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*Research Article*

# Elementary proofs of recent congruences for overpartitions wherein non-overlined parts are not divisible by 6

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**Abstract**

We define $\overline{R_l^*}(n)$ as the number of overpartitions of $n$ in which non-overlined parts are not divisible by $l$. In a recent work, Nath, Saikia, and the second author [*arXiv*:2503.12145v2 [math.NT], (2025)] established several families of congruences for $\overline{R_l^*}(n)$. In the concluding remarks of their paper, they conjectured that $\overline{R_6^*}(n)$ satisfies an infinite family of congruences modulo $128$. In this paper, we confirm their conjectures using elementary methods. Additionally, we provide elementary proofs of two congruences for $\overline{R_6^*}(n)$ previously proven via the machinery of modular forms by Alanazi, Munagi, and Saikia [*arXiv*:2412.18938 [math.NT], (2024)].

**Keywords:** partition; overpartition; generating function; congruence.

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## 1. Introduction

A *partition* of a positive integer $n$ is a finite non–increasing sequence of positive integers $(\lambda_1, \lambda_2, \dots, \lambda_k)$ whose sum equals $n$. The integers $\lambda_1, \lambda_2, \dots, \lambda_k$ are called the *parts* of the partition. As an example, the number of partitions of the integer $n = 4$ is $5$, and the partitions in question are

$$(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad (1,1,1,1).$$

More information about integer partitions can be found in [3, 4].

One generalization of an integer partition is an *overpartition* of $n$ [5] which is a partition of $n$ wherein the first occurrence of a part may be overlined. As an example, there are 14 overpartitions of $n = 4$:

$$(4), \quad (\overline{4}), \quad (3,1), \quad (\overline{3},1), \quad (3,\overline{1}), \quad (\overline{3},\overline{1}), \quad (2,2), \quad (\overline{2},2),$$
$$(2,1,1), \quad (\overline{2},1,1), \quad (2,\overline{1},1), \quad (\overline{2},\overline{1},1), \quad (1,1,1,1), \quad (\overline{1},1,1,1).$$

The number of overpartitions of $n$ is often denoted $\overline{p}(n)$; from the above we see that $\overline{p}(4) = 14$.

Since the work of Corteel and Lovejoy [5], a variety of restricted overpartition functions have been defined and analyzed. As an example, Alanazi, Alenazi, Keith, and Munagi [1] considered the family of functions $\overline{R_\ell^*}(n)$ which counts the number of overpartitions of weight $n$ wherein non-overlined parts are not allowed to be divisible by $\ell$ while there are no restrictions on the overlined parts. For example, there are 12 overpartitions counted by $\overline{R_3^*}(4)$:

$$(4), \quad (\overline{4}), \quad (\overline{3},1), \quad (\overline{3},\overline{1}), \quad (2,2), \quad (\overline{2},2),$$
$$(2,1,1), \quad (\overline{2},1,1), \quad (2,\overline{1},1), \quad (\overline{2},\overline{1},1), \quad (1,1,1,1), \quad (\overline{1},1,1,1).$$

One can readily see that two overpartitions counted by $\overline{p}(4)$, namely $(3,1)$ and $(3,\overline{1})$, do not appear in the list above. This is true because they contain a non-overlined part which is divisible by $\ell = 3$.

In [1], Alanazi et al. proved a number of congruence properties satisfied by the functions $\overline{R_\ell^*}(n)$ which, for each $\ell$, satisfies the generating function identity

$$\sum_{n=0}^{\infty} \overline{R_\ell^*}(n) q^n = \frac{f_2 f_\ell}{f_1^2}$$

where

$$f_k = \prod_{m=1}^{\infty} (1 - q^{km}).$$

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Subsequently, additional work on this family of functions has been completed; see [2, 11–13] for examples of such work.

Our goal in this brief paper is to utilize truly elementary means to prove two different sets of results for the function $\overline{R_6^*}(n)$. First, we note the following theorem which combines the statements of two conjectures that recently appeared in the work of Nath, Saikia, and the second author [11].

**Theorem 1.1** (Conjecture 8.1 and Conjecture 8.2 in [11])**.** *For all $n \geq 0$ and $k \geq 0$, we have*

$$\overline{R_6^*}\left(18 \cdot 3^{2k+1}n + \frac{153 \cdot 3^{2k} - 1}{4}\right) \equiv 0 \pmod{128}. \tag{1}$$

Next, we mention a pair of congruences given by Alanazi, Munagi, and Saikia [2, Theorem 4.4]. It is important to note that the authors proved these properties via an automated approach which relies on the machinery of modular forms; our goal here is to provide a classical proof for each of these congruences.

**Theorem 1.2.** *For $n \geq 0$, we have*

$$\overline{R_6^*}(27n + 11) \equiv 0 \pmod{64}, \tag{2}$$
$$\overline{R_6^*}(81n + 47) \equiv 0 \pmod{24}. \tag{3}$$

In order to prove Theorems 1.1 and 1.2, we will need a few foundational results which already appear in the literature. We gather all of the necessary results here. We begin with a well–known identity of Jacobi.

**Lemma 1.1** (Jacobi)**.** *We have*

$$f_1^3 = \sum_{m \geq 0} (-1)^m (2m+1) q^{m(m+1)/2}. \tag{4}$$

**Proof.** See Equation (1.7.1) in [7]. □

Next, we share a pair of 2–dissection identities that will be useful in our work below.

**Lemma 1.2.** *We have*

$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q \frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}, \tag{5}$$
$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q \frac{f_2^2 f_{12}^3}{f_4 f_6^2}. \tag{6}$$

**Proof.** Equations (5) and (6) correspond to (29) and (30), respectively, in [6, Lemma 1]. □

In an analogous fashion, we also require several 3–dissection results which will be used in our generating function manipulations below.

**Lemma 1.3.** *We have*

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q \frac{f_3 f_{18}^2}{f_6 f_9}, \tag{7}$$
$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9}, \tag{8}$$
$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6}, \tag{9}$$
$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}, \tag{10}$$
$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}, \tag{11}$$
$$\frac{1}{f_1^3} = \frac{f_6^2 f_9^{15}}{f_3^{14} f_{18}^6} + 3q \frac{f_6 f_9^{12}}{f_3^{13} f_{18}^3} + 9q^2 \frac{f_9^9}{f_3^{12}} + 8q^3 \frac{f_9^6 f_{18}^3}{f_3^{11} f_6} + 12q^4 \frac{f_9^3 f_{18}^6}{f_3^{10} f_6^2} + 16q^6 \frac{f_{18}^{12}}{f_3^8 f_6^4 f_9^3}. \tag{12}$$

**Proof.** Equations (7) and (8) appear as (14.3.2) and (14.3.3) in [7], respectively. Identity (9) was proven in [8], and [9] contains a proof of (10). The identities (11) and (12) can be found in [10, Lemma 3]. □

Lastly, we need the following well-known fact which basically follows from the Binomial Theorem and divisibility properties of certain binomial coefficients.

**Lemma 1.4.** *For a prime $p$ and positive integers $k$ and $l$,*

$$f_l^{p^k} \equiv f_{lp}^{p^{k-1}} \pmod{p^k}. \tag{13}$$

## 2. Proof of Theorem 1.1

We begin by recalling the generating function

$$\sum_{n=0}^{\infty}\overline{R_6^*}(n)q^n = \frac{f_2f_6}{f_1^2} = \left(\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_6^3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^3}{f_3^6}\right)f_6 \quad \text{(thanks to (9))}.$$

Extracting the terms in which the exponents of $q$ are of the form $3n+2$, dividing both sides by $q^2$, and then replacing $q^3$ by $q$, we get

$$\sum_{n=0}^{\infty}\overline{R_6^*}(3n+2)q^n = 4\frac{f_2^3f_6^3}{f_1^6} \tag{14}$$
$$= 4\frac{f_2^3f_6^3}{f_1^{32}}f_1^{26}$$
$$\equiv 4\frac{f_2^3f_6^3}{f_2^{16}}f_1^{26} \pmod{128} \quad \text{(thanks to (13))}$$
$$= 4f_6^3\left(\frac{f_1^2}{f_2}\right)^{13}$$
$$= 4f_6^3\left(\frac{f_9^2}{f_{18}}-2q\frac{f_3f_{18}^2}{f_6f_9}\right)^{13}.$$

Observing that $4(a-2b)^{13}\equiv 4a^{13}+24a^{12}b+96a^{11}b^2+64a^{10}b^3+64a^9b^4 \pmod{128}$, extracting the terms in which the exponents of $q$ are of the form $3n$, we get

$$\sum_{n=0}^{\infty}\overline{R_6^*}(9n+2)q^{3n}\equiv f_6^3\left(4\frac{f_9^{26}}{f_{18}^{13}}+64\frac{f_9^{20}}{f_{18}^{10}}\cdot q^3\frac{f_3^3f_{18}^6}{f_6^3f_9^3}\right) \pmod{128}.$$

Replacing $q^3$ by $q$ gives

$$\sum_{n=0}^{\infty}\overline{R_6^*}(9n+2)q^n\equiv\sum_{n=0}^{\infty}T_1(9n+2)q^n+\sum_{n=0}^{\infty}T_2(9n+2)q^n \pmod{128}, \tag{15}$$

where

$$\sum_{n=0}^{\infty}T_1(9n+2)q^n = 4\frac{f_2^3f_3^{26}}{f_6^{13}}, \tag{16}$$
$$\sum_{n=0}^{\infty}T_2(9n+2)q^n = 64q\frac{f_1^3f_3^{17}}{f_6^4}\equiv 64qf_1^3f_3^9 \pmod{128} \quad \text{(thanks to (13))}. \tag{17}$$

For given $n\geq 0$ and $k\geq 0$, setting

$$l_{n,k} := 18\cdot 3^{2k+1}n+\frac{153\cdot 3^{2k}-1}{4},$$

we have $l_{n,k}\equiv 2 \pmod 9$. To show that $\overline{R_6^*}(l_{n,k})\equiv 0 \pmod{128}$, by (15), it suffices to prove the following two congruences

$$T_1(l_{n,k})\equiv 0 \pmod{128}, \tag{18}$$
$$T_2(l_{n,k})\equiv 0 \pmod{128}. \tag{19}$$

**Proof of (18):** Using (4) in (16), we get

$$\sum_{n=0}^{\infty}T_1(9n+2)q^n = 4\frac{f_3^{26}}{f_6^{13}}\left(\sum_{m\geq 0}(-1)^m(2m+1)q^{m(m+1)}\right). \tag{20}$$

We now check whether $m(m+1)+3k=6n+4$ for some $m,n$ and $k$. Equivalently, $(2m+1)^2+12k=24n+17$. This is not possible since 5 is a quadratic nonresidue modulo 12. Thus, the right-hand side of (20) does not contain terms in which the exponents of $q$ are of the form $6n+4$, and hence

$$T_1(54n+38)\equiv 0 \pmod{128},$$

which implies that $T_1(l_{n,k})\equiv 0 \pmod{128}$ when $k=0$.

In order to show $T_1(l_{n,k}) \equiv 0 \pmod{128}$ for $k \geq 1$, we first establish the following claim.

**Claim 2.1.** *For $k \geq 1$, we have*

$$\sum_{n=0}^{\infty} T_1\left(3^{2k+2}n + \frac{3^{2k+2}-1}{4}\right) q^n \equiv \pm 32 q f_1^3 f_3^9 - (16a+12)\frac{f_1^8 f_3^{18}}{f_2 f_6^9} \pmod{128} \tag{21}$$

*for some integer $a$. Here, $\pm$ indicates that (21) takes either the $+$ or the $-$ sign, not both at once.*

**Proof of Claim 2.1.** We prove this by induction on $k$. Applying (13) to (16), we get

$$\begin{aligned}\sum_{n=0}^{\infty} T_1(9n+2) q^n &\equiv 4\frac{f_2^3 f_6^3}{f_3^6} \pmod{128}\\ &= 4\left(\frac{f_{12} f_{18}^6}{f_6 f_{36}^3} - 3q^2 f_{18}^3 + 4q^6 \frac{f_6^2 f_{36}^6}{f_{12}^2 f_{18}^3}\right)\frac{f_6^3}{f_3^6} \quad \text{(thanks to (11))}.\end{aligned}$$

Extracting the terms which contain the form $q^{3n+2}$, dividing by $q^2$, and replacing $q^3$ by $q$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} T_1(27n+20) q^n &\equiv -12 f_6^3 \left(\frac{f_2}{f_1^2}\right)^3 \pmod{128}\\ &= -12 f_6^3 \left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6}\right)^3 \quad \text{(using (9))}.\end{aligned}$$

Extracting the terms that contain exponents of $q$ of the form $3n$ gives

$$\sum_{n=0}^{\infty} T_1(81n+20) q^{3n} \equiv -12 f_6^3 \left(\frac{f_6^{12} f_9^{18}}{f_3^{24} f_{18}^9} + 56q^3 \frac{f_6^9 f_9^9}{f_3^{21}}\right) \pmod{128}.$$

We replace $q^3$ by $q$ to obtain

$$\sum_{n=0}^{\infty} T_1\left(3^4 n + \frac{3^4-1}{4}\right) q^n \equiv -32q\frac{f_2^{12} f_3^9}{f_1^{21}} - 12\frac{f_2^{15} f_3^{18}}{f_1^{24} f_6^9} \pmod{128} \equiv -32q f_1^3 f_3^9 - 12\frac{f_2^{15} f_3^{18}}{f_1^{24} f_6^9} \pmod{128},$$

where the last congruence follows on applying (13). This establishes the claim for $k=1$.

Suppose that (21) holds for a fixed $k$. Then, we show that (21) holds for $k+1$. From (21), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_1\left(3^{2k+2}n + \frac{3^{2k+2}-1}{4}\right) q^n &\equiv \pm 32 q f_1^3 f_3^9 - (16a+12)\frac{f_1^8 f_3^{18}}{f_2 f_6^9} \pmod{128}\\ &= \pm 32 q f_1^3 f_3^9 - (16a+12)(f_1^3)^2 \frac{f_1^2}{f_2}\frac{f_3^{18}}{f_6^9}\\ &= \pm 32 q f_3^9 \left(\frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}\right)\\ &\quad - (16a+12)\left(\frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}\right)^2 \left(\frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}\right)\frac{f_3^{18}}{f_6^9},\end{aligned}$$

where the last equality follows using (11) and (7). We extract the terms in which the exponents of $q$ are of the form $3n+2$ to obtain

$$\sum_{n=0}^{\infty} T_1\left(3^{2k+2}(3n+2) + \frac{3^{2k+2}-1}{4}\right) q^{3n+2} \equiv \pm 32 q^2 f_3^9 f_9^3 - (16a+12)\left(21q^2 \frac{f_3^{18} f_9^8}{f_6^9 f_{18}} + 48q^5 \frac{f_3^{21} f_{18}^8}{f_6^{12} f_9}\right) \pmod{128}.$$

Dividing by $q^2$ and replacing $q^3$ by $q$ yields

$$\begin{aligned}\sum_{n=0}^{\infty} T_1\left(3^{2k+3}n + \frac{3^{2k+4}-1}{4}\right) q^n &\equiv \pm 32 f_1^9 f_3^3 - 21(16a+12)\frac{f_1^{18} f_3^8}{f_2^9 f_6} + 64q\frac{f_1^{21} f_6^8}{f_2^{12} f_3} \pmod{128}\\ &\equiv \pm 32 f_1^9 f_3^3 - 21(16a+12)\left(\frac{f_1^2}{f_2}\right)^9 \frac{f_3^8}{f_6} + 64q\frac{f_6^8}{f_1^3 f_3} \pmod{128} \quad \text{(thanks to (13))}\\ &\equiv \pm 32 f_3^3 \left(\frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3\right)^3 - 21(16a+12)\left(\frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}\right)^9 \frac{f_3^8}{f_6}\\ &\quad + 64q\frac{f_6^8}{f_3}\left(\frac{f_6^2 f_9^{15}}{f_3^{14} f_{18}^6} + 3q\frac{f_6 f_9^{12}}{f_3^{13} f_{18}^3} + 9q^2 \frac{f_9^9}{f_3^{12}}\right) \pmod{128} \quad \text{(using (7), (11), (12))}.\end{aligned}$$

We observe that $12(x-2y)^9 \equiv 12(x^9 - 18x^8y + 16x^7y^2) \pmod{128}$. So, extracting the terms that contain the form $q^{3n}$, we get

$$\sum_{n=0}^{\infty} T_1\left(3^{2k+3}(3n) + \frac{3^{2k+4}-1}{4}\right) q^{3n} \equiv \pm 32\left(\frac{f_6^3 f_9^{18}}{f_{18}^9} - 27q^3 f_3^3 f_9^9\right) - 21(16a+12)\frac{f_3^8 f_9^{18}}{f_6 f_{18}^9} + 64q^3\frac{f_6^8 f_9^9}{f_3^{13}} \pmod{128}.$$

We replace $q^3$ by $q$ to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} T_1\left(3^{2k+4}n + \frac{3^{2k+4}-1}{4}\right) q^{n} &\equiv \pm 32\frac{f_2^3 f_3^{18}}{f_6^9} \pm 32q f_1^3 f_3^9 - 21(16a+12)\frac{f_1^8 f_3^{18}}{f_2 f_6^9} + 64q\frac{f_2^8 f_3^9}{f_1^{13}} \pmod{128}\\ &\equiv \pm 32\frac{f_1^8 f_3^{18}}{f_2 f_6^9} \pm 32q f_1^3 f_3^9 - 21(16a+12)\frac{f_1^8 f_3^{18}}{f_2 f_6^9} + 64q f_1^3 f_3^9 \pmod{128} \quad \text{(by (13))}\\ &= \mp 32q f_1^3 f_3^9 - \big(21(16a+12) \mp 32\big)\frac{f_1^8 f_3^{18}}{f_2 f_6^9} \pmod{128}.\end{aligned}$$

Note that $21(16a+12)+32 \equiv 16(5a+1)+12 \pmod{128}$ and $21(16a+12)-32 \equiv 16(5a+5)+12 \pmod{128}$. This completes both the induction and proof of the claim. □

From (21), for $k \geq 1$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_1\left(3^{2k+2}n + \frac{3^{2k+2}-1}{4}\right) q^{n} &\equiv \pm 32q f_1^3 f_3^9 - (16a+12)(f_1^3)^2 \frac{f_1^2}{f_2}\frac{f_3^{18}}{f_6^9} \pmod{128}\\ &= \pm 32q f_3^9\left(\frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3\frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}\right)\\ &\quad - (16a+12)\left(\frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3\frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}\right)^2\left(\frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}\right)\frac{f_3^{18}}{f_6^9} \quad \text{(by (7) and (11))}.\end{aligned}$$

Extracting the terms that contain the form $q^{3n+1}$, we get

$$\sum_{n=0}^{\infty} T_1\left(3^{2k+2}(3n+1) + \frac{3^{2k+2}-1}{4}\right) q^{3n+1} \equiv \pm 32q\frac{f_3^8 f_6 f_9^6}{f_{18}^3} - (16a+12)\left(-40q^4\frac{f_3^{20} f_9^2 f_{18}^5}{f_6^{11}} - 8q\frac{f_3^{17} f_9^{11}}{f_6^8 f_{18}^4}\right) \pmod{128}.$$

Dividing by $q$ and replacing $q^3$ by $q$ gives

$$\begin{aligned}\sum_{n=0}^{\infty} T_1\left(3^{2k+3}n + \frac{5\cdot 3^{2k+2}-1}{4}\right) q^{n} &\equiv \pm 32\frac{f_1^8 f_2 f_3^6}{f_6^3} - 32q\frac{f_1^{20} f_3^2 f_6^5}{f_2^{11}} - 32\frac{f_1^{17} f_3^{11}}{f_2^8 f_6^4} \pmod{128}\\ &\equiv 32 f_3^2\left(\pm\frac{f_2^5}{f_6} - q\frac{f_6^5}{f_2} - \frac{f_1}{f_3^3} f_6^2\right) \pmod{128} \quad \text{thanks to (13)}\\ &= 32 f_3^2\left(\pm\frac{f_2^5}{f_6} - q\frac{f_6^5}{f_2} - \left(\frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}\right) f_6^2\right) \pmod{128} \quad \text{(using (5))}\\ &\equiv 32 f_3^2\left(\pm\frac{f_2^5}{f_6} - \frac{f_2^5}{f_6}\right) \pmod{128} \quad \text{(thanks to (13))}\\ &\equiv \begin{cases} 0 \pmod{128} & \text{when taking positive sign,}\\ -64 f_2^5 \pmod{128} & \text{when taking negative sign and applying (13).}\end{cases}\end{aligned}$$

Observe that the right-hand side of the last congruence contains no terms that contain odd powers of $q$. Thus, for $n \geq 0$ and $k \geq 1$, we have

$$T_1\left(3^{2k+3}(2n+1) + \frac{5\cdot 3^{2k+2}-1}{4}\right) \equiv 0 \pmod{128},$$

where

$$3^{2k+3}(2n+1) + \frac{5\cdot 3^{2k+2}-1}{4} = 18\cdot 3^{2k+1}n + \frac{153\cdot 3^{2k}-1}{4} = l_{n,k}.$$

**Proof of (19):** We now establish the following claim for $T_2$.

**Claim 2.2.** *For $n \geq 0$ and $k \geq 0$, we have*

$$\sum_{n=0}^{\infty} T_2\left(2\cdot 3^{2k+2}n + \frac{3^{2k+2}-1}{4}\right) q^{n} \equiv \lambda 64 f_1^3 + 64q f_3^3 f_6^3 \pmod{128}, \tag{22}$$

*where $\lambda = 0$ or $1$.*

**Proof of Claim 2.2.** We prove the claim by induction on $k$. Applying (13) to (17), we get

$$\sum_{n=0}^{\infty} T_2(9n+2)q^n \equiv 64q\frac{f_1^3}{f_3}f_6^5 \pmod{128}$$
$$= 64q\left(\frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}\right) f_6^5 \quad \text{(using (6))}.$$

We extract the terms that contain even powers of $q$ and then replace $q^2$ by $q$ to obtain

$$\sum_{n=0}^{\infty} T_2\left(2\cdot 3^2 n + \frac{3^2-1}{4}\right) q^n \equiv 64q\frac{f_1^2 f_3^3 f_6^3}{f_2} \pmod{128}$$
$$\equiv 64q f_3^3 f_6^3 \pmod{128} \quad \text{(thanks to (13))}.$$

This establishes the claim for $k=0$.

Suppose that (22) holds for a fixed $k$. We show that it also holds for $k+1$. From (22), we have

$$\sum_{n=0}^{\infty} T_2\left(2\cdot 3^{2k+2} n + \frac{3^{2k+2}-1}{4}\right) q^n \equiv \lambda 64 f_1^3 + 64q f_3^3 f_6^3 \pmod{128}$$
$$= \lambda 64\left(\frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3\frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}\right) + 64q f_3^3 f_6^3 \quad \text{(thanks to (11))}.$$

Extracting the terms in which the exponents of $q$ are of the form $3n+1$, dividing by $q$ and replacing $q^3$ by $q$, we get

$$\sum_{n=0}^{\infty} T_2\left(2\cdot 3^{2k+3} n + \frac{3^{2k+4}-1}{4}\right) q^n \equiv \lambda 64 f_3^3 + 64 f_1^3 f_2^3 \pmod{128}$$
$$= \lambda 64 f_3^3 + 64\left(\frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}\right)^3 \quad \text{(thanks to (10))}.$$

Observe that $64(a-b-2c)^3 \equiv 64(a^3+a^2b+ab^2+b^3) \pmod{128}$. We extract the terms that contain exponents of $q$ of the form $3n$ and replace $q^3$ by $q$ to get

$$\sum_{n=0}^{\infty} T_2\left(2\cdot 3^{2k+4} n + \frac{3^{2k+4}-1}{4}\right) q^n \equiv \lambda 64 f_1^3 + 64\left(\frac{f_2^3 f_3^{12}}{f_1^3 f_6^6} + q f_3^3 f_6^3\right) \pmod{128}$$
$$\equiv \lambda 64 f_1^3 + 64\left(f_1^3 + q f_3^3 f_6^3\right) \pmod{128}$$
$$\text{(using (13))}$$
$$\equiv \begin{cases} 64q f_3^3 f_6^3 \pmod{128} & \text{if } \lambda = 1, \\ 64 f_1^3 + 64q f_3^3 f_6^3 \pmod{128} & \text{if } \lambda = 0. \end{cases}$$

This shows that (22) holds for $k+1$ and completes the proof of the claim. □

Since, from (11),

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3\frac{f_3^2 f_{18}^6}{f_6^2 f_9^3},$$

the right-hand sides in (22) do not contain terms in which the exponents of $q$ are of the form $3n+2$. Therefore, for all $n \geq 0$ and $k \geq 0$, we have

$$T_2\left(2\cdot 3^{2k+2}(3n+2) + \frac{3^{2k+2}-1}{4}\right) \equiv 0 \pmod{128},$$

where

$$2\cdot 3^{2k+2}(3n+2) + \frac{3^{2k+2}-1}{4} = 18\cdot 3^{2k+1} n + \frac{153\cdot 3^{2k}-1}{4} = l_{n,k}.$$

□

## 3. Proof of Theorem 1.2

We close this work by quickly providing elementary proofs of the congruences in Theorem 1.2. These rely on our generating function manipulations above, and follow from a straightforward analysis of the dissections in question.

**Proof of (2):** From (14), we have

$$\sum_{n=0}^{\infty}\overline{R_6^*}(3n+2)q^n = 4\frac{f_1^{10}f_2^3f_6^3}{f_1^{16}}$$
$$\equiv 4\frac{f_6^3f_1^{10}}{f_2^5} \pmod{64} \quad \text{(applying (13))}$$
$$= 4f_6^3\left(\frac{f_9^2}{f_{18}}-2q\frac{f_3f_{18}^2}{f_6f_9}\right)^5 \quad \text{(thanks to (7))}.$$

As $4(a-2b)^5 \equiv 4(a^5+6a^4b+8a^3b^2) \pmod{64}$, extracting the terms with exponents divisible by 3 gives

$$\sum_{n=0}^{\infty}\overline{R_6^*}(9n+2)q^{3n} \equiv 4f_6^3\frac{f_9^{10}}{f_{18}^5} \pmod{64}.$$

Replacing $q^3$ by $q$ yields

$$\sum_{n=0}^{\infty}\overline{R_6^*}(9n+2)q^n \equiv 4f_2^3\frac{f_3^{10}}{f_6^5} \pmod{64}$$
$$= 4\left(\sum_{m\geq 0}(-1)^m(2m+1)q^{m(m+1)}\right)\frac{f_3^{10}}{f_6^5} \quad \text{(using (4))}.$$

The proof will be completed by showing that there exist no integers $m$ and $n$ satisfying

$$m(m+1) = 3n+1,$$

or,

$$(2m+1)^2 = 12n+5.$$

Since 5 is not a quadratic residue modulo 12, no such integers exist. Therefore, we know that, for all $n\geq 0$,

$$\overline{R_6^*}(9(3n+1)+2) = \overline{R_6^*}(27n+11) \equiv 0 \pmod{64}.$$

**Proof of (3):** Again from (14), we have

$$\sum_{n=0}^{\infty}\overline{R_6^*}(3n+2)q^n = 4\frac{f_2^3f_6^3}{f_1^6}$$
$$\equiv 4\frac{f_2^3f_6^3}{f_2^3} \pmod 8 \quad \text{(applying (13))}$$
$$= 4f_6^3.$$

By extracting the terms of the form $q^{3n}$ and replacing $q^3$ by $q$, we obtain

$$\sum_{n=0}^{\infty}\overline{R_6^*}(9n+2)q^n \equiv 4f_2^3 \pmod 8$$
$$= \sum_{m\geq 0}(-1)^m(2m+1)q^{m(m+1)} \quad \text{(thanks to (4))}.$$

We claim that there exist no integers $m$ and $n$ satisfying

$$9n+5 = m(m+1),$$

or equivalently,

$$(2m+1)^2 = 36n+21.$$

Since 21 is not a quadratic residue modulo 36, no such integers exist. It follows that

$$\overline{R_6^*}(81n+47) \equiv 0 \pmod 8. \tag{23}$$

Next, we apply (13) to (14) to deduce that

$$\sum_{n=0}^{\infty}\overline{R_6^*}(3n+2)q^n \equiv 4\frac{f_6^4}{f_3^2} \pmod 3.$$

Then,

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{R_6^*}(9n+2)q^n &\equiv 4\frac{f_2^4}{f_1^2} \pmod 3\\ &= 4\left(\frac{f_6 f_9^2}{f_3 f_{18}} + q\,\frac{f_{18}^2}{f_9}\right)^2 \quad \text{(using (8))}.\end{aligned}$$

Extracting the terms with exponents of $q$ are of the form $3n+2$, dividing by $q^2$, and replacing $q^3$ by $q$ gives

$$\sum_{n=0}^{\infty} \overline{R_6^*}(27n+20)q^n \equiv 4\frac{f_6^4}{f_3^2} \pmod 3.$$

Since the resulting series is expressed in terms of $q^3$, and therefore cannot contain any terms of the form $q^{3n+1}$, we conclude that

$$\overline{R_6^*}(27(3n+1)+20) = \overline{R_6^*}(81n+47) \equiv 0 \pmod 3. \tag{24}$$

Combining (23) and (24) completes the proof of (3). □

As we close, it is worth noting that the proof above can be modified in a straightforward fashion to prove that, for all $n \geq 0$,

$$\overline{R_6^*}(27(3n+2)+20) = \overline{R_6^*}(81n+74) \equiv 0 \pmod{24},$$

a result which was not mentioned in [2].

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