,j}$ the number of those edges of a graph whose one endvertex has degree i and the other one has degree j.

Theorem 2.1.2. *[29] In the class of all n-order chemical trees,*

(i) *only the graphs with degree set $\{1, 4\}$ maximize the ESO index for $n \geq 5$ provided that $n \equiv 2 \pmod 3$;*

(ii) *only the graphs with degree set $\{1, 3, 4\}$, such that $m_{1,3} = 2$ and $n_3 = 1$, maximize the ESO index for $n \geq 7$ provided that $n \equiv 1 \pmod 3$;*

(iii) *only the graphs with degree set $\{1, 2, 4\}$, such that $m_{1,2} = 1$ and $n_2 = 1$, maximize the ESO index for $n \geq 6$ provided that $n \equiv 0 \pmod 3$.*

A tree containing exactly one vertex of degree larger than 2 is known as a starlike tree. So, every starlike tree has at least three pendent paths. We denote by $S(n; r_1, r_2, \ldots, r_k)$ the n-order starlike tree with $k \geq 3$ pendent paths of lengths $r_1, r_2, \ldots, r_k$ such that $r_1 \geq r_2 \geq \ldots \geq r_k$, where $\sum_{i=1}^{k} r_i = n - 1$.

Theorem 2.1.3. *[70] If G is a graph with the maximum ESO index among all n-order trees with k pendent vertices, then $G \cong S(n; r_1, r_2, \ldots, r_k)$. Also, if $k \geq \lfloor n/2 \rfloor$ then $1 \leq r_i \leq 2$ and if $k < \lfloor n/2 \rfloor$ then $r_i \geq 2$.*

A tree of order at least three is said to be a caterpillar tree if removing its all pendent vertices results in a path graph. Note that an n-order tree with a diameter of 3 must have $n - 2$ pendent vertices. Hence, by Theorem 2.1.3, the starlike tree $S(n; 2, 1, \ldots, 1)$ (which is also a caterpillar tree) uniquely maximizes the ESO index in the class of all n-order trees with diameter 3 for $n \geq 6$ (see [70]). For the case $4 \leq d \leq n - 2$, we have the following result.

Theorem 2.1.4. *[70] A graph G maximizes the ESO index in the class of all n-order trees with diameter d, with $4 \leq d \leq n-2$, if and only if $G \cong S(n; r_1, r_2, 1, \ldots, 1)$ such that $r_1 \geq r_2 \geq 2$.*

Theorem 2.1.4 also follows from a result established in [13].

A set of pairwise non-adjacent edges in a graph G is called a matching of G. A matching with the maximum possible edges within a graph G is referred to as a maximum matching of G; the number of elements in such a matching is known as the matching number of G. A matching M in G is said to be a perfect matching if every vertex of G is incident to exactly one edge of M.

Theorem 2.1.5. *[70] A graph G maximizes the ESO index in the class of all n-order trees with a matching number β if and only if G is obtained from the star graph $S_{n-\beta+1}$ by subdividing its $\beta - 1$ edges.*

Since every even-order path graph has the perfect matching, the minimal version of Theorem 2.1.5 for $n = 2\beta$ follows from Theorem 2.1.1. This leads to the following problem.

Problem 2.1.1. *Characterize graphs that minimize the ESO index in the class of all n-order trees with a matching number β for $n > 2\beta$.*

In [20], it was proved that the graph minimizing the ESO index among all n-order trees with the maximum degree Δ must be a starlike tree for $3 \leq \Delta \leq n - 2$; in [1, 26], it was independently shown that such a tree must be isomorphic to $S(n; n - \Delta, 1, \ldots, 1)$.

Theorem 2.1.6. *[1, 26] A graph G minimizes the ESO index in the class of all n-order trees with the maximum degree Δ if and only if G is isomorphic to $S(n; n - \Delta, 1, \ldots, 1)$, where $3 \leq \Delta \leq n - 2$.*

We remark here that the maximal version of Theorem 2.1.6 is unknown [26]. Since $\mathcal{ESO}(S(n; n - \Delta, 1, \ldots, 1))$ is strictly increasing in $\Delta \geq 2$ (see [1]), the following result is a consequence of Theorem 2.1.6.

Corollary 2.1.1. *A graph G minimizes the ESO index in the class of all n-order trees with the maximum degree at least k if and only if G is isomorphic to $S(n; n-k, 1, \ldots, 1)$, where $3 \le k \le n-2$.*

Since the graph $S(n; n-\Delta, 1, \ldots, 1)$ has only one vertex of maximum degree Δ, this graph also minimizes the ESO index among all n-order trees with one vertex of maximum degree Δ for $3 \le \Delta \le n-2$ (see [1]). This observation leads to the following problem.

Problem 2.1.2. *Characterize graphs that minimize/maximize the ESO index among all fixed-order trees with a given number of vertices of maximum degree.*

We remark here that Problem 2.1.2 has recently been addressed in [3] for the case of chemical trees.

A vertex of degree at least 3 is known as a branching vertex. The problem of determining graphs minimizing the ESO index in the class of all fixed-order (chemical) trees with a given number of branching vertices was solved in [1]; the maximal version of this problem is still open.

A non-trivial path $P : v_1v_2 \ldots v_k$ in a graph G is said to be a segment if $d_G(v_1), d_G(v_2) \notin \{2\}$ and $d_G(v_i) = 2$ when $2 \le i \le k-1$; additionally, if $\max\{d_G(v_1), d_G(v_2)\} \ge 3$ and $\min\{d_G(v_1), d_G(v_2)\} = 1$, then P is called a pendent path of G. The problem of determining graphs minimizing/maximizing the ESO index in the class of all fixed-order trees with a given number of segments was solved in [1] (see also [64] for the case regarding the maximum ESO index); when this class is restricted to chemical trees, the corresponding problem regarding the maximum ESO index remains open.

Theorem 2.1.7. *[70] Among all n-order unicyclic graphs, the cycle graph (the unicyclic graph with maximum degree $n-1$) minimizes (maximizes, respectively) the ESO index for every $n \ge 4$.*

Theorem 2.1.8. *[53] Among all n-order biicyclic graphs, the graph with maximum degree $n-1$ and second maximum degree 3 uniquely maximizes the ESO index for every $n \ge 7$.*

Let $H_{n,\ell}$ denote the graph obtained from the n-order star graph S_n by inserting ℓ edge(s) between a fixed pendent vertex and ℓ other pendent vertex/vertices, where $n \ge \ell + 2 \ge 3$; if $\ell \ge 3$, then we denote by $H'_{n,\ell}$ the graph obtained from $H_{n,\ell-1}$ by inserting an edge between two vertices of degree two. Finally, denote by $H''_{n,5}$ the graph obtained from $H'_{n,4}$ by inserting an edge between a vertex of degree 2 and a vertex of degree 3.

Remark 2.1.1. *We note that $\mathcal{ESO}(H_{n,\ell}) > \mathcal{ESO}(H'_{n,\ell})$ for $n \ge \ell + 2$ with $\ell \in \{3, 4, 5\}$, and also $\mathcal{ESO}(H_{n,5}) > \mathcal{ESO}(H''_{n,5})$ for $n \ge 7$.*

Define a function Ψ on $[1, \infty) \times [1, \infty)$ as $\Psi(x_1, x_2) = (x_1 + x_2)\sqrt{x_1^2 + x_2^2}$. We note that Ψ and its derivative function Ψ_{x_i}, for $i \in \{1, 2\}$, with respect to x_i are strictly increasing. Also, we note that $\Psi(x_1 + k, x_2 - k) - \Psi(x_1, x_2) > 0$ for $x_1 \ge x_2 \ge k + 1 \ge 2$. Therefore, from Theorem 7 of [14] and Remark 2.1.1, the following result follows.

Corollary 2.1.2. *The graph $H_{n,\ell}$ uniquely maximizes the ESO index in the class of all n-order ℓ-cyclic graphs for every $n \ge \ell + 2$ and $\ell \in \{3, 4, 5\}$.*

The problem of characterizing the graphs maximizing the ESO index among all fixed-order 3-cyclic graphs was also independently addressed in [48]. Based on Corollary 2.1.2, we propose the following conjecture.

Conjecture 2.1.1. *In the class of all n-order ℓ-cyclic graphs, $H_{n,\ell}$ uniquely maximizes the ESO index for every $n \ge \ell + 2 \ge 8$.*

Recall that $m_{i,j}$ is the number of those edges in a graph whose one endvertex has degree i and the other has degree j.

Theorem 2.1.9. *[14] In the class of all n-order ℓ-cyclic graphs,*

- **(i)** *only the graphs with degree set $\{2, 3\}$, provided that $m_{2,2} = 0$, minimize the ESO index for $2(\ell - 1) < n < 5(\ell - 1)$ and $\ell \ge 3$;*
- **(ii)** *only the graphs with degree set $\{2, 3\}$, provided that $m_{2,2} = 0 = m_{3,3}$, minimize the ESO index for $n = 5(\ell - 1)$ and $\ell \ge 2$;*
- **(iii)** *only the graphs with degree set $\{2, 3\}$, provided that $m_{3,3} = 0$, minimize the ESO index for $n > 5(\ell - 1)$ and $\ell \ge 2$.*

The problem of characterizing the graphs minimizing the ESO index among all fixed-order 3-cyclic graphs was also independently addressed in [48].

Theorem 2.1.10. *[29] A graph G in the class of all n-order chemical graphs maximizes the ESO index if and only if G is 4-regular, where $n \ge 5$.*

Theorem 2.1.10 also follows from Proposition 4 of [17].

The problem of characterizing graphs minimizing the ESO index among all fixed-order unicyclic graphs of a given maximum degree was addressed in [26], where the maximal version of the aforementioned problem was proposed, along with a similar problem for bicyclic graphs.

Theorem 2.1.11. *[16] A graph G maximizes the ESO index among all those n-order connected graphs whose vertex/edge connectivity does not exceed k if and only if G is constructed from the $(n-1)$-order complete graph K_{n-1} by adding a new vertex and making it adjacent to exactly k vertices of K_{n-1}, where $n \geq 5$ and $1 \leq k \leq n-1$.*

Additional extremal results concerning the ESO index can be found in [52, 55]. Several extremal results with respect to the ESO index regarding hexagonal systems can be found in [23, 54].

We end this subsection with the following remarks:

1. None of the extremal results of [9] is applicable to the ESO index, and hence, characterizing graphs maximizing/minimizing the ESO index among all fixed-order triangular chains remains open.
2. Theorem 2.10(i) and Theorem 2.12(i) of [18] cover the ESO index.
3. Theorem 3.6 of [19] covers the ESO index.

2.2. Bounds on Elliptic Sombor Index

In this subsection, we summarize the known lower and upper bounds for the ESO index established in the literature, together with the graph classes and structural parameters under which these bounds are attained.

A bipartite graph with exactly two distinct vertex degrees, where each part of the bipartition consists of vertices of equal degree, is called a semiregular bipartite graph; for instance, see Figure 2.1.

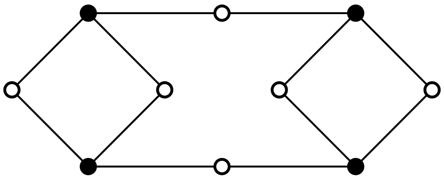

Figure 2.1: A semiregular bipartite graph of order 10.

Theorem 2.2.1. *[38] If G is a non-trivial connected graph, then*

$$\mathcal{F} < \mathcal{ESO}(G) \leq \sqrt{2}\,\mathcal{F} \tag{1}$$

and

$$\frac{2\mathcal{Z}_2 + \mathcal{F}}{\sqrt{2}} \leq \mathcal{ESO}(G) < 2\mathcal{Z}_2 + \mathcal{F}, \tag{2}$$

where the right equality in (1) or left equality in (2) holds if and only if G is regular. Also, it holds that

$$\mathcal{ESO}(G) \leq \sqrt{\mathcal{F}(2\mathcal{Z}_2 + \mathcal{F})}, \tag{3}$$

where the equality is attained if and only if G is either a regular graph or a bipartite semiregular graph.

Theorem 2.2.2. *[36] Let G be a graph with size m, maximum degree Δ and minimum degree δ. Then,*

$$2\mathcal{Z}_2 + \mathcal{F} - 2(2-\sqrt{2})\Delta^2 m \leq \mathcal{ESO}(G) \leq \frac{1}{\sqrt{2}}\left[2\mathcal{Z}_2 + \mathcal{F}\right] + \left[(\Delta+\delta)\sqrt{\Delta^2+\delta^2} - \frac{1}{\sqrt{2}}(\Delta+\delta)^2\right] m, \tag{4}$$

where the equality on the left-hand side holds if and only if G is regular, while the equality on the right-hand side holds if and only if G is either a regular graph or a bipartite semiregular graph. If, in addition, the graph G has order n. Then,

$$(n-2)\mathcal{Z}_1 + 4m^2 - 2(2-\sqrt{2})\Delta^2 m \leq \mathcal{ESO}(G) + \overline{\mathcal{ESO}}(G) \leq \frac{1}{\sqrt{2}}\left[(n-2)\mathcal{Z}_1 + 4m^2\right] + \left[(\Delta+\delta)\sqrt{\Delta^2+\delta^2} - \frac{1}{\sqrt{2}}(\Delta+\delta)^2\right] m,$$

where the conditions for equality cases are the same as in (4).

Theorem 2.2.3. *[36] If G is a graph of order n and size m, then*

$$\frac{1}{\sqrt{2}}\left[(n-2)\mathcal{Z}_1+4m^2\right] \le \mathcal{ESO}(G)+\overline{\mathcal{ESO}}(G) < (n-2)\mathcal{Z}_1+4m^2,$$

where the equality on the left-hand side holds if and only if the graph G is regular.

Theorem 2.2.4. *[17] Let G be a connected graph of maximum degree Δ.*

(i). *If G has $m \ge 2$ edges and p pendent vertices, then*

$$3\sqrt{5}\,p+8\sqrt{2}(m-p) \le \mathcal{ESO}(G) \le p(\Delta+1)\sqrt{\Delta^2+1}+2\sqrt{2}(m-p)\Delta^2,$$

where the right equality holds if and only if either G is Δ-regular or the degree set of G is $\{1,\Delta\}$, while the left equality holds if and only if G is either the cycle graph or the path graph.

(ii). *If G is a tree of order n, then*

$$\mathcal{ESO}(G) \le \Delta(\Delta+1)\sqrt{\Delta^2+1}+2\sqrt{2}(n-\Delta-1)\Delta^2,$$

with equality if and only if G is the star graph.

(iii). *If G has size m and minimum degree δ, then*

$$2\sqrt{2}\,m\delta^2 \le \mathcal{ESO}(G) \le 2\sqrt{2}\,m\Delta^2,$$

where either of the equalities holds if and only if G is regular.

Several bounds on the ESO index can be found in [55] as special cases of general results.

Upper and lower bounds for the ESO index involving the α-Sombor index, the symmetric division deg index, the Gutman–Milovanović index, the general sum-connectivity index, and the hyperbolicity constant, can be found in [32].

Several bounds on the ESO index in terms of some other graph invariants can be found in [51,52], whereas bounds on the ESO index for graph operations can be found in [31,62].

3. Euler–Sombor Index

This section is organized into two subsections. The first subsection presents the known extremal studies concerning the EUSO index, whereas the second subsection focuses on the various lower and upper bounds established for this index.

3.1. Extremal Results on Euler–Sombor Index

If we replace the text “ESO index” in Theorem 2.1.1 with “EUSO index”, then the revised statement of the mentioned theorem remains true:

Theorem 3.1.1. *[71] The path graph P_n (respectively, complete graph K_n) uniquely minimizes (respectively, maximizes) the EUSO index in the class of all n-order connected graphs for every $n \ge 3$. If the aforementioned class of graphs is restricted to trees, then the star graph S_n uniquely maximizes the considered index.*

The problem of characterizing graphs with the first six minimum values of the EUSO index among all fixed-order trees was addressed in [43] (see also [25], where such a problem for the case of the second minimum and second maximum values of the EUSO index was addressed).

Theorem 3.1.2. *[5, 71] In the class of all n-order chemical trees, with $n \ge 13$,*

(i) *only the graphs with degree set $\{1,4\}$ maximize the EUSO index for $n \equiv 2 \pmod 3$;*

(ii) *only the graphs with degree set $\{1,3,4\}$, such that $m_{3,4}=3$ and $n_3=1$, maximize the EUSO index for $n \equiv 1 \pmod 3$;*

(iii) *only the graphs with degree set $\{1,2,4\}$, such that $m_{2,4}=2$ and $n_2=1$, maximize the EUSO index for $n \equiv 0 \pmod 3$.*

Theorem 3.1.2 was also established in [43,46] independently.

The graphs that maximize the EUSO index among all fixed-order chemical ℓ-cyclic graphs, under certain constraints, may be obtained from Theorem 2 of [5].

The problem of characterizing graphs that maximize the EUSO index among all fixed-order chemical trees with a given number of pendent vertices was addressed in [63]; see also [75], where a similar problem was addressed.

If we replace the text “ESO index” in Theorem 2.1.10 with “EUSO index”, then the revised statement of the mentioned theorem remains true, which follows from Theorem 1 of [29]:

Theorem 3.1.3. *[29] A graph G in the class of all n-order chemical graphs maximizes the EUSO index if and only if G is 4-regular, where $n \geq 5$.*

Theorem 3.1.3 was also established in [43, 46] independently.

Theorem 3.1.4. *[20, 58] A graph G maximizes the EUSO index among all n-order trees with k pendent vertices if and only if G is isomorphic to $S(n; n-k, 1, \dots, 1)$, where $3 \leq k \leq n-2$.*

In Theorem 3.1.4, what if we restrict the considered trees to n-order chemical trees? This question was answered in [75]. Let $\mathcal{T}^*_{n,k}$ denote the class of all n-order trees of maximum degree 3 with k pendent vertices such that $m_{1,2} = m_{2,3} = k$, $m_{1,3} = 0$, $m_{2,2} = n - 3k + 2$ and $m_{3,3} = k - 3$, where $3 \leq k \leq \lfloor (n+2)/3 \rfloor$.

Theorem 3.1.5. *[65] A graph G minimizes the EUSO index in the class of all n-order trees with k pendent vertices if and only if $G \in \mathcal{T}^*_{n,k}$, where $3 \leq k \leq \lfloor (n+2)/3 \rfloor$.*

Since all the graphs of the class $\mathcal{T}^*_{n,k}$ are chemical trees, if we replace the text "trees" with "chemical trees" in Theorem 3.1.5 then the revised statement remains true; see also [75].

Based on Theorems 3.1.4 and 3.1.5, we propose the following problem:

Problem 3.1.1. *Characterize graphs that minimize the EUSO index in the class of all n-order trees with k pendent vertices for $\lfloor (n+2)/3 \rfloor < k \leq n-2$.*

The problem of determining graphs minimizing/maximizing the EUSO index in the class of all fixed-order trees with a given number of segments was solved in [64]; when this class is restricted to chemical trees, the corresponding problem regarding the maximum EUSO index remains open.

Theorem 3.1.6. *[13, 56, 58] A graph G maximizes the EUSO index in the class of all n-order trees with diameter d, with $3 \leq d \leq n-2$, if and only if $G \cong S(n; d-1, 1, \dots, 1)$.*

The problem of determining graphs that attain the second-maximum value of the EUSO index in the class of all fixed-order trees with a given diameter was also addressed in [56].

The following problem was proposed in [59]:

Problem 3.1.2. *[59] Among all fixed-order trees with a given diameter, characterize graphs that attain*

(i) *the first four minimum values of the EUSO index,*

(ii) *the third and fourth maximum values of the EUSO index.*

If we replace the text "ESO index" in Theorem 2.1.5 with "EUSO index", then the revised statement of the mentioned theorem remains true:

Theorem 3.1.7. *[20, 58] A graph G maximizes the EUSO index in the class of all n-order trees with a matching number β if and only if G is obtained from the star graph $S_{n-\beta+1}$ by subdividing its $\beta - 1$ edges.*

A dominating set of a graph G is a subset D of $V(G)$ such that every vertex $u \in V(G) \setminus D$ has at least one neighbor in D. A minimum dominating set of G is a dominating set of G with the smallest possible number of elements. The domination number of G is the cardinality of a minimum dominating set of G. The problem of characterizing graphs that minimize/maximize the EUSO index among all fixed-order trees with a given domination number was addressed in [25].

Theorem 3.1.8. *[20] If a graph G minimizes the EUSO index over the class of all n-order trees with the maximum degree Δ, then $G \cong S(n; r_1, r_2, \dots, r_k)$. Also, if $3 \leq \Delta \leq \lfloor (n-1)/2 \rfloor$ then $r_i \geq 2$ and if $\lceil (n-1)/2 \rceil \leq \Delta \leq n-2$ then $1 \leq r_i \leq 2$.*

A nontrivial connected graph that contains a vertex whose removal results in a tree is called a quasi-tree. A quasi-tree that is not a tree is called a non-trivial quasi-tree. The problem of characterizing graphs that minimize/maximize the EUSO index among all fixed-order non-trivial quasi-trees was addressed in [25].

Theorem 3.1.9. *[4, 14, 43, 66, 69] The cycle graph uniquely minimizes the EUSO index among all n-order unicyclic graphs for every $n \geq 4$.*

The problem of characterizing graphs with the first three maximum and minimum values of the EUSO index among all fixed-order unicyclic graphs was addressed in [69] (see also [43] for the case of the first three minimum values of the mentioned index).

Theorem 3.1.10. *[14] Among all n-order ℓ-cyclic graphs, $H_{n,\ell}$ uniquely maximizes the EUSO index for every $\ell \in \{1, 2, 3, 4, 5\}$ and $n \geq \ell + 2$ provided that $n \geq 4$.*

Some of the cases of Theorem 3.1.10 were also established in [4, 44, 66] independently.

Khanra and Das [43] asked to determine the extremal values and extremal graphs with respect to the EUSO index over the "class of all chemical unicyclic graphs"; such a class contains infinitely many graphs, and hence there is no graph maximizing the EUSO index (over the considered class), whereas the graph minimizing the EUSO index over this class is the 3-order cycle graph. If, in the aforementioned problem, we fix the order, then the solution to its minimal part follows from Theorem 3.1.9, whereas the solution to the maximal part of this modified problem follows from Theorem 2 of [5] and the proof of Corollary 3.12 in [71] (see also [2] where this problem was solved independently and where also the extremal graphs for small values of n can be found); see the next theorem.

Theorem 3.1.11. *[5, 71] In the class of all n-order chemical unicyclic graphs, with $n \geq 13$,*

(i) *only the graphs with degree set $\{1, 4\}$ maximize the EUSO index for $n \equiv 0 \pmod 3$;*

(ii) *only the graphs with degree set $\{1, 3, 4\}$, such that $m_{3,4} = 3$ and $n_3 = 1$, maximize the EUSO index for $n \equiv 2 \pmod 3$;*

(iii) *only the graphs with degree set $\{1, 2, 4\}$, such that $m_{2,4} = 2$ and $n_2 = 1$, maximize the EUSO index for $n \equiv 1 \pmod 3$.*

Theorem 3.1.12. *[6] A graph G maximizes the EUSO index among all n-order unicyclic graphs with k pendent vertices if and only if G is the graph obtained by attaching k pendent vertices to one vertex of the $(n-k)$-order cycle graph C_{n-k}, where $2 \leq k \leq n-3$.*

Let C_4 be the 4-order cycle graph and let $u, v \in V(C_4)$ be its non-adjacent vertices. Denote by $U_{n,d}$ the graph obtained from C_4 by attaching a pendent path of length $d-3$ at vertex u and attaching $n-d-1$ pendent vertices at vertex v.

Theorem 3.1.13. *[12, 56, 59] A graph G maximizes the EUSO index among all n-order unicyclic graphs with diameter d if and only if G is isomorphic to $U_{n,d}$, where $4 \leq d \leq n-2$.*

The problem of characterizing graphs that minimize the EUSO index among all fixed-order unicyclic graphs with a given diameter was addressed in [45]. Now, we state the following open problem posed in [59]:

Problem 3.1.3. *[59] Characterize graphs that attain the second, third and fourth minimum / maximum values of the EUSO index among all fixed-order unicyclic graphs with a given diameter.*

The girth of a graph G containing at least one cycle is the length of a smallest cycle in G. The problem of characterizing graphs with the extremum values of the EUSO index among all fixed-order unicyclic graphs of a given girth was addressed in [67] (for an extension concerning the minimum case, see [61]); see also [24], where the minimal version of this problem as well as the problem of characterizing graphs minimizing the EUSO index among all fixed-order connected graphs with a given girth was solved, thus providing a solution to an open problem posed in [43]. Here, we state a related problem.

Problem 3.1.4. *[24, 43] Characterize graphs maximizing the EUSO index among all fixed-order connected graphs with a given girth.*

Theorem 3.1.14. *[6] A graph G maximizes the EUSO index in the class of all n-order unicyclic graphs with a matching number $\beta \geq 2$ if and only if G is obtained from the starlike tree $S(n; \underbrace{2, \ldots, 2}_{\beta-2}, 1, \ldots, 1)$ by inserting an edge between two pendent vertices.*

Theorem 3.1.15. *[8] Let G be a graph that minimizes the EUSO index among all n-order unicyclic graphs of maximum degree Δ, where $3 \leq \Delta \leq n-2$. Then, the following statements hold.*

(i). *If $3 \leq \Delta \leq \lfloor (n+1)/2 \rfloor$, then G is obtained by attaching $\Delta - 2$ pendent paths of length at least 2 to a single vertex of a cycle.*

(ii). *If $\lceil (n+1)/2 \rceil \leq \Delta \leq n-2$, then G is the unique graph obtained from the 3-order cycle graph C_3 by attaching $2\Delta - n - 1$ pendent vertices and $n - \Delta - 1$ pendent paths of length 2 to a single vertex.*

Let $\mathcal{B}_n$ denote the class of all those n-order bicyclic graphs that have the degree set $\{2, 3\}$ such that $m_{3,3} = 1$.

Theorem 3.1.16. *[66] Only the member(s) of the class $\mathcal{B}_n$ minimize(s) the EUSO index among all n-order bicyclic graphs for every $n \geq 5$.*

Theorem 3.1.17. *[7] A graph G maximizes the EUSO index in the class of all n-order bicyclic graphs with a matching number $\beta \geq 3$ if and only if G is obtained from the starlike tree $S(n; \underbrace{2, \ldots, 2}_{\beta-3}, 1, \ldots, 1)$ by inserting two non-adjacent edges between pendent vertices.*

Let $\mathcal{G}_{n,\ell}$ be the class of all those n-order ℓ-cyclic graphs that have degree set $\{2, 3\}$ such that $m_{2,3} = 2,\ m_{2,2} = n - 2\ell + 1$ and $m_{3,3} = 3\ell - 4$.

Theorem 3.1.18. *[14, 49] In the class of all n-order ℓ-cyclic graphs, only the member(s) of the class $\mathcal{G}_{n,\ell}$ minimize(s) the EUSO index for every $\ell \geq 3$ and $n \geq 5(\ell - 1)$.*

Theorem 3.1.18 for $\ell = 3$ was also given in [44] independently.

A planar graph having a plane embedding such that all vertices lie on the boundary of the outer face is called an outerplanar graph. An outerplanar graph is said to be maximal if no edge can be inserted without affecting its outerplanar property. The problem of characterizing graphs maximizing/minimizing the EUSO index over the class of all fixed-order maximal outerplanar graphs was addressed in [42].

The following result (follows from Proposition 5 of [76] and was derived independently in [24]) provides a solution to a problem posed in [43]:

Theorem 3.1.19. *[24, 76] Let G be a graph that maximizes the EUSO index among all n-order connected graphs with k pendent vertices such that $n \geq 5$ and $0 \leq k \leq n - 2$.*

(i). *If $k = n - 2$, then G is formed by subdividing an edge of the star graph S_{n-1}.*

(ii). *If $k \leq n - 3$, then G is obtained from the complete graph K_{n-k} by attaching k pendent vertices to a single vertex of K_{n-k}.*

Here, we state a problem related to Theorem 3.1.19.

Problem 3.1.5. *[24, 43] Characterize graphs minimizing the EUSO index among all fixed-order connected graphs with a given number of pendent vertices.*

The following problem is a slightly modified version of the one posed in [43] (particularly, here we fix the order of the considered graphs).

Problem 3.1.6. *Characterize graphs minimizing / maximizing the EUSO index among all fixed-order connected graphs with a given (i) diameter or (ii) domination number.*

In Theorem 2.1.11, if we replace the text “ESO index” with “EUSO index”, then the revised statement remains true:

Theorem 3.1.20. *[16] A graph G maximizes the EUSO index among all those n-order connected graphs whose vertex / edge connectivity does not exceed k if and only if G is constructed from the $(n - 1)$-order complete graph K_{n-1} by adding a new vertex and making it adjacent to exactly k vertices of K_{n-1}, where $n \geq 5$ and $1 \leq k \leq n - 1$.*

Extremal results with respect to the EUSO index regarding hexagonal systems can be found in [43].

We conclude this subsection with the following remarks:

1. Theorem 2.10(i) and Theorem 2.12(i) of [18] cover the EUSO index
2. Corollary 3.2(ii) and Corollary 3.3(ii) of [9] cover the EUSO index.
3. None of the extremal results of [19] is applicable to the EUSO index, and hence, characterizing graphs maximizing/minimizing the EUSO index among all fixed-order pentagonal chains remains open.

3.2. Bounds on Euler–Sombor Index

This subsection is devoted to the known lower and upper bounds for the EUSO index reported in the literature. In particular, we summarize the existing inequalities involving this index, together with the corresponding extremal graphs and equality conditions whenever available.

Theorem 3.2.1. *[17] Let G be a connected graph of maximum degree Δ.*

(i). *If G has $m \geq 2$ edges and p pendent vertices, then*

$$\sqrt{7}\,p + 2\sqrt{3}(m-p) \leq \mathcal{EUSO}(G) \leq p\sqrt{\Delta^2+\Delta+1} + \sqrt{3}(m-p)\Delta,$$

where the right equality holds if and only if either G is Δ-regular or the degree set of G is $\{1,\Delta\}$, while the left equality holds if and only if G is either the cycle graph or the path graph.

(ii). *If G is a tree of order n, then*

$$\mathcal{EUSO}(G) \leq \Delta\sqrt{\Delta^2+\Delta+1} + \sqrt{3}(n-\Delta-1)\Delta,$$

with equality if and only if G is the star graph.

(iii). *[71] If G has size m and minimum degree δ, then*

$$\sqrt{3}\,m\delta \leq \mathcal{EUSO}(G) \leq \sqrt{3}\,m\Delta,$$

where either of the equalities holds if and only if G is regular.

Theorem 3.2.2. *[35] Let G be a non-trivial connected graph. Then,*

$$\mathcal{SO} < \mathcal{EUSO}(G) \leq \sqrt{\frac{3}{2}}\mathcal{SO},$$

where the equality on the right-hand side holds if and only if G is regular. (The right-hand side inequality was also established in [71].) In addition, if G has size $m \geq 2$, then

$$\mathcal{EUSO}(G) \geq \mathcal{SO} + (\sqrt{7}-\sqrt{5})m,$$

where the equality holds if and only if $m = 2$. Also, if $m \geq 3$ and the minimum degree of G is at least 2, then

$$\mathcal{EUSO}(G) \geq \mathcal{SO} + (\sqrt{12}-\sqrt{8})m,$$

where the equality holds if and only if G is the m-order cycle graph C_m. Additionally, if the minimum degree of G is δ, then

$$\mathcal{EUSO}(G) \geq \mathcal{SO} + (\sqrt{3}-\sqrt{2})\delta m,$$

where the equality holds if and only if G is δ-regular.

Theorem 3.2.3. *[71] Let G be a graph. Then*

$$\mathcal{EUSO}(G) \geq \sqrt{3}\,\mathcal{RR}(G)$$

with equality if and only if every component of G is regular, where

$$\mathcal{RR}(G) = \sum_{uv\in E}(d_u d_v)^{1/2}$$

is the reciprocal Randić index [37]. Also, if the size of G is m, then

$$\mathcal{EUSO}(G) \leq \sqrt{m\,(\mathcal{F}+\mathcal{Z}_2)}$$

with equality if and only if $d_u^2 + d_v^2 + d_u d_v$ is a fixed positive integer for every edge $uv \in E(G)$.

Additional bounds on the EUSO index can be found in the recent paper [68]. Also, some bounds on the EUSO index for graphs under some operations can be found in [39].

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