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Research Article

Cozonality in Plane Graphs of Maximum Degree Four

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Abstract

Let G be a plane graph. A cozonal labeling of G is a face labeling using the labels $\{1,2\}$ such that for each vertex v, the labels of regions having v on their boundary sum to $0 \pmod 3$. Cozonal labelings in connected graphs with maximum degree three have been characterized. In this paper, we define a nonconsecutive walk as one where no two consecutive edges along the walk are on the boundary of the same region, and where every internal vertex of the walk is incident with exactly four regions. By considering these walks, cozonal labelings of graphs with maximum degree four are characterized.

Keywords: cozonal graph labeling; plane graph; four-color problem.

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1. Introduction

A graph is *planar* if it can be drawn in the plane (or on the sphere) in such a way that no two edges cross. Any such embedding is called a *plane graph*. Let G be a plane graph with vertex, edge and region sets $V(G), E(G)$ and $F(G)$, respectively. Motivated by an interest in the Four-Color Theorem, the Australian physicist Cooroo Egan devised (in 2014) a vertex labeling of plane graphs known as a zonal labeling [5]. A *zonal labeling* of a plane graph G is a labeling $\ell : V(G) \to \{1,2\}$ such that for every region $R \in F(G)$ with boundary B_R,

$$\sum_{v \in V(B_R)} \ell(v) = 0 \pmod 3.$$

If G has a zonal labeling, we say that G is *zonal*. Furthermore, a planar graph is zonal if at least one of its embeddings admits a zonal labeling. While the original motivation for zonal labelings is deeply rooted in the Four-Color Theorem, there are many interesting results on zonal labelings that are not directly connected to this famous theorem [1,4,6].

In this paper, we study a related labeling known as a *cozonal labeling*. Given a vertex v, let X_v be the set of regions having v on their boundary. A *cozonal labeling* of a plane graph G is a labeling $\ell : F(G) \to \{1,2\}$ such that for all $v \in V(G)$,

$$\sum_{R \in X_v} \ell^*(R) = 0 \pmod 3.$$

We call the value $\sum_{R\in X_v} \ell(R)$ the *label of* v, and denote this by $\ell(v)$. We say G is *cozonal* if G has a cozonal labeling. Furthermore, a planar graph is cozonal if at least one of its embeddings admits a cozonal labeling. Cozonal labelings were introduced in [2] to provide a new perspective on zonal labelings. In [3], it is proven that a plane graph G is zonal if and only if its dual G^* is cozonal. In [3], the cozonal plane graphs of maximum degree 3 were characterized. Here, we extend this result to cozonal plane graphs of maximum degree 4.

2. Terminology and Previous Results

Many of our results are based on connectivity and regularity. A *cut vertex* is a vertex whose deletion disconnects a component of a graph. As the graphs here may be multigraphs containing parallel edges and loops, we should note that a vertex incident with a loop is considered a cut vertex (unless that vertex and loop make up the entire component). A connected graph with at least three vertices and without a cut vertex is 2-*connected*. A *bridge* is an edge whose deletion disconnects a component of a graph. Lastly, a graph is k-*regular* if every vertex has degree k. For further terminology not defined here, see [7].

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Our proofs utilize several previous results from [3], which we restate below. We use $\deg^*(v)$ to denote the number of regions having v on their boundary and $\Delta(G)$ to denote the maximum degree of a vertex in G.

Proposition 2.1. *Let G be a nonempty connected plane graph. Then for $v \in V(G)$,*

1. $\deg^*(v) \leq \deg(v)$.

2. *v is a cut vertex if and only if* $\deg^*(v) < \deg(v)$.

Theorem 2.1. *Let G be a plane graph with set B of bridges. Then, G is cozonal if and only if each component of $G - B$ is cozonal.*

Next, the plane graphs with maximum degree 3 are characterized in [3].

Theorem 2.2. *Let G be a connected graph with $\Delta(G) \leq 3$. Then, G is cozonal if and only if one of the following is true:*

- *G is a cycle C_n, with $n \geq 1$.*

- *The graph formed by deleting every bridge of G is 2-regular.*

- *G is a cubic map (that is, a bridgeless 3-regular plane graph).*

In the 2-connected case, this is summarized as follows:

Corollary 2.1. *Let G be a 2-connected plane graph with $\Delta(G) \leq 3$. Then, G is cozonal if and only if G is regular.*

We now proceed to extend these results to a characterization of cozonal plane graphs with maximum degree 4. This analysis uses the following result from [3], which is a dualization of a result in [4].

Theorem 2.3. *If G is a 2-connected Eulerian plane graph, then G is cozonal.*

This has two immediate corollaries:

Corollary 2.2. *If G is a 2-connected 4-regular plane graph, then G is cozonal.*

Corollary 2.3. *Let G be a 2-connected plane graph with $\Delta(G) = 4$. If G has no vertices of degree 3, then G is cozonal.*

Thus, the main obstacle to cozonality in 2-connected plane graphs with $\Delta(G) = 4$ lies in the vertices of degree 3. We give a generalization which allows us to characterize cozonal bridgeless plane graphs with $\Delta(G) = 4$.

3. Nonconsecutive Walks

Let G be a bridgeless plane graph. A *nonconsecutive walk* W is a walk in G where no two consecutive edges along the walk are on the boundary of the same region, and for every internal vertex $v \in W$ (that is, any vertex other than those on the ends of the walk), $\deg^*_G(v) = 4$. When $\Delta(G) = 4$, this means that for every internal vertex $v \in W$, v is a non-cut vertex of degree 4. While we will primarily focus on the case where $\Delta(G) = 4$, these walks can still give some insight on graphs with higher maximum degrees, and therefore we will keep our definition appropriately general.

Nonconsecutive walks beginning with a vertex incident with exactly 3 regions are of particular interest. If $\Delta(G) = 4$, then this includes all vertices of degree 3 (which are necessarily non-cut vertices, since G is bridgeless) and all cut vertices of degree 4.

We begin in Figure 3.1 with an example of a 2-connected graph G_1 and two nonconsecutive walks beginning and ending with vertices of degree 3. We will see that the intersecting nature of these walks is sufficient to justify that G_1 is not cozonal.

Next, we consider an example of a graph G_2 having a cut vertex of degree 4, with a nonconsecutive walk beginning and ending at that vertex. The only other nonconsecutive walk starting and ending at a vertex u with $\deg^*(u) = 3$ is symmetric to the one shown. These walks do not contain any of the features that contradict cozonality, so G_2 is cozonal. A cozonal labeling is presented along with the nonconsecutive walk in in Figure 3.2.

We now establish several properties of certain nonconsecutive walks belonging to a cozonal bridgeless plane graph. The first property provides motivation for all the other properties we devise.

Proposition 3.1. *Let G be a cozonal bridgeless plane graph, ℓ be a cozonal labeling of G, and W be a nonconsecutive walk beginning with a vertex v with $\deg^*(v) = 3$. Then each edge of W is incident with two regions of the same label under ℓ, and these labels alternate along W.*

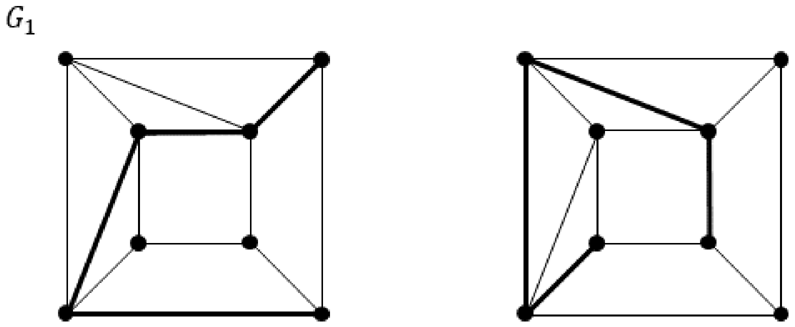

Figure 3.1: A 2-connected graph G_1 with two nonconsecutive walks beginning at vertices of degree three that cross at internal vertices.

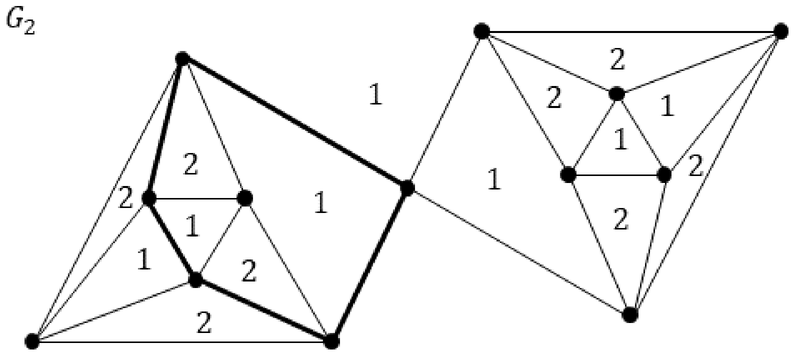

Figure 3.2: A connected graph G_2 with a nonconsecutive walk beginning and ending at the cut vertex of degree four, along with a cozonal labeling.

Proof. First, note that all regions incident with v have the same label. Therefore, the two regions incident with any edge incident with v have the same label.

Now, assume that edges e_1 through e_k of W have the property that each edge is incident with two regions of the same label, and these labels alternate. Suppose the regions incident with e_k each have label a. Consider e_{k+1}. Edges e_k and e_{k+1} are joined by a vertex v_{k+1}, which must have $\deg^*(v_{k+1}) = 4$. Since e_k, e_{k+1} are not on the boundary of any shared regions, we see that two of the regions incident with v_{k+1} are those incident with e_k, and the other two are incident with e_{k+1}. Denote these four regions by R_0, R_1, R_2, R_3, with e_k incident with R_0, R_1 and e_{k+1} incident with R_2, R_3. The label of v_{k+1} is equal to $\sum_{i=0}^{3} \ell(R_i) = 2a + \ell(R_2) + \ell(R_3)$. The only way this sum can be 0 in $\mathbb{Z}_3$ is if $\ell(R_2) = \ell(R_3) = 2a$. Therefore, we see that the labels incident with an edge alternate. □

This strong pattern in nonconsecutive walks beginning with a vertex v with $\deg^*(v) = 3$ leads to the following properties.

Proposition 3.2. *Let G be a cozonal bridgeless plane graph, $\mathcal{W}$ be the set of all nonconsecutive walks in G containing a vertex incident with exactly three regions, $E_{\mathcal{W}}$ be the set of all edges belonging to some walk in $\mathcal{W}$, and $u \in V(G)$ be a non-cut vertex of degree 4. Then, exactly zero or two edges incident with u belong to $E_{\mathcal{W}}$.*

Proof. Let G have a cozonal labeling ℓ. By Proposition 3, for all $W \in \mathcal{W}$, each edge of W is incident with two regions of the same label. Therefore, each edge in $E_{\mathcal{W}}$ is incident with two regions having the same label. Let u be a non-cut vertex with $\deg(u) = 4$ incident with four regions R_0, R_1, R_2, R_3. If three edges incident with u belong to $E_{\mathcal{W}}$, then by transitivity the four regions incident with u have the same label, and $\ell(u) = 4\ell(R_0) \neq 0$. This contradicts the cozonality of G, and therefore at most two edges incident with u belong to $E_{\mathcal{W}}$.

To show that we cannot have exactly one edge, suppose there is a non-cut vertex u having $\deg(u) = 4$ and exactly one edge e incident with u belongs to $E_{\mathcal{W}}$. Then there is some nonconsecutive walk $W \in \mathcal{W}$ ending in u, and thus whose last edge is e. However, this walk W can be extended to a walk W' by adding the edge e' incident with u that is not consecutive to e (that is, the edge which is not on the boundary of any shared regions with e), and thus both e and e' belong to $E_{\mathcal{W}}$. Therefore, we cannot have exactly one edge incident with u that belongs to $E_{\mathcal{W}}$. □

Note that we are restricted specifically to non-cut vertices of degree 4, and not general vertices incident with exactly four regions. The situation for vertices incident with exactly four regions is more complex and requires different techniques to analyze.

Proposition 3.3. *Let G be a cozonal bridgeless plane graph, $\mathcal{W}$ be the set of all nonconsecutive walks in G containing a vertex incident with exactly three regions, $E_{\mathcal{W}}$ be the set of all edges belonging to some walk in $\mathcal{W}$, and $u \in V(G)$ with $\deg(u) = 2$. Then, u is not incident with any edges of $E_{\mathcal{W}}$.*

Proof. Let G have a cozonal labeling ℓ and assume to the contrary that some walk $W \in \mathcal{W}$ contains an edge e incident with a vertex v of degree 2. Then, e is incident with two regions having the same label. These are the only two regions incident with v and therefore, $\ell(v) \neq 0$. This contradicts the cozonality of G. Thus, W has no vertex of degree 2. □

We have shown two properties of nonconsecutive walks in cozonal bridgeless plane graphs. In fact, these properties characterize all bridgeless cozonal plane graphs with maximum degree 4.

4. Cozonal Bridgeless Plane Graphs G with $\Delta(G) = 4$

To simplify the forthcoming proof, we first prove a lemma involving labelings of plane graphs with $\Delta(G) = 4$ in which all components are Eulerian. This labeling will not be a strictly cozonal labeling, but it will be very similar to one.

Lemma 4.1. *Let G be a plane graph with $\Delta(G) \leq 4$ such that each component is Eulerian with no isolated vertices. Then, G has a labeling $\ell : F(G) \to \{1, 2\}$ such that for all $v \in V(G)$,*

- *each edge incident with v is on the boundary of two regions having different labels, and*
- *if v is not a cut vertex of a component of G, then $\ell(v) = \sum_{R \in X_v} \ell(R) = 0 \pmod 3$.*

Proof. The first condition is equivalent to saying that ℓ is a region 2-coloring of G. Since each component of G is Eulerian, the dual of each component is bipartite, and therefore each component of G has a 2-coloring of its regions. As each component of G is region 2-colorable, a straightforward induction argument (on the number of components) shows that G is region 2-colorable. Now, let $\ell : V(G) \to \{1, 2\}$ be a region 2-coloring of G. This meets the first condition of the labeling.

To see that this also meets the second condition, note that if v is not a cut vertex, it is incident with an even number of regions equal to its degree. Furthermore, the region labels alternate about v. Thus, $\deg(v)/2$ regions have the label 1 and $\deg(v)/2$ regions have the label 2, and thus $\ell(v) = \deg(v)/2 + 2(\deg(v)/2) = 3(\deg(v)/2) = 0$. We have shown that for each vertex v that is not a cut vertex of a component, $\ell(v) = 0$ and both conditions on our labeling have been met. □

We are now ready to prove our primary result on bridgeless plane graphs with $\Delta(G) = 4$.

Theorem 4.1. *Let G be a bridgeless plane graph with $\Delta(G) = 4$, $\mathcal{W}$ be the set of all nonconsecutive walks in G containing a vertex incident with exactly three regions, and $E_{\mathcal{W}}$ consist of all edges belonging to a walk in $\mathcal{W}$. Then, G is cozonal if and only if every non-cut vertex v with $\deg(v) = 4$ is incident with exactly zero or two edges of $E_{\mathcal{W}}$ and no vertex w with $\deg(w) = 2$ is incident with an edge of $E_{\mathcal{W}}$.*

Proof. First, let every non-cut vertex v with $\deg(v) = 4$ be incident with exactly 0 or 2 edges of $E_{\mathcal{W}}$ and no vertex w with $\deg(w) = 2$ be incident with an edge of $E_{\mathcal{W}}$. Form G' by deleting all vertices incident with exactly 3 regions and all edges in $E_{\mathcal{W}}$. Observe that each vertex in G' was either a vertex of degree 4 in G that had 0 or 2 incident edges removed, or otherwise was a vertex of degree 2 in G that had no incident edges removed. Therefore, each component of G' is Eulerian. Construct the labeling $\ell' : F(G') \to \{1, 2\}$ from Lemma 4.1. This labeling has the following two properties: For all $x' \in V(G')$, each edge incident with x' is on the boundary of regions having different labels, and if x' is not a cut vertex of a component of G', then $\ell'(x') = 0$. Form $\ell : F(G) \to \{1, 2\}$, where $\ell(R)$ is given by $\ell'(R')$ for the unique region of G' where $R \subset R'$. We now examine the induced label of a vertex $x \in V(G)$.

Case 1: $\deg^*(x) = \deg(x) = 2$: Then, x does not belong to a nonconsecutive walk, the edges incident with x in G are the same as those in G', and the sum of labels of regions incident with x under ℓ is the same as that under ℓ'. Since x is not of degree 4 and therefore not a cut vertex of G', this sum is 0.

Case 2: $\deg^*(x) = 3$: Then, x is interior to a region $R' \in F(G')$. Therefore, all regions incident with x in G are also interior to R', and thus all have the same label. Therefore, the sum of these region labels is 0.

Case 3: $\deg^*(x) = 4$: There are three subcases here.

Subcase 3.1: x is incident with 2 edges of $E_{\mathcal{W}}$: In this case, the two edges $e, e' \in E_{\mathcal{W}}$ incident with x belong to some walk W and are interior to regions of G'. Therefore, $\deg_{G'}(x) = 2$, and since e and e' are not on the boundary of the same region in G, they are interior to different regions of G'. This is illustrated in Figure 4.1. Since each edge in G' is incident with two regions having different labels under ℓ', the two regions incident with x in G' have labels 1 and 2 respectively under ℓ'. Thus in the labeling ℓ of G, v is incident with two consecutive regions labeled 1 and two consecutive regions labeled 2, which sums to 0.

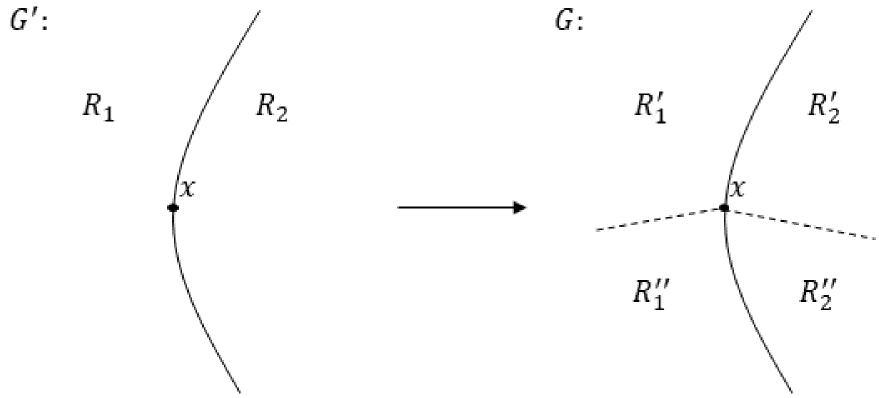

Figure 4.1: Edges incident with x belonging to $E_{\mathcal{W}}$ are interior to different regions of G'.

Subcase 3.2: x is incident with no edges of $E_{\mathcal{W}}$ and is not a cut vertex of G': As in Case 1, the edges incident with x in G are the same as those in G', and the sum of labels of regions incident with x under ℓ is the same as that under ℓ', which is 0.

Subcase 3.3: x is incident with no edges of $E_{\mathcal{W}}$ and is a cut vertex of G': Again, the edges incident with x in G are the same as those in G'. So, the labels of regions incident with x under ℓ are the same as those under ℓ', with the exception that x is incident with four distinct regions instead of three. We noted previously that each edge incident with x is incident with regions having different labels. Therefore, the regions incident with x under ℓ' in G' alternate in label $1, 2, 1, 2$. Since these regions are each distinct in G, $\ell(x) = 0$.

Therefore, the sum of regions incident with each vertex in $V(G)$ is 0 and ℓ is a cozonal labeling of G.

Next, assume that G has a cozonal labeling ℓ. By Proposition 3.2, every non-cut vertex v with $\deg(v) = 4$ is incident with exactly zero or two edges of $E_{\mathcal{W}}$. By Proposition 3.3, no vertex w with $\deg(w) = 2$ is incident with any edge of $E_{\mathcal{W}}$. □

While the properties used to characterize the bridgeless plane graphs with $\Delta(G) = 4$ may seem obscure, they are purely structural and do not rely on a labeling of G. In addition, this means that it is relatively simple to show that a bridgeless plane graph with $\Delta(G) = 4$ is not cozonal, as this only requires finding $E_{\mathcal{W}}$ and verifying that some vertex contradicts Propositions 3.2 or 3.3. If the graph has bridges, we can reduce to the bridgeless case using Theorem 2.1.

As an additional note, one may observe that the conditions in Theorem 4.1 generalize to conditions for all connected bridgeless plane graphs with $\Delta(G) \leq 4$. If $\Delta(G) = 2$, then $\mathcal{W}$ is empty and the result holds trivially. If $\Delta(G) = 3$, then $E_{\mathcal{W}}$ consists of all edges incident with a vertex of degree 3, and the conditions simplify to requiring that no vertex of degree 3 is adjacent to a vertex of degree 2. By connectivity, this then implies that G must be 3-regular.

5. Concluding Remarks

In [3], there was a brief discussion on the concept of absolute cozonality, a dualization of absolute zonality. A planar graph G is *absolutely cozonal* if every planar embedding of G is cozonal. It was noted in [3] that the characterization for cozonal graphs with maximum degree 3 did not depend on the embedding. Thus for any planar graph G with $\Delta(G) \leq 3$, G is absolutely cozonal if and only if at least one embedding of G is cozonal. This is not the case for $\Delta(G) = 4$, as exhibited by the two planar embeddings of the planar graph G in Figure 5.1. In fact, one can see that in the embedding G_2, every single edge belongs to some nonconsecutive walk containing a vertex v with $\deg^*(v) = 3$. A graph G where at least one embedding is cozonal and one is not is called *conditionally cozonal*, following the terminology for zonal graphs described in [4]. This leads to a natural question:

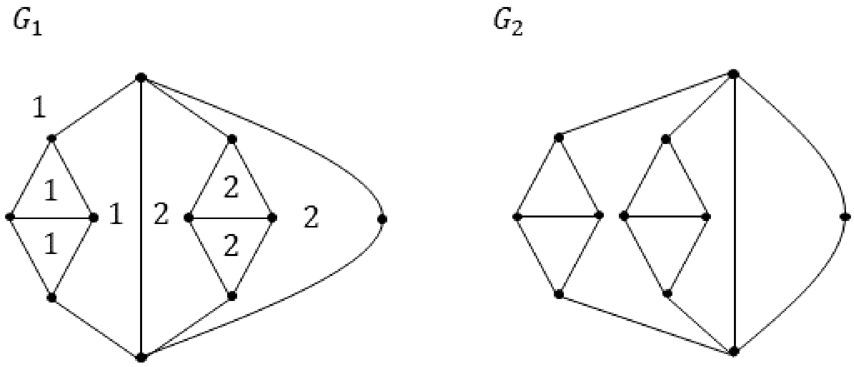

Figure 5.1: Two embeddings G_1, G_2, one of which is cozonal, and one of which is not.

Problem 5.1. *Which planar graphs with $\Delta(G) = 4$ are conditionally cozonal?*

Note that different embeddings of a planar graph may have duals whose underlying graphs are not isomorphic. For instance, the plane graph G_2 in Figure 5.1 has a region having 8 edges on its boundary, while G_1 has no such region. Therefore, the dual of G_2 has a vertex of degree 8, while the dual of G_1 does not. Thus, the underlying graphs of these duals are not isomorphic. Furthermore, the fact that the underlying graph G of G_1 and G_2 is conditionally cozonal does not necessarily tell us whether the underlying graphs of either dual are conditionally zonal. Therefore, we see that the study of absolute zonality and absolute cozonality are in fact substantially different questions. This leads to the following question:

Problem 5.2. *Is there a plane graph H with plane dual H^* such that the underlying graph G of H is absolutely zonal, but the underlying graph G' of H^* is only conditionally cozonal (or vice versa)?*

Lastly, while the tools here can be extended to graphs with higher maximum degree, the general results are much less elegant in this new setting. Still, examining nonconsecutive walks can show how certain region labels force the labeling of other regions, especially when the walk begins with a vertex v with $\deg^*(v) = 3$. This would be especially useful when there are a small number of vertices of high degree.

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