Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **16** (2025) 43–50
DOI: 10.47443/dml.2025.089

Research Article

On the limitations of ROBDDs in deciding the evasiveness of Boolean functions

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(Received: 30 April 2025. Received in revised form: 18 July 2025. Accepted: 3 September 2025. Published online: 17 September 2025.)

Abstract

In this paper, we prove that examining all the Reduced Ordered Binary Decision Diagrams (ROBDDs) of a Boolean monotone function is not enough to determine the evasiveness of the function. Working in the framework of simplicial topology, we introduce the notion of *ordered non evasive simplicial complex*, corresponding to our ROBDDs quest, and we prove that there exist non evasive simplicial complexes which are not ordered non evasive.

Keywords: evasiveness; binary decision diagram; simplicial complex.

2020 Mathematics Subject Classification: 05E45, 94D10.

1. Introduction

Fast evaluation of Boolean functions is of great practical importance and the problem is also foundational in complexity theory. In this context, a Boolean function is called *evasive* if it is necessary to evaluate each of its arguments to know its output. So, to determine whether a Boolean function is or is not evasive is an interesting problem. Another relevant practical problem is to find the best way of representing a Boolean function to solve a certain kind of task. Bryant [4] in 1986 introduced the notion of *Binary Decision Diagram* (or *BDD*, in short) as a compact way of representing Binary Decision Trees (and therefore Boolean functions). A BDD is a directed acyclic graph obtained from a binary decision tree by means of two kinds of reductions (see [4] for details). Each BDD defines a Boolean function, and a Boolean function can be represented by many BDDs. A Boolean function is non evasive if and only if it can be represented by means of a BDD with *depth* strictly less than n, being n the number of variables of the Boolean function. Among the zoo of BDDs, one class that is specially efficient for certain type of Boolean operations is that of *Reduced Ordered BDDs* (*ROBDDs*, in short). In this variant, a permutation of the Boolean function variables is fixed, and in each branch of the BDD the variables are questioned in that order (but some of the variables may be skipped). Each binary decision tree with a fixed order in the variables is canonically associated to a *truth table* for the Boolean function. In these terms, following the classical approach by Knuth [6], the search for *beads* in a truth table is the key to studying the non evasiveness of the Boolean function. So, in this setting, it is natural to pose the following question: are ROBDDs enough to determine evasiveness? That is to say, if all the ROBDDs for a Boolean function are of maximal depth, then the Boolean function is evasive? In other words: the absence of "bead decompositions" in all the truth tables of a Boolean function implies its evasiveness? We know (see [2]) that if the Boolean function is non monotonic, the answer is negative. So, we can place ourselves in the case of Boolean monotone functions to explore that problem. Since there is a bijective correspondence between Boolean monotone functions and simplicial complexes, we translate our problem to the language of simplicial topology [8]. A non evasive simplicial complex is contractible, so we can understand our quest as looking for criteria helping to detect the homotopical triviality of simplicial complexes. The presence of a bead in a truth table is equivalent to finding a variable that has no influence in the output of the corresponding Boolean sub-function. In topological terms, this amounts to detect a simplicial sub-complex which is a cone with peek in the variable that can be skipped. Guided by this intuition, we introduce in this paper the concept of *ordered non evasive simplicial complex*. A simplicial complex is ordered non evasive if and only if there exists a ROBDD representig it (as a Boolean function) with depth less than the number of variables/vertices. In [2], we proved that *dismantlable* (or 0-collapsible; see [3]) simplicial complexes are ordered non evasive (in that paper [2], ordered non evasive simplicial complexes were called *ligneous*). Since dismantlable complexes are non evasive, we know that the class of ordered non evasive simplicial complexes is large enough.

In summary, our objective is to prove that the set of ordered non evasive simplicial complexes is different from the set of non evasive simplicial complexes. To this aim, it is enough to find an example of a non evasive simplicial complex which

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is not ordered non evasive. Nevertheless, since the number of permutations increases exponentially with the number of vertices, an enumerative approach, based on an exhaustive search, is not possible in practice. So, our tactic was to find a simplicial property hold by every ordered non evasive simplicial complex but not by all the non evasive simplicial complexes. Our inspiration came from two sources. First, convex union representable simplicial complexes collapse to its stars [5]. Second, there exist non evasive simplicial complexes with exactly two free faces [1]. So, our idea was to find a simplicial property of each ordered non evasive simplicial complex, related to collapses over stars, and implying that there are enough free faces in it (more than two). This property is reflected in Theorem 4.1, the main result of this paper.

2. Preliminaries

Given any set X, we will denote by $\mathcal{P}(X)$ the power set of X and by $\#X$ its cardinality.

Definition 2.1. *Given a* finite *set* V*, a* simplicial complex *on* V *is a pair* (K,V)*, where* $K \in \mathcal{P}(\mathcal{P}(V))$ *and it is "closed by subsets" (in other words: for any* $s \in K$*,* $\mathcal{P}(s) \subseteq K$*). The set* V *is called the* vertex set *of* (K,V)*. Each element* $s \in K$ *is called a* simplex *of* (K,V)*. If* $s = \{v_{i_0}, \ldots, v_{i_d}\}$*, it is said that the* dimension *of the simplex* s *is* d *(note that the enumeration of vertices starts at* 0*). If* $V \neq \emptyset$*, the simplicial complex* $(\mathcal{P}(V),V)$ *is called* standard simplex of dimension $\#V - 1$*, and it is denoted by* $\Delta^{(\#V-1)}$.

Remark 2.1. **(i).** *Let us note that if* (K,V) *is a simplicial complex and* $V \subsetneq W$*, then* (K,W) *is also a simplicial complex, a* different *one. Conversely, if* $v \in V$ *and* v *is not a member of any simplex in* K*,* $(K, V \setminus \{v\})$ *is also a simplicial complex. Sometimes, we will denote a simplicial complex* (K,V) *simply by* K*, but in general we will write the complete description (mainly in definitions and statements), because most of the concepts in this paper depend heavily on the vertex set* V *that is considered in each situation.*

(ii). *The empty set* $\emptyset$ *is considered of dimension* -1*, coherently with the dimension definition, and it is a simplex of any* $K \neq \emptyset$*. The singleton simplicial complex* $K = \{\emptyset\}$ *admits any vertex set (including* $V = \emptyset$*, as* $K = \emptyset$ *does), and it has a somehow* pathological *nature in most of our definitions and results. Sometimes, in the literature, given* $v \in V$*, it is written* $v \in K$*, but it is necessary to understand that the "vertex complex" associated to* v *is not a singleton, the notation refers to the subcomplex* $\{\emptyset, \{v\}\} \subseteq K$*.*

Definition 2.2. *Let* (K,V) *be a simplicial complex and let* $s \neq \emptyset$ *be a simplex of dimension* d *of* (K,V)*. Any subset* s' *of* s *of dimension* $d-1$ *is called a* face *of* s *(note that* s' *could be equal to* $\emptyset$*); in this situation,* s *is called a* coface *of* s'*. A* facet *of* (K,V) *is a simplex of* (K,V) *with no coface.*

Definition 2.3. *Let* (K,V) *be a simplicial set, and let* v *be a vertex of* V*. The* costar *of* K *with respect to* v *is defined by* $\mathrm{cost}(K,v) = \{s \in K \mid v \notin s\} \in \mathcal{P}(\mathcal{P}(V \setminus \{v\}))$*. Then, when referring to the* costar of a simplicial complex (K,V) *with respect to a vertex* $v \in V$ *we mean the simplicial complex* $(\mathrm{cost}(K,v), V \setminus \{v\})$.

Definition 2.4. *Let* (K,V) *be a simplicial set, and let* v *be a vertex of* V*. The* link *of* K *with respect to* v *is defined by* $\mathrm{link}(K,v) = \{s \in \mathrm{cost}(K,v) \mid \{v\} \cup s \in K\} \in \mathcal{P}(\mathcal{P}(V \setminus \{v\}))$*. Then, when referring to the* link of a simplicial complex (K,V) *with respect to a vertex* $v \in V$ *we mean the simplicial complex* $(\mathrm{link}(K,v), V \setminus \{v\})$.

Remark 2.2. *Let us note that, by its very definition,* $\mathrm{link}(K,v) \subseteq \mathrm{cost}(K,v)$*. When* $\mathrm{link}(K,v) = \mathrm{cost}(K,v)$ *it is the case of* K *being a* cone *(see Definition 2.5). The study of the relationship* $\mathrm{link}(K,v) \subset \mathrm{cost}(K,v)$ *from a topological point of view is central in this paper.*

Definition 2.5. *Let* (K,V) *be a simplicial complex and let* $v \notin V$*. The* cone *of* K *with* peek v *is defined by* $v * K = K \cup \{\{v\} \cup s \mid s \in K\} \in \mathcal{P}(\mathcal{P}(V \cup \{v\}))$*. Then,* K *is called the* basis *of the cone, and when referring to the* cone of a simplicial complex (K,V) *with peek* $v \notin V$ *we mean the simplicial complex* $(v * K, V \cup \{v\})$.

Remark 2.3. **(i).** *Let* (K,V) *be a simplicial set, and let* v *be a vertex of* V*. The* star *of* K *with respect to* v *is defined by* $\mathrm{star}(K,v) = \{s \in K \mid v \in s\} \in \mathcal{P}(\mathcal{P}(V))$*. This concept explains why we call* costar *(see Definition 2.3) the complement in* K *of the star. Note that, in general,* $\mathrm{star}(K,v)$ *is not a simplicial complex (it is not "closed by subsets"). The minimal simplicial complex containing* $\mathrm{star}(K,v)$ *is called* closed star *and it is equal to* $v * \mathrm{link}(K,v)$*, and this is the notation preferred by us along the paper.*

(ii). *Let us note that given* (K,V) *and* $v \in V$ *such that* $K = v * (K \setminus \mathrm{star}(K,v))$*, then* (K,V) *is a cone with peek* v*. When the peek is unimportant, we simply write "*(K,V) *is a cone", meaning that there exists a vertex* $v \in V$ *such that* (K,V) *is a cone with peek* v*.*

(iii). *According to our definition, the simplicial complex $(\emptyset, V)$ is a cone with peek v for all $v \in V$ (assuming $V \neq \emptyset$). It is a convention to declare if $\emptyset$ is or is not a cone, but it is convenient for us to consider so.*

The announced characterization of a cone by means of costar and link follows:

Proposition 2.1. *A simplicial complex (K, V) is a cone with peek v if and only if* $\mathrm{link}(K, v) = \mathrm{cost}(K, v)$.

The following decomposition is an essential tool in our approach; its proof is direct from the definitions, too.

Proposition 2.2. *Let $V \neq \emptyset$, (K, V) be a simplicial complex and $v \in V$. Then $K = \mathrm{cost}(K, v) \cup (v * \mathrm{link}(K, v))$.*

Remark 2.4. *Let us observe that it is also true that $K = \mathrm{cost}(K, v) \cup \mathrm{star}(K, v)$, being even a disjoint union. However, since $\mathrm{star}(K, v)$ is not in general a simplicial complex, this decomposition is not so useful for reasoning by induction as that of Proposition 2.2.*

Now, we introduce the main concept about evasiveness (see, for instance, [3]).

Definition 2.6. *Let $V \neq \emptyset$ and (K, V) be a simplicial complex. Then (K, V)* is non evasive *if $K = \emptyset$, $K = \{\emptyset, \{v\}\}$ for some $v \in V$, or $\exists v \in V$ such that $(\mathrm{cost}(K, v), V \setminus \{v\})$ and $(\mathrm{link}(K, v), V \setminus \{v\})$ are non evasive.*

Definition 2.7. *Let (K, V) be a simplicial complex. A simplex $f \in K$ is called a* free face *if it has one, and only one, coface. Since the coface of a free face $f \in K$ is unique, we can refer unambiguously to it by the notation $co(f) \in K$.*

Remark 2.5. *According to the definition, if the dimension of a free face $f \in K$ is d, that of its coface $co(f)$ is $d + 1$. Furthermore, $co(f)$ must be a facet.*

Definition 2.8. *Given a simplicial complex (K, V) and a free face $f \in K$, the pair $(f, co(f))$ is called an* elementary collapse datum*. From an elementary collapse datum $(f, co(f))$, we define a new simplicial complex $(K \setminus \{f, co(f)\}, W_f)$, where $W_f = V \setminus \{v\}$ when $(f, co(f)) = (\emptyset, \{v\})$ or $(f, co(f)) = (\{v\}, co(\{v\}))$; otherwise $W_f = V$. The simplicial complex $(K \setminus \{f, co(f)\}, W_f)$ is called the* elementary collapse *of (K, V) through $(f, co(f))$, and we also say that (K, V)* elementarily collapses *to $(K \setminus \{f, co(f)\}, W_f)$.*

A collapse *from a simplicial complex (K, V) to another one (L, W) is defined by a sequence of elementary collapses $((f_1, co(f_1)), \ldots, (f_r, co(f_r)))$ where the last simplicial complex obtained is (L, W). We say then that (K, V)* collapses to *(L, W). The case $r = 0$ is included, so that (K, V) is considered a (trivial) collapse of itself.*

A simplicial complex (K, V) is called collapsible *if it collapses to a simplicial complex of the form $(\emptyset, W)$ for some vertex set $W \subseteq V$ (including $W = \emptyset$).*

Remark 2.6. *Let us stress that, contrary to the situation about evasiveness, the vertex set V plays no role when considering collapses (they are related to the geometrical/simplicial nature of a simplicial complex and not to its combinatorial one as a Boolean function). So, sometimes, when dealing with collapses we will simply write K instead of (K, V).*

3. Cones and collapses

The following property is well-known.

Proposition 3.1. *A cone is non evasive.*

Proof. Let (K, V) be a cone and let $v \in V$ be one of its possible peeks. The proof is by induction over $n = \#V$, the cardinality of V. If $n \leq 1$, $K = \emptyset$ or $K = \{\emptyset, \{v\}\}$ and in both cases K is non evasive. Assuming that $n > 1$, we can choose a vertex $w \in V$ such that $w \neq v$. By direct inspection, we check that $(\mathrm{cost}(K, w), V \setminus \{w\})$ and $(\mathrm{link}(K, w), V \setminus \{w\})$ are both cones with peek v. Then, by induction hypothesis, both are non evasive, and the proof is completed. □

A series of lemmas relating cones and collapses, which will be used in the sequel, follows. We omit the proofs, they are simple and repetitive.

Lemma 3.1. *Let (B_1, V_{B_1}) and (B_2, V_{B_2}) be simplicial complexes such that $B_2 \subseteq B_1$ and $V_{B_2} \subseteq V_{B_1}$, and $v \notin V_{B_1}$. Then $(v * B_1, V_{B_1} \cup \{v\})$ collapses to $(v * B_2, V_{B_2} \cup \{v\})$.*

Corollary 3.1. *A cone is collapsible.*

Lemma 3.2. *Let (B, V_B) and (C, V_C) be simplicial complexes such that (B, V_B) collapses to (C, V_C), and $v \notin V_B$. Then $v * B$ collapses to $B \cup (v * C)$.*

Lemma 3.3. *Let* (A, V_A)*,* (B, V_B) *and* (C, V_C) *be simplicial complexes such that* $B \subseteq A$*,* $V_B \subseteq V_A$ *and* (B, V_B) *collapses to* (C, V_C)*. If* $v \notin V_A$*, then* $v * A$ *collapses to* $B \cup (v * C)$*.*

Lemma 3.4. *Let* (A, V_A) *and* (B, V_B) *be simplicial complexes such that* $B \subseteq A$ *and* $V_B \subseteq V_A$*, with* (B, V_B) *collapsible, and* $v \notin V_A$*. Then* $(A \cup (v * B), V_A \cup \{v\})$ *collapses to* (A, V_A)*.*

The following proposition, needed in the sequel, is well-known (see, for instance, [8]), but we include an original proof based on Lemma 3.4.

Proposition 3.2. *A non evasive simplicial complex is collapsible.*

Proof. Let (K, V) be a non evasive simplicial complex. We organize the proof by induction on n, the cardinality of V. If $n = 0$, $K = \emptyset$ is collapsible. If $n = 1$, $V = \{v_1\}$ and there are two non evasive simplicial complexes: (1) $K = \emptyset$ is collapsible, and (2) $K = \{\emptyset, \{v_1\}\}$ is collapsible, too (because $\emptyset$ is a free face, and so $(\emptyset, \{v_1\})$ defines an elementary collapse to $\emptyset$).

In the inductive case, applying the decomposition from Proposition 2.2 on a vertex v producing a costar and a link which are non evasive and instantiating Lemma 3.4 with $A := \mathrm{cost}(K, v)$ and $B := \mathrm{link}(K, v)$ (since the number of vertices in the link is less than n and it is non evasive, it is collapsible by induction hypothesis), we get a collapse from K to $\mathrm{cost}(K, v)$. Again by induction hypothesis, $\mathrm{cost}(K, v)$ collapses to $\emptyset$, and composing both collapses, we get that (K, V) is collapsible. □

We introduce a corollary that leads us to Lemma 3.5, a generalisation of the previous lemmas (we have included the previous lemmas since it is the way we got to Lemma 3.5, and, in addition, they are the base cases to prove it).

Corollary 3.2. *Let* (A, V_A) *be a collapsible simplicial complex and* $v \notin V_A$*. Then* $(v * A, V_A \cup \{v\})$ *collapses to* (A, V_A)*.*

Proof. One can choose to instantiate Lemma 3.4 with $A := B$, or Lemma 3.2 with $A := B$ and $C := \emptyset$. □

Lemma 3.5. *Let* $(A, V_A), (B, V_B), (C, V_C), (D, V_D)$ *be simplicial complexes such that* $C \subseteq A$*,* $V_C \subseteq V_A$*,* $D \subseteq B$*,* $V_D \subseteq V_B$*,* (A, V_A) *collapses to* (B, V_B) *and* (C, V_C) *collapses to* (D, V_D)*. Let* $v \notin V_A$*. Then* $A \cup (v * C)$ *collapses to* $B \cup (v * D)$*.*

Proof. First, we manipulate the collapsing sequence from (C, V_C) to (D, V_D) as in the proofs of Lemma 3.2 or Lemma 3.4 to get a collapse from $A \cup (v * C)$ to $A \cup (v * D)$. Then, we check that the sequence of elementary collapses from (A, V_A) to (B, V_B), is also a valid sequence from $A \cup (v * D)$ to $B \cup (v * D)$. Composing both collapses we get the result. □

The following result is symmetrical to Lemma 3.3.

Corollary 3.3. *Let* (A, V_A)*,* (B, V_B) *and* (C, V_C) *be simplicial complexes such that* $C \subseteq B$*,* $V_C \subseteq V_B$*, and* (A, V_A) *collapses to* (B, V_B)*. Let* $v \notin V_A$*. Then* $A \cup (v * B)$ *collapses to* $v * C$*.*

Proof. Renaming in Lemma 3.5 $B := C$, $C := B$ and $D := C$, we get that $A \cup (v * B)$ collapses to $C \cup (v * C) = v * C$. □

Definition 3.1. *Let* $V \neq \emptyset$ *and* $\#V = n$*,* (K, V) *be a simplicial complex, and let* $(v_1, \ldots, v_n)$ *be an ordering of the elements of* V*. Then* (K, V) is ordered non evasive with respect to $(v_1, \ldots, v_n)$ *if either* (K, V) *is a cone with peek* v_1*, or* $(\mathrm{cost}(K, v_1), V \setminus \{v_1\})$ *and* $(\mathrm{link}(K, v_1), V \setminus \{v_1\})$ *are ordered non evasive with respect to* $(v_2, \ldots, v_n)$*.*

Proposition 3.3. *Let* (K, V) *be an ordered non evasive simplicial complex with respect to some ordering of* V*. Then* (K, V) *is non evasive.*

Proof. Since a cone is non evasive (Proposition 3.1), the proof follows easily by induction on the cardinality of V. □

Now, we mimic, in the ordered setting, the layered approach by Barmak and Minian (see [3]).

Definition 3.2. *Let* $V \neq \emptyset$ *and* $\#V = n$*,* (K, V) *be a simplicial complex, and let* $(v_1, \ldots, v_n)$ *be an ordering of the elements of* V*. Then* (K, V) is ordered 0-collapsible with respect to $(v_1, \ldots, v_n)$ *if either* (K, V) *is a cone with peek* v_1*, or* $(\mathrm{cost}(K, v_1), V \setminus \{v_1\})$ *is ordered* 0*-collapsible with respect to* $(v_2, \ldots, v_n)$ *and* $(\mathrm{link}(K, v_1), V \setminus \{v_1\})$ *is a cone with peek* v_2*.*

Remark 3.1. *Let us note that, in the previous definition, if* $V = \{v_1\}$ *(a singleton), and* (K, V) *is not a cone (this implies that* $K = \{\emptyset\}$*, because the other two possibilities, namely* $K = \emptyset$ *and* $K = \{\emptyset, \{v_1\}\}$*, define cones), then* $(\mathrm{cost}(K, v_1), V \setminus \{v_1\})$ *is not ordered* 0*-collapsible (because its vertex set is empty). Then, when asking about* $(\mathrm{link}(K, v_1), V \setminus \{v_1\})$ *we know that* $n > 1$*, and we can ask whether* v_2 *is a peek.*

Definition 3.3. *Let* $V \neq \emptyset$ *and* $\#V = n$*,* (K, V) *be a simplicial complex, and let* $(v_1, \ldots, v_n)$ *be an ordering of the elements of* V*. Let* m *be a natural number (*$m > 0$*). Then* (K, V) is ordered m-collapsible with respect to $(v_1, \ldots, v_n)$ *if either* (K, V) *is a cone with peek* v_1*, or* $(\mathrm{cost}(K, v_1), V \setminus \{v_1\})$ *is* m*-collapsible with respect to* $(v_2, \ldots, v_n)$ *and* $(\mathrm{link}(K, v_1), V \setminus \{v_1\})$ *is* $(m-1)$*-collapsible with respect to* $(v_2, \ldots, v_n)$*.*

The proofs of the following two lemmas are straightforward and are therefore omitted.

Lemma 3.6. *Let (K,V) be a simplicial complex such that $\#V \leq 2$. The following properties are equivalent: (1) (K,V) is a cone; (2) (K,V) is non evasive; (3) (K,V) is ordered non evasive with respect to any permutation of V (there are at most 2); (4) (K,V) is ordered m-collapsible with respect to any permutation of V and $m \geq 0$.*

Lemma 3.7. *Let (K,V) be a cone. Then (K,V) is ordered m-collapsible with respect to any ordering of V and for $m \geq 0$.*

Proposition 3.4. *Let $m \geq 0$ and (K,V) be an ordered m-collapsible simplicial complex with respect to $(v_1,\ldots,v_n)$. Then (K,V) is ordered $(m+1)$-collapsible with respect to $(v_1,\ldots,v_n)$.*

Proof. We divide the proof into two parts. First, we prove that 0-collapsible implies 1-collapsible. Second, taking the previous step as base case, we prove the statement by induction over m.

To prove that 0-collapsible implies 1-collapsible we apply induction over $n = \#V$. Thanks to Lemma 3.6, we can go directly to the inductive case. If (K,V) is a cone with peek v_1, the proof is finished by Lemma 3.7. Otherwise, $(\mathrm{cost}(K,v_1), V \setminus \{v_1\})$ is 1-collapsible with respect to $(v_2,\ldots,v_n)$ by induction hypothesis, and $(\mathrm{link}(K,v_1), V \setminus \{v_1\})$ is 0-collapsible again by Lemma 3.6. And this ends the proof that 0-collapsible implies 1-collapsible.

The second part follows the same pattern, taking into account that $(\mathrm{link}(K,v_1), V \setminus \{v_1\})$ $(m-1)$-collapsible implies $(\mathrm{link}(K,v_1), V \setminus \{v_1\})$ m-collapsible, by the hypothesis induction over m. □

The proof of the following proposition follows a similar pattern to the previous one (first, prove that 0-collapsible implies non evasive; then, taking the previous step as base case, the statement is proved by induction over m).

Proposition 3.5. *Let $m \geq 0$ and (K,V) be an ordered m-collapsible simplicial complex with respect to $(v_1,\ldots,v_n)$ (with $V \neq \emptyset$ and $\#V = n$). Then (K,V) is ordered non evasive with respect to $(v_1,\ldots,v_n)$.*

Proposition 3.6. *Let (K,V) be an ordered non evasive simplicial complex with respect to $(v_1,\ldots,v_n)$ (with $V \neq \emptyset$ and $\#V = n$). Then there exits $m \geq 0$ such that (K,V) is ordered m-collapsible with respect to $(v_1,\ldots,v_n)$.*

Proof. We define $m := n$, the cardinality of V, and prove the statement by induction over n. By Lemma 3.6, we can go ahead with the inductive case, with $n > 2$. If (K,V) is a cone, it is ordered m-collapsible with respect to any permutation of V and for all non-negative integer number m (by Lemma 3.7). Therefore, we can assume that $(\mathrm{cost}(K,v_1), V \setminus \{v_1\})$ and $(\mathrm{link}(K,v_1), V \setminus \{v_1\})$ are ordered non evasive with respect to $(v_2,\ldots,v_n)$. By induction hypothesis, $(\mathrm{link}(K,v_1), V \setminus \{v_1\})$ and $(\mathrm{cost}(K,v_1), V \setminus \{v_1\})$ are ordered $(n-1)$-collapsible with respect to $(v_2,\ldots,v_n)$. By Proposition 3.4, $(\mathrm{cost}(K,v_1), V \setminus \{v_1\})$ also is ordered n-collapsible with respect to $(v_2,\ldots,v_n)$, ending the proof. □

Remark 3.2. *Let us note that n is an upper bound of m, but it is a strict one. For instance, a cone with peek the first vertex in the ordering is obviously ordered 0-collapsible, independently of the cardinality of its vertex set.*

4. Main theorem

As explained in the introduction, our goal is to find a simplicial property hold by every ordered non evasive simplicial complex but not by all the non evasive simplicial complexes, and our inspiration comes from two sources. First, convex union representable simplicial complexes collapse to its stars [5] (and K collapses to $v * \mathrm{link}(K,v)$ if and only if $\mathrm{cost}(K,v)$ collapses to $\mathrm{link}(K,v)$). Second, there exist non evasive simplicial complexes with exactly two free faces [1]. So, our idea is to find a simplicial property of each ordered non evasive simplicial complex K ensuring that there are enough free faces in K. This is the property we found:

Theorem 4.1. *Let (K,V) be an ordered non evasive simplicial complex with respect to $(v_1,\ldots)$ (with $V \neq \emptyset$). Then $(\mathrm{cost}(K,v_1), V \setminus \{v_1\})$ collapses to $(\mathrm{link}(K,v_1), V \setminus \{v_1\})$.*

According to the propositions of the previous section, it is enough to prove the theorem for ordered m-collapsible simplicial complexes, for all $m \geq 0$. The central part of the proof is by induction over the measure $(m,n) < (m',n')$ if and only if $m < m'$, or $m = m'$ and $n < n'$, with $n = \#V$.

First, we need some auxiliary lemmas. Particularly, we need to prove that, under good circumstances, an ordered non evasive simplicial complex with respect to a permutation $(v_1,v_2,v_3,\ldots,v_n)$ is also ordered non evasive with respect to $(v_2,v_1,v_3,\ldots,v_n)$. The result is based on the following properties that can be proved by direct inspection.

Lemma 4.1. *Let (K,V) be a simplicial complex and $\#V \geq 2$. If $v_1 \in V, v_2 \in V$ and $v_1 \neq v_2$, then $\mathrm{cost}(\mathrm{cost}(K,v_1),v_2) = \mathrm{cost}(\mathrm{cost}(K,v_2),v_1)$, $\mathrm{cost}(\mathrm{link}(K,v_1),v_2) = \mathrm{link}(\mathrm{cost}(K,v_2),v_1)$, and $\mathrm{link}(\mathrm{link}(K,v_1),v_2) = \mathrm{link}(\mathrm{link}(K,v_2),v_1)$.*

Lemma 4.2. *Let (K, V) be an ordered m-collapsible simplicial complex with respect to $(v_1, v_2, v_3, \ldots, v_n)$ such that $m > 0$ and $n = \#V > 2$. If K is not a cone with peek v_1 and neither $\mathrm{cost}(K, v_1)$ nor $\mathrm{link}(K, v_1)$ are cones with peek v_2, then (K, V) is ordered m-collapsible with respect to $(v_2, v_1, v_3, \ldots, v_n)$.*

Proof. First, let us note that K is not a cone with peek v_2 (otherwise, $\mathrm{cost}(K, v_1)$ and $\mathrm{link}(K, v_1)$ would also be cones with peek v_2, contrary to the hypotheses of the lemma). In the same vein, $\mathrm{cost}(K, v_2)$ and $\mathrm{link}(K, v_2)$ are not both cones with peek v_1 (in that case, K would also be a cone with peek v_1). Let us observe that, even if these cases could exist, (K, V) would be ordered m-collapsible with respect to $(v_2, v_1, v_3, \ldots, v_n)$. This observation will be relevant in a later proof.

Remark that, according to the hypotheses, $\mathrm{cost}(\mathrm{cost}(K, v_1), v_2)$ is ordered m-collapsible with respect to $(v_3, \ldots, v_n)$ and $\mathrm{cost}(\mathrm{link}(K, v_1), v_2)$ and $\mathrm{link}(\mathrm{cost}(K, v_1), v_2)$ are ordered $(m-1)$-collapsible with respect to the same permutation. If $m = 1$ then $\mathrm{link}(\mathrm{link}(K, v_1), v_2)$ is a cone with peek v_3, and if $m > 1$, $\mathrm{link}(\mathrm{link}(K, v_1), v_2)$ is ordered $(m-2)$-collapsible with respect to $(v_3, \ldots, v_n)$ (when $m > 1$, this includes also the case when $\mathrm{link}(\mathrm{link}(K, v_1), v_2)$ is a cone with peek v_3). Applying Lemma 4.1, we get that $\mathrm{cost}(\mathrm{cost}(K, v_2), v_1)$ is ordered m-collapsible with respect to $(v_3, \ldots, v_n)$ and $\mathrm{cost}(\mathrm{link}(K, v_2), v_1)$ and $\mathrm{link}(\mathrm{cost}(K, v_2), v_1)$ are ordered $(m-1)$-collapsible with respect to the same permutation. If $m = 1$ then $\mathrm{link}(\mathrm{link}(K, v_2), v_1)$ is a cone with peek v_3, and if $m > 1$, $\mathrm{link}(\mathrm{link}(K, v_2), v_1)$ is ordered $(m-2)$-collapsible with respect to $(v_3, \ldots, v_n)$.

Going back to the decomposition of K corresponding to the permutation $(v_2, v_1, v_3, \ldots, v_n)$, the possible cases are:

1. $\mathrm{link}(K, v_2)$ is a cone with peek v_1 but $\mathrm{cost}(K, v_2)$ is not a cone with peek v_1.
2. $\mathrm{cost}(K, v_2)$ is a cone with peek v_1 but $\mathrm{link}(K, v_2)$ is not a cone with peek v_1.
3. The case in which neither $\mathrm{cost}(K, v_2)$ nor $\mathrm{link}(K, v_2)$ are cones with peek v_1 is dealt with using similar reasoning.

In the first case, according to Lemma 3.7, $\mathrm{link}(K, v_2)$ is ordered $(m-1)$-collapsible with respect to $(v_1, v_3, \ldots, v_n)$. Now, we know that $\mathrm{cost}(\mathrm{cost}(K, v_2), v_1)$ is ordered m-collapsible and $\mathrm{link}(\mathrm{cost}(K, v_2), v_1)$ is ordered $(m-1)$-collapsible, both with respect to the same ordering $(v_3, \ldots, v_n)$; therefore $\mathrm{cost}(K, v_2)$ is ordered m-collapsible with respect to $(v_1, v_3, \ldots, v_n)$. Joining both properties, we get the result in this case. The remaining two cases are dealt with using a similar reasoning. □

Before starting the proof of Theorem 4.1 by induction over (m, n), we prove the base case when $m = 0$.

Proposition 4.1. *Let (K, V) be an ordered 0-collapsible simplicial complex with respect to $(v_1, \ldots)$ (with $V \neq \emptyset$). Then $(\mathrm{cost}(K, v_1), V \setminus \{v_1\})$ collapses to $(\mathrm{link}(K, v_1), V \setminus \{v_1\})$.*

Proof. The proof is by induction over $n = \#V$. If $n \leq 2$ a direct inspection (see Lemma 3.6) proves the result. Otherwise, if K is a cone with peek v_1, $\mathrm{cost}(K, v_1) = \mathrm{link}(K, v_1)$ (Lemma 2.1) and the proof is completed. In the remaining case, $\mathrm{link}(K, v_1)$ is a cone with peek v_2 and $\mathrm{cost}(K, v_1)$ is ordered 0-collapsible with respect to $(v_2, \ldots, v_n)$. By induction hypothesis, we know that $\mathrm{cost}(\mathrm{cost}(K, v_1), v_2)$ collapses to $\mathrm{link}(\mathrm{cost}(K, v_1), v_2)$. Now, we can instantiate Corollary 3.3, with $A = \mathrm{cost}(\mathrm{cost}(K, v_1), v_2)$, $B = \mathrm{link}(\mathrm{cost}(K, v_1), v_2)$ and $C = \mathrm{link}(\mathrm{link}(K, v_1), v_2)$ (= $\mathrm{cost}(\mathrm{link}(K, v_1), v_2)$, because $\mathrm{link}(K, v_1)$ is a cone with peek v_2), obtaining that $\mathrm{cost}(\mathrm{cost}(K, v_1), v_2) \cup (v_2 * \mathrm{link}(\mathrm{cost}(K, v_1), v_2))$ collapses to $v_2 * \mathrm{link}(\mathrm{link}(K, v_1), v_2) = \mathrm{link}(K, v_1)$. Taking into account the main decomposition from Lemma 2.2, $\mathrm{cost}(\mathrm{cost}(K, v_1), v_2) \cup (v_2 * \mathrm{link}(\mathrm{cost}(K, v_1), v_2)) = \mathrm{cost}(K, v_1)$, and the proof is completed. □

We present now the proof of the main theorem, by induction over (m, n).

Proof of Theorem 4.1. The base case when $m = 0$ is proved by the previous proposition. The case $m = 1$ is proved by a similar argument. So, given a simplicial complex (K, V) ordered m-collapsible with respect to $(v_1, \ldots, v_n)$, we can assume that the result is true for any other simplicial complex m'-collapsible with $m' < m$ and we proceed by induction over $n = \#V$. In the cases where $n \leq 2$ or K is a cone with peek v_1, we know that the theorem holds. So, let $m > 2$, $n > 2$ and K not a cone with peek v_1. We proceed by distinguishing cases depending on the reasons why $\mathrm{cost}(K, v_1)$ is ordered m-collapsible and $\mathrm{link}(K, v_1)$ is ordered $(m-1)$-collapsible (both with respect to $(v_2, \ldots, v_n)$).

1. If $\mathrm{cost}(K, v_1)$ and $\mathrm{link}(K, v_1)$ are both cones with peek v_2, by applying Lemma 3.1 with $B_1 = \mathrm{cost}(\mathrm{cost}(K, v_1), v_2)$ ($= \mathrm{link}(\mathrm{cost}(K, v_1), v_2)$) and $B_2 = \mathrm{cost}(\mathrm{link}(K, v_1), v_2)$ ($=\mathrm{link}(\mathrm{link}(K, v_1), v_2)$), we get the result.
2. If $\mathrm{cost}(K, v_1)$ is a cone with peek v_2 and $\mathrm{link}(K, v_1)$ is not, by induction hypothesis, we know that $\mathrm{cost}(\mathrm{link}(K, v_1), v_2)$ collapses to $\mathrm{link}(\mathrm{link}(K, v_1), v_2)$. By applying Lemma 3.3 with $A = \mathrm{cost}(K, v_1)$, $B = \mathrm{cost}(\mathrm{link}(K, v_1), v_2)$ and $C = \mathrm{link}(\mathrm{link}(K, v_1), v_2)$, and then Lemma 2.2, the result holds.
3. If $\mathrm{link}(K, v_1)$ is a cone with peek v_2 and $\mathrm{cost}(K, v_1)$ is not, by induction hypothesis, we know that $\mathrm{cost}(\mathrm{cost}(K, v_1), v_2)$ collapses to $\mathrm{link}(\mathrm{cost}(K, v_1), v_2)$. By applying Corollary 3.3 with $A = \mathrm{cost}(\mathrm{cost}(K, v_1), v_2)$, $B = \mathrm{link}(\mathrm{cost}(K, v_1), v_2)$ and $C = \mathrm{link}(\mathrm{link}(K, v_1), v_2) (= \mathrm{cost}(\mathrm{link}(K, v_1), v_2))$, and then Lemma 2.2, the result holds.

4. In this case, neither $\text{cost}(K, v_1)$ nor $\text{link}(K, v_1)$ are cones with peek v_2, and Lemma 4.2 can be applied: (K, V) is ordered m-collapsible with respect to $(v_2, v_1, v_3, \ldots, v_n)$. Next we will prove that, in this case, $\text{cost}(\text{cost}(K, v_1), v_2)$ collapses to $\text{cost}(\text{link}(K, v_1), v_2)$ and $\text{link}(\text{cost}(K, v_1), v_2)$ collapses to $\text{link}(\text{link}(K, v_1), v_2)$. We can apply Lemma 3.5 with $A = \text{cost}(\text{cost}(K, v_1), v_2)$, $B = \text{cost}(\text{link}(K, v_1), v_2)$, $C = \text{link}(\text{cost}(K, v_1), v_2)$ and $D = \text{link}(\text{link}(K, v_1), v_2)$, and applying Lemma 2.2 twice the result follows.

In the fourth case, we know (as seen in the proof of Lemma 4.2) that K is not a cone with peek v_2. So, we can consider the four subsimplicial complexes $\text{cost}(\text{cost}(K, v_2), v_1)$, $\text{link}(\text{cost}(K, v_2), v_1)$, $\text{cost}(\text{link}(K, v_2), v_1)$ and $\text{link}(\text{link}(K, v_2), v_1)$. Either for being $\text{cost}(K, v_2)$ or $\text{link}(K, v_2)$ cones with peek v_1 or by induction hypothesis, we obtain that $\text{cost}(\text{cost}(K, v_2), v1)$ collapses to $\text{link}(\text{cost}(K, v_2), v_1)$ and $\text{cost}(\text{link}(K, v_2), v_1)$ collapses to $\text{link}(\text{link}(K, v_2), v_1)$ (trivial collapses in the case of a cone). Applying Lemma 4.1, we complete the proof of the main theorem. □

5. Consequences of the main theorem

Let (K, V) be a simplicial complex, we define the set $V(K) := \{v \in V \mid \{v\} \in K\}$. Note that, given K, $V(K)$ is the smallest vertex set such that $(K, V(K))$ is a simplicial complex.

Definition 5.1. *A simplicial complex is called* pure of dimension d *(or d-*complex*) if all its facets are of dimension d.*

The proof of the following lemma is straightforward and is therefore omitted.

Lemma 5.1. *Let (K, V) be a d-complex with $d > 0$, and let $v \in V$ such that $\{v\} \in K$. Then $\text{link}(K, v)$ is a $(d-1)$-complex.*

Now, let us state and prove the following well-known proposition (see, for instance, [1]). We include a proof, because it is important the fact that, if the simplicial complex is not a tree, the two free faces contain the *pivotal* vertex v.

Proposition 5.1. *Let $(K, V(K))$ be a non evasive d-complex with $d > 0$. Then $(K, V(K))$ has at least two free faces.*

Proof. The proof is by induction on d, the *dimension of* K. If d is 1, $(K, V(K))$ must be a tree, and from the hypothesis of the statement, it has at least one edge, so it has at least two free vertices, as claimed in the statement.

If d is greater than 1, since $(K, V(K))$ is non evasive there exits a vertex $v \in V$ such that the corresponding costar and link are non evasive. We fix our attention on $\text{link}(K, v)$. It is non evasive and it is a $(d-1)$-complex (according to Lemma 5.1), so by induction hypothesis there are at least two free faces f_1 and f_2 in the link. Now it is easy to check that $\{v\} \cup f_1$ and $\{v\} \cup f_2$ are different free faces of K. □

We are now ready to introduce the first consequence of Theorem 4.1.

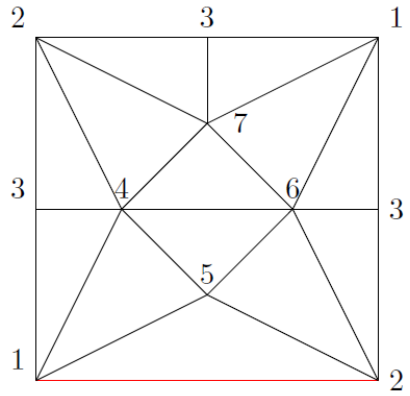
Corollary 5.1. *Let $(K, V(K))$ be an ordered non evasive d-complex with respect to $(v_1, \ldots)$ such that it is not a cone with peek v_1, and let $d > 1$. Then $(K, V(K))$ has at least three free faces.*

Proof. Applying Theorem 4.1, we know that $\text{cost}(K, v_1)$ collapses to $\text{link}(K, v_1)$. Since $\text{link}(K, v_1) \neq \text{cost}(K, v_1)$ (by Proposition 2.1), the collapse from $\text{cost}(K, v_1)$ to $\text{link}(K, v_1)$ is not trivial, and therefore there exists a free face f_3 of $\text{cost}(K, v_1)$ which is not a simplex of $\text{link}(K, v_1)$; therefore the coface of f_3 *in* $\text{cost}(K, v_1)$ is not a simplex of $\text{link}(K, v_1)$. Now, we claim that this free face f_3 of $\text{cost}(K, v_1)$ is also a free face of K. First, let us note that $co(f_3)$, which is a facet in $\text{cost}(K, v_1)$ must also be a facet in K. Otherwise, $\{v_1\} \cup co(f_3) \in K$, contradicting that $co(f_3) \notin \text{link}(K, v_1)$. Let us assume that f_3 has a coface in K different from $co(f_3)$. Then that new coface should be $\{v_1\} \cup f_3 \in K$, contradicting that $f_3 \notin \text{link}(K, v_1)$. Therefore, f_3 is a free face of K such that $v_1 \notin f_3$. Since the dimension of $(K, V(K))$ is strictly greater than 1, according to the proof of Proposition 5.1, it has two free faces, both containing v_1, and so both are different from f_3. □

Adiprasito, Benedetti, and Lutz [1] constructed a non evasive 2-complex with exactly two free faces (see Figure 5.1(a)). Since that example is not a cone, that same simplicial complex proves that ROBDDs are not enough to decide evasiveness:

Corollary 5.2. *There exists a non evasive simplicial complex (K, V) which is not ordered non evasive with respect to any permutation of V.*

The 2-complex with exactly two free faces described in [1] is constructed as the barycentric subdivision of the 2-complex Σ_2 represented in Figure 5.1(a); we denote it by $\text{sd}\Sigma_2$. Let $\sigma = (1, 2)$ be the only free face in Σ_2 (displayed in red in Figure 5.1(a)) and v_σ the corresponding vertex in $\text{sd}\Sigma_2$ (see Figure 5.1(b)). Note that v_σ is the only vertex of $\text{sd}\Sigma_2$ such that $\text{link}(\text{sd}\Sigma_2, v_\sigma)$ is non evasive (the rest of links are not collapsible complexes). Moreover, the two free faces of $\text{cost}(\text{sd}\Sigma_2, v_\sigma)$ belong to $\text{link}(\text{sd}\Sigma_2, v_\sigma)$ (see Figure 5.1(b)). Then, any sequence of elementary collapses from $\text{cost}(\text{sd}\Sigma_2, v_\sigma)$ does not preserve $\text{link}(\text{sd}\Sigma_2, v_\sigma)$, illustrating that Theorem 4.1 does not hold in this example (since its premises are not satisfied).

(a): 

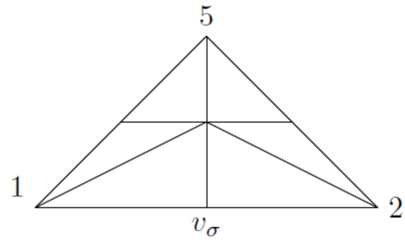
(b): 

Figure 5.1: (a): The 2-complex Σ_2. **(b):** Zoom on the barycentric subdivision of the bottom triangle of 2-complex Σ_2 of **(a)**.

From Theorem 4.1 we can establish that any ordered non evasive simplicial complex is anti-collapsible, describing explicitly how an ordered non evasive simplicial complex (K, V) anti-collapses to the standard simplex $\Delta^{(\#V-1)}$. Before proving that, we state the following property of ordered non evasive simplicial complexes.

Corollary 5.3. *Let $V \neq \emptyset$ and (K, V) be an ordered non evasive simplicial complex with respect to $(v_1, \ldots)$. Then K collapses to $v_1 * \mathrm{link}(K, v_1)$, the closed star of K with respect to v_1.*

Proof. The sequence of elementary collapses from $\mathrm{cost}(K, v_1)$ to $\mathrm{link}(K, v_1)$ shows that K collapses to $v_1 * \mathrm{link}(K, v_1)$ (the decomposition in Proposition 2.2 can be used and then Corollary 3.3). □

Corollary 5.4. *Let $V \neq \emptyset$ and (K, V) be an ordered non evasive simplicial complex with respect to $(v_1, \ldots)$. Then Δ^{n-1} collapses to K and K collapses to $v_1 * \mathrm{link}(K, v_1)$.*

Proof. As the second collapse is that of Corollary 5.3, we focus on the first one. The standard simplex Δ^{n-1} is a cone with peek v_1 and let A be its basis. We apply Lemma 3.3 with $B := \mathrm{cost}(K, v_1)$ and $C := \mathrm{link}(K, v_1)$ (the hypotheses of Lemma 3.3 are satisfied thanks to Theorem 4.1). Now, from the thesis of Lemma 3.3 and using the decomposition in Proposition 2.2, we get the result. □

6. Conclusions and further work

In this paper we have answered, in the negative, whether ROBDDs are enough to determine the evasiveness of Boolean functions. This has been achieved by looking for a simplicial property satisfied by all the ordered non evasive simplicial complexes: namely, its costar collapses to its link. Several questions remain open. First, the relationship among m-collapsible complexes and its ordered counterpart should be elucidated. Second, it would be also interesting to know if there exist non-trivial ordered non evasive simplicial complexes with exactly three free faces. Furthermore, in a parallel project, we are formalizing all our developments in the Isabelle/HOL proof assistant [7]. The reason is that the problem, being very combinatorial in nature, has shown itself to be very prone to our mistakes, and then we prefer to ensure all our steps by means of a mechanical tool. This project is almost finished and is a continuation of the library reported in [2].

Acknowledgement

This work has been partially supported by projects PID2024-155834NB-I00 and PID2024-157733NB-I00 funded by MICIU/AEI/10.13039/501100011033 and by ERDF/EU, and by project AFIANZA 2024/03 funded by La Rioja Government.

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