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Research Article

Hamiltonicity in directed Toeplitz graphs with $s_1 = 1$ and $s_2 = 3$

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Abstract

A directed Toeplitz graph $T_n\langle s_1,\dots,s_p;t_1,\dots,t_q\rangle$ with vertices $1,2,\dots,n$, where an edge $(i,\,j)$ occurs if and only if $j-i=s_l$ or $i-j=t_k$ for some $1\le l\le p$ and $1\le k\le q$, is a digraph whose adjacency matrix is a Toeplitz matrix. In this paper, we study Hamiltonicity in directed Toeplitz graphs with $s_1=1$, $s_2=3$ and $s_3\le 7$.

Keywords: adjacency matrix; Toeplitz graph; Hamiltonian graph.

2020 Mathematics Subject Classification: 05C20, 05C45.

1. Introduction

A Toeplitz graph, whether directed or undirected, is a graph whose adjacency matrix is a Toeplitz matrix—a square matrix in which all elements along each diagonal parallel to the main diagonal are constant. This structure implies that the presence or absence of edges in the graph follows a repetitive pattern based on vertex differences. Toeplitz graphs offer an interesting blend of algebraic structure and combinatorial properties, making them valuable for studying Hamiltonian cycles, connectivity, and spectral graph theory. Various properties of Toeplitz graphs, including colorability, planarity, bipartiteness, connectivity, cycle discrepancy, edge irregularity strength, decomposition, labeling, and metric dimension, have been extensively studied; for example, see [1–12, 14, 28].

In this paper, we consider only finite simple directed graphs. A directed Toeplitz graph, denoted $T_n\langle s_1,\dots,s_p;t_1,\dots,t_q\rangle$, is a digraph of order $n>\max\{s_p,t_q\}$, with vertices labeled $1,2,\dots,n$, where an edge $(i,\,j)$ occurs if and only if $j-i=s_l$ or $i-j=t_k$ for some $1\le l\le p$ and $1\le k\le q$.

The study of Hamiltonian properties in Toeplitz graphs was initiated by van Dal et al. in [29] and later explored further in [13, 27, 30]. Research on Hamiltonicity in directed Toeplitz graphs was first conducted by Malik and Zamfirescu in [26], followed by Malik [15–22], Malik and Qureshi [23, 24], and Malik and Ramezani [25].

In [19, 20], the Hamiltonicity of Toeplitz graphs $T_n\langle 1,3,s_3;t\rangle$ was fully examined for $s_3=4$, whereas in [24], a complete investigation was conducted for $s_3=5$. The cases $s_3=6$ and $s_3=7$ were explored in [22] and [25], respectively, where some conjectures were proposed. In this paper, we address most of these conjectures, refining or extending the previous investigations. We give a complete characterization of Hamiltonicity for $s_3=6$, resolving all previously unsettled cases, and we provide new results for $s_3=7$, where many conjectures are confirmed and several unresolved configurations are reduced to a small number of open cases. Our contributions combine constructive proofs of Hamiltonian cycles with computational verification using a Python-based algorithm.

2. Preliminaries

For a vertex a in $T_n\langle 1,3,7;t\rangle$ and $r\in\{4,8\}$, we define the path $A_a(r)$ in $T_n\langle 1,3,7;t\rangle$ as $A_a(r)=(a,a+1,a+r)$. We also define the path $B_a(8)$ in $T_n\langle 1,3,7;t\rangle$ as $B_a(8)=A_a(4)\cup(a+4,a+7,a+8)$. These paths are illustrated in Figures 2.1 and 2.2.

Remark 2.1. If the Toeplitz graph $T_n\langle 1,3,s_3;t\rangle$ has a Hamiltonian cycle that includes the edge $(n-2,n-1)$, then the graph $T_{n+(t-1)}\langle 1,3,s_3;t\rangle$ also possesses the same property. This is because a Hamiltonian cycle in $T_n\langle 1,3,s_3;t\rangle$ can be transformed into a Hamiltonian cycle in $T_{n+(t-1)}\langle 1,3,s_3;t\rangle$ by replacing the edge $(n-2,n-1)$ with the path $(n-2,n+1,n+2,\dots,n+(t-3),n+(t-2),n+(t-1),n-1)$, which preserves the same property. For instance, as illustrated in Figure 2.3, a Hamiltonian cycle in $T_{10}\langle 1,3,6;5\rangle$ is transformed into a Hamiltonian cycle in $T_{14}\langle 1,3,6;5\rangle$ by replacing the edge (8, 9) with the path (8, 11, 12, 13, 14, 9). This process can be repeated to extend the Hamiltonian cycle to $T_{18}\langle 1,3,6;5\rangle$, and so forth.

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Figure 2.1: Paths $A_a(r)$ in $T_n\langle 1,3,7;t\rangle$, where $r \in \{4,8\}$.

Figure 2.2: Path $B_a(8)$ in $T_n\langle 1,3,7;t\rangle$.

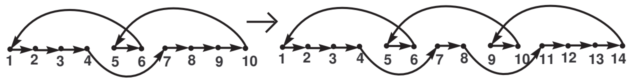

Figure 2.3: Hamiltonian cycles in $T_{10}\langle 1,3,6;5\rangle$ and $T_{14}\langle 1,3,6;5\rangle$.

3. Toeplitz graphs $T_n\langle 1,3,6;t\rangle$

In [22], it was established that the graph $T_n\langle 1,3,6;t\rangle$ is Hamiltonian for all n when $t \geq 12$ or $t \in \{5,10\}$. Additionally, for $t \in \{3,4,6,7,8,9,11\}$, it was shown that $T_n\langle 1,3,6;t\rangle$ is Hamiltonian for all n, except for a finite set of values. These exceptional cases were proposed as conjectures in [22]. In this section, we verify the non-Hamiltonicity of these conjectured exceptions using a Python-based algorithm, described in Algorithm 1.

Algorithm 1 Finding Hamiltonian Cycle in a Toeplitz Graph

```
Require: Graph adjacency matrix G of size n × n, starting vertex s
Ensure: Hamiltonian cycle if exists, otherwise an empty list
Initialize path P with −1 (unvisited), set P[0] = P[n] = s
function IsValidConnection(G, v, index, P)
    if G[P[index − 1]][v] = 0 or v is in P then
        return False
    end if
    return True
end function
function UtilHamiltonianCycle(G, P, index)
    if index = n then
        return G[P[index − 1]][P[0]] = 1
    end if
    for each vertex v in G do
        if IsValidConnection(G, v, index, P) then
            P[index] ← v
            if UtilHamiltonianCycle(G, P, index + 1) then
                return True
            end if
            P[index] ← −1                                    ▷ Backtrack
        end if
    end for
    return False
end function
function FindHamiltonianCycle(G, s)
    Initialize path P of size n + 1 with −1
    Set P[0] = P[n] = s
    if UtilHamiltonianCycle(G, P, 1) then
        return P
    else
        return ∅
    end if
end function
```

Through computational verification, we confirm the non-Hamiltonicity of the graph for these small values of n. In each case, the algorithm returns an empty list, indicating the absence of a Hamiltonian cycle. This verification leads to the following refined summary of the results reported in [22]:

- $T_n\langle 1,3,6;3\rangle$ is Hamiltonian if and only if $n \notin \{7,8,9,12,14,16\}$.
- $T_n\langle 1,3,6;4\rangle$ is Hamiltonian if and only if $n \neq 12$.
- $T_n\langle 1,3,6;6\rangle$ is Hamiltonian if and only if $n \notin \{10,14\}$.
- $T_n\langle 1,3,6;7\rangle$ is Hamiltonian if and only if $n \notin \{9,12\}$.
- $T_n\langle 1,3,6;8\rangle$ is Hamiltonian if and only if $n \neq 10$.
- $T_n\langle 1,3,6;9\rangle$ is Hamiltonian if and only if $n \neq 13$.
- $T_n\langle 1,3,6;11\rangle$ is Hamiltonian if and only if $n \neq 13$.

Also, in [22], it was established that $T_n\langle 1,3,6;2\rangle$ is Hamiltonian for all $n \equiv 0,1,3 \pmod 4$, while for $n \equiv 2 \pmod 4$, a conjecture has been proposed. In this section, we also prove the conjecture that $T_n\langle 1,3,6;2\rangle$ is non-Hamiltonian for $n \equiv 2 \pmod 4$ (see Theorem 3.1). This result allows us to restate the conclusion as follows:

- $T_n\langle 1,3,6;2\rangle$ is Hamiltonian if and only if $n \not\equiv 2 \pmod 4$.

Theorem 3.1. *$T_n\langle 1,3,6;2\rangle$ is non-Hamiltonian for $n \equiv 2 \pmod 4$.*

Proof. Assume, for contradiction, that $T_n\langle 1,3,6;2\rangle$ is Hamiltonian for $n \equiv 2 \pmod 4$. Then there exists a Hamiltonian cycle H, which must be expressible as the union of two directed paths: $H = H_{1\to n} \cup H_{n\to 1}$, where $H_{1\to n}$ visits some vertices from 1 to n, while the remaining vertices are visited by $H_{n\to 1}$. In $T_n\langle 1,3,6;2\rangle$, each vertex a is connected to $a+1$, $a+3$, $a+6$ (increasing edges), and $a-2$ (the only decreasing edge), providing the indices lie within valid bounds. Since $n \equiv 2 \pmod 4$, we can write $n = 4k+2$ for some integer k, which means the total number of vertices is even. To construct $H_{n\to 1}$, we attempt to descend from n to 1 using decreasing edge of length 2. The only consistent path is $(n, n-2, n-4, \ldots, 2, 3, 1)$, which covers all even vertices in descending order, then jumps to 3 and finally to 1. Next, we construct $H_{1\to n}$, avoiding all vertices (except 1 and n) already used in $H_{n\to 1}$, namely the vertices $2, 3, \ldots, n-2, n-4$ in a descending pattern. The only viable strategy for $H_{1\to n}$ is to use a pattern that alternates between increasing edges of length 6 and decreasing edges of length 2. This creates a repeating pattern of the form: $(1,7,5)$, $(5,11,9)$, $\ldots$, $(n-7, n-1, n-3)$, $(n-3, n)$, where each subpath $(a, a+6, a+4)$ consists of a forward step of length 6 followed by a backward step of length 2. However, this sequence successfully reaches $n-3$ only if $n-3 \equiv 0 \pmod 4$. But since $n \equiv \pmod 4$, it follows that $n-3 \equiv 3 \pmod 4$, which makes it impossible to reach $n-3$ using this pattern. Therefore, we cannot complete the path from 1 to n, and therefore no Hamiltonian cycle can exist. Thus, we conclude that the graph $T_n\langle 1,3,6;2\rangle$ is non-Hamiltonian for all $n \equiv 2 \pmod 4$. □

The computational verification of our theoretical argument further solidifies the result in the theorem. By applying Algorithm 1 implemented in Python, we confirm the non-Hamiltonicity of the graph for small values of $n \equiv 2 \pmod 4$. In each of these cases, the function consistently returns empty lists, thereby verifying that the graph $T_n\langle 1,3,6;2\rangle$ does not contain a Hamiltonian cycle for $n \equiv 2 \pmod 4$.

4. Toeplitz graphs $T_n\langle 1,3,7;t\rangle$ with odd t

In [25], it was shown that for odd values of t where $3 \le t \le 15$, the graph $T_n\langle 1,3,7;t\rangle$ is Hamiltonian if and only if n is even. Furthermore, for odd $t \ge 17$, it was shown that $T_n\langle 1,3,7;t\rangle$ is Hamiltonian for all even n, except in the following cases: (i) if $n \equiv 6, 10, \ldots, t-5 \pmod{(t-1)}$ when $t \equiv 3 \pmod 4$, or (ii) if $n \equiv 8, 12, \ldots, t-5 \pmod{(t-1)}$ when $t \equiv 1 \pmod 4$.

These two cases were stated as conjecture. In this section, we aim to prove this conjecture in Theorem 4.1. Specifically, for odd t, $T_n\langle 1,3,7;t\rangle$ is bipartite, and since a bipartite graph cannot contain a Hamiltonian cycle unless n is even, we can conclude the result for all odd $t \ge 3$ as follows: for odd $t \ge 3$, $T_n\langle 1,3,7;t\rangle$ is Hamiltonian if and only if n is even.

Theorem 4.1. *For odd $t \ge 17$, if $t \equiv 3 \pmod 4$, then $T_n\langle 1,3,7;t\rangle$ is Hamiltonian for all even $n \equiv 6, 10, \ldots, t-5 \pmod{(t-1)}$. For odd $t \ge 17$, if $t \equiv 1 \pmod 4$, then $T_n\langle 1,3,7;t\rangle$ is Hamiltonian for all even $n \equiv 8, 12, \ldots, t-5 \pmod{(t-1)}$.*

Proof. Let t be an odd number such that $t \ge 17$, and let n be an even number.
Case 1. $t \equiv 1 \pmod 4$ and $n \equiv 8, 12, \ldots, t-5 \pmod{(t-1)}$. It follows that, $t+7 \le n \le 2t-6$. For $n = 2t-6$, a Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ is $(1,4) \cup A_4(4) \cup A_8(4) \cup \cdots \cup A_{n-t-3}(4) \cup (n-t+1, n-t+2, n-t+9, n-t+10, \ldots, n-1, n, n-t, n-t+3, n-t+4, n-t+5, n-t+8{=}t+2, 2) \cup A_2(4) \cup A_6(4) \cup \ldots \cup A_{n-t-5}(4) \cup (n-t-1, n-t+6{=}t, t+1, 1)$, see Figure 4.1.

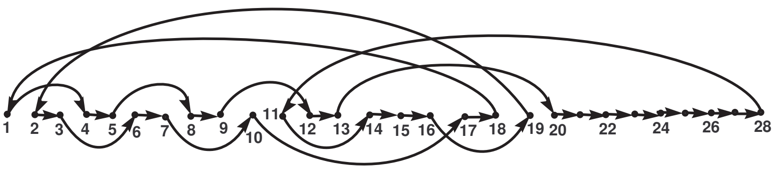

Figure 4.1: A Hamiltonian cycle in $T_{28}\langle 1,3,7;17\rangle$.

For $n = 2t-14$. Clearly here $t \ge 21$ as $t+7 \le 2t-14$. A Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ is $(1,4) \cup A_4(4) \cup A_8(4) \cup \cdots \cup A_{n-t-3=t-17}(4) \cup (n-t+1 = t-13, t-10, t-7, t-4, t-1, t+2, 2) \cup A_2(4) \cup A_6(4) \cup \cdots \cup A_{n-t-5}(4) \cup (n-t-1 = t-15, t-12, t-9, t-6, t-3, t, t+3, t+4, t+5, \ldots, n-1, n, n-t, n-t+3 = t-11, t-8, t-5, t-2, t+1, 1)$, see Figure 4.2.

For $n = 2t-22$. Clearly here $t \ge 29$ as $t+7 \le 2t-22$. A Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ is $A_1(4) \cup A_1(5) \cup \cdots \cup A_{n-t-3=t-17}(4) \cup (n-t+1 = t-21, t-18, t-15, \ldots, t+3, t+4, \ldots, n-1, n, n-t = t-22, t-19, t-16, \ldots, t+2, 2) \cup A_2(4) \cup A_6(4) \cup \cdots \cup A_{n-t-5}(4) \cup (n-t-1 = t-23, t-20, t-17, \ldots, t+1, 1)$, see Figure 4.3.

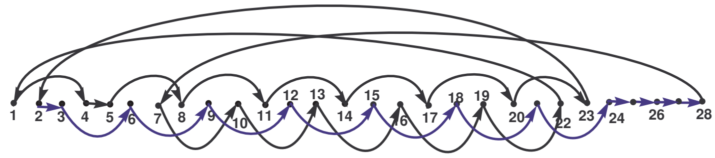

Figure 4.2: A Hamiltonian cycle in $T_{28}\langle 1,3,7;21\rangle$.

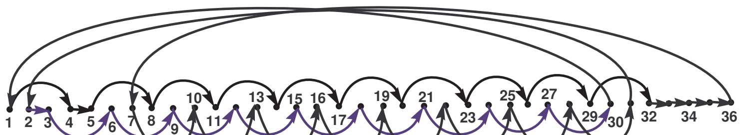

Figure 4.3: A Hamiltonian cycle in $T_{36}\langle 1,3,7;29\rangle$.

For $n \notin \{2t-6, 2t-14, 2t-22\}$. We consider two cases based on whether $2t-n$ is congruent to 2 modulo 8.
(i) If $2t-n \equiv 2 \pmod 8$, then a Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ is $A_1(4) \cup A_5(4) \cup \cdots \cup A_{n-t-10}(4) \cup (n-t-6, n-t-5, n-t+2, n-t+3, n-t+4) \cup A_{n-t+4}(8) \cup A_{n-t+12}(8) \cup \cdots \cup A_{t-6}(8) \cup (t+2, t+3, 3) \cup A_3(4) \cup A_7(4) \cup \cdots \cup A_{n-t-8}(4) \cup (n-t-4, n-t-3, n-t-2, , n-t-1, n-t+6) \cup A_{n-t+6}(4) \cup A_{n-t+10}(4) \cup \cdots \cup A_{t-8}(4) \cup (t-4, t-3, t+4, t+5, \ldots, n-1, n, n-t) \cup A_{n-t}(8) \cup A_{n-t+8}(8) \cup \cdots \cup A_{t-10}(8) \cup (t-2, t-1, t, t+1, 1)$, see Figure 4.4.

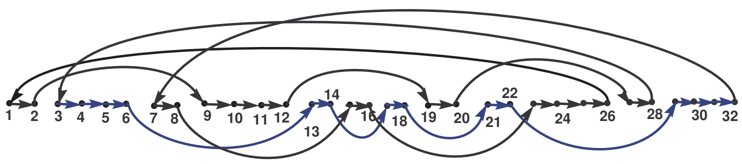

Figure 4.4: A Hamiltonian cycle in $T_{32}\langle 1,3,7;25\rangle$.

(ii) If $2t-n \not\equiv 2 \pmod 8$, then a Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ is $A_1(4) \cup A_5(4) \cup \cdots \cup A_{n-t-10}(4) \cup (n-t-6, n-t-5, n-t+2, n-t+3, n-t+10) \cup A_{n-t+10}(4) \cup A_{n-t+14}(4) \cup \cdots \cup A_{t-20}(4) \cup (t-16, t-15, t-8, t-7, t-6, t-5, t+2, t+3, 3) \cup A_3(4) \cup A_7(4) \cup \cdots \cup A_{n-t-8}(4) \cup (n-t-4, n-t-3, n-t-2, , n-t-1, n-t+6, n-t+7, n-t+8) \cup A_{n-t+8}(8) \cup A_{n-t+16}(8) \cup \cdots \cup A_{t-22}(8) \cup (t-14, t-13, t-12, t-11, t-4, t-3, t+4, t+5, \ldots, n-1, n, n-t, n-t+1, n-t+4) \cup A_{n-t+4}(8) \cup A_{n-t+12}(8) \cup \cdots \cup A_{t-10}(8) \cup (t-2, t-1, t, t+1, 1)$, see Figure 4.5.

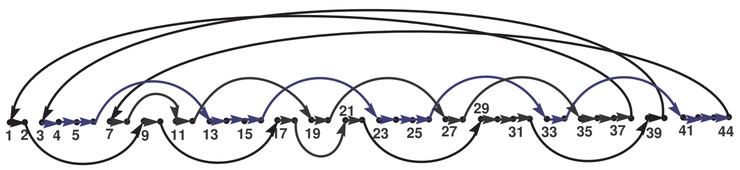

Figure 4.5: A Hamiltonian cycle in $T_{44}\langle 1,3,7;37\rangle$.

Case 2. $t \equiv 3 \pmod 4$ and $n \equiv 6, 10, \ldots, t-5 \pmod{(t-1)}$. Clearly, $t+5 \le n \le 2t-6$. For $n = 2t-6$, a Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ is $(1,4) \cup A_4(4) \cup A_8(4) \cup \cdots \cup A_{n-t-9}(4) \cup (n-t-5, n-t-4, n-t+3, n-t+6 = t, t+3) \cup A_{t+3}(4) \cup A_{t+7}(4) \cup \cdots \cup A_{n-6}(4) \cup (n-2, n-1, n-t-1, n-t+2, n-t+5, n-t+8 = t+2, 2) \cup A_2(4) \cup A_6(4) \cup \cdots \cup A_{n-t-7}(4) \cup (n-t-3, n-t-2, n-t+1, n-t+4, n-t+11 = t+5) \cup A_{t+5}(4) \cup A_{t+9}(4) \cup \cdots \cup A_{n-4}(4) \cup (n, n-t, n-t+7 = t+1, 1)$, see Figure 4.6.

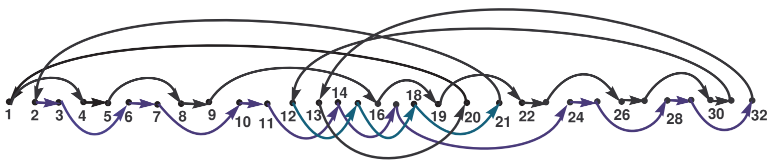

Figure 4.6: A Hamiltonian cycle in $T_{32}\langle 1,3,7;19\rangle$.

For $n = 2t-14$ (excluding $n = 24$ when $t = 19$), a Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ is $(1, 2, \ldots, n-t-4 = t-18) \cup A_{t-18}(8) \cup A_{t-10}(8) \cup A_{t-2}(8) \cup A_{t+6}(4) \cup A_{t+10}(4) \cup \cdots \cup A_{n-7}(4) \cup (n-3, n-2, n-t-2, n-t-1, n-t+6, n-t+7, n-t+8, n-t+9, n-t+16 = t+2, t+3, t+4) \cup A_{t+4}(4) \cup A_{t+8}(4) \cup \cdots \cup A_{n-5}(4) \cup (n-1, n, n-t, n-t+1, n-t+2, n-t+3, n-t+10 = t-4, t-3, t, t+1, 1)$, refer to Figure 4.7.

A Hamiltonian cycle in $T_{24}\langle 1,3,7;19\rangle$ is (1, 8, 9, 16, 17, 24, 5, 12, 19, 22, 23, 4, 11, 14, 15, 18, 21, 2, 3, 6, 7, 10, 13, 20, 1).

For $n \notin \{2t-6, 2t-14\}$. We consider two cases based on whether $2t-n$ is congruent to 2 modulo 8.
(i) If $2t-n \equiv 2 \pmod 8$, a Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ is $(1, 2, \ldots, n-t-4) \cup A_{n-t-4}(8) \cup A_{n-t+4}(8) \cup A_{n-t+12}(8) \cup A_{n-t+20}(4) \cup A_{n-t+24}(4) \cup \cdots \cup A_{n-7}(4) \cup (n-3, n-2, n-t-2, n-t-1, n-t+6, n-t+7, n-t+8, n-y+9, n-t+16, n-t+17, n-t+18) \cup A_{n-t+18}(8) \cup A_{n-t+26}(8) \cup \cdots \cup A_{t-4}(8) \cup A_{t+4}(4) \cup A_{t+8}(4) \cup \cdots \cup A_{n-5}(4) \cup (n-1, n, n-t, n-t+1, n-t+2, n-t+3, n-t+10, n-t+11, n-t+14) \cup A_{n-t+14}(8) \cup A_{n-t+22}(8) \cup \cdots \cup A_{t-8}(8) \cup (t, t+1, 1)$, see Figure 4.8.

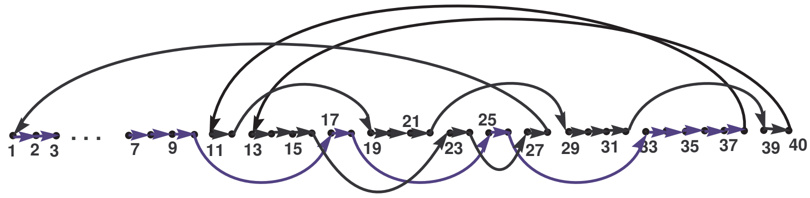

Figure 4.7: A Hamiltonian cycle in $T_{40}\langle 1,3,7;27\rangle$.

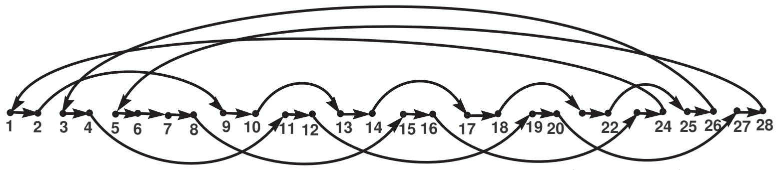

Figure 4.8: A Hamiltonian cycle in $T_{28}\langle 1,3,7;23\rangle$.

(ii) If $2t-n \not\equiv 2 \pmod 8$, then a Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ is $(1,2,\ldots,n-t-3,n-t+4)\cup A_{n-t+4}(4)\cup A_{n-t+8}(4)\cup\cdots\cup A_{n-7}(4)\cup(n-3,n-2,n-t-2,n-t-1,n-t+6,n-t+7,n-t+8,n-t+9,n-t+16,n-t+17,n-t+18)\cup A_{n-t-18}(8)\cup A_{n-t-10}(8)\cup\cdots\cup A_{t-4}(8)\cup A_{t+4}(4)\cup A_{t+8}(4)\cup\cdots\cup A_{n-5}(4)\cup(n-1,n,n-t,n-t+1,n-t+2)\cup A_{n-t+2}(8)\cup A_{n-t+10}(8)\cup\cdots\cup A_{t-8}(8)\cup(t,t+1,1)$, see Figure 4.9.

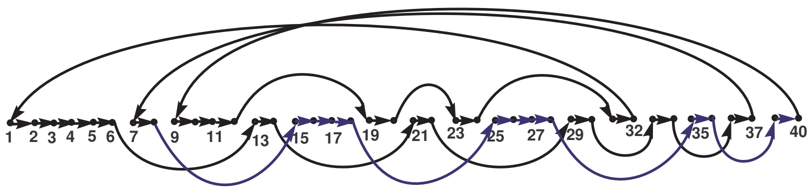

Figure 4.9: A Hamiltonian cycle in $T_{40}\langle 1,3,7;31\rangle$.

Thus, for any odd $t\geq 7$, the Toeplitz graph $T_n\langle 1,3,7;t\rangle$ is Hamiltonian for all even values of n that satisfy the conditions: $n\equiv 6,10,\ldots,t-5 \pmod{(t-1)}$ when $t\equiv 3 \pmod 4$, or $n\equiv 8,12,\ldots,t-5 \pmod{(t-1)}$ when $t\equiv 1 \pmod 4$. □

5. Toeplitz graphs $T_n\langle 1,3,7;t\rangle$ with even t

In [25], it was shown that for $t=6$, $T_n\langle 1,3,7;t\rangle$ is Hamiltonian for all n. Additionally, for even values of $t\leq 14$, that is for $t\in\{2,4,8,10,12,14\}$, it was shown that $T_n\langle 1,3,7;t\rangle$ is Hamiltonian for all n, except for a finite number of cases, which were stated as conjectures. In this section, we examine these conjectures and verify the conjectures' non-Hamiltonicity using Algorithm 1 in Python. This ultimately leads to a summary of the results as follows:

- $T_n\langle 1,3,7;2\rangle$ is Hamiltonian if and only if $n\notin\{8,11,12,13,14,18,19,20,25,26,32\}$.
- $T_n\langle 1,3,7;4\rangle$ is Hamiltonian if and only if $n\neq 12$.
- $T_n\langle 1,3,7;8\rangle$ is Hamiltonian if and only if $n\neq 10$.
- $T_n\langle 1,3,7;10\rangle$ is Hamiltonian if and only if $n\neq 14$.
- $T_n\langle 1,3,7;12\rangle$ is Hamiltonian if and only if $n\notin\{14,18\}$.
- $T_n\langle 1,3,7;14\rangle$ is Hamiltonian if and only if $n\neq 19$.

Furthermore, for even $t\geq 16$, a conjecture was stated as:
(a). If $t\equiv 0 \pmod 4$, then $T_n\langle 1,3,7;t\rangle$ is Hamiltonian for all n different from $t+2$.
(b). If $t\equiv 2 \pmod 4$, then $T_n\langle 1,3,7;t\rangle$ is Hamiltonian for all n.

In this section, we also prove some theorems that partially address the conjecture. The remaining cases, due to their irregular patterns of Hamiltonian cycles, are presented as conjectures. Finding a general pattern for the Hamiltonian cycle in these cases is challenging. However, computational verification for small values of n supports our conjecture.

Theorem 5.1. *The graph $T_{t+2}\langle 1,3,7;t\rangle$ is non-Hamiltonian when $t\equiv 0 \pmod 4$.*

Proof. Let $t\equiv 0 \pmod 4$. Assume, for contradiction, that the graph $T_{t+2}\langle 1,3,7;t\rangle$ is Hamiltonian, and let H be a Hamiltonian cycle in this graph.

The Hamiltonian cycle H can be decomposed into two paths: $P_{1\to t+2}$ (path from vertex 1 to vertex $t+2$) and $P_{t+2\to 1}$ (path from vertex $t+2$ to vertex 1). Since every vertex in H must have both in-degree and out-degree equal to one, and the only decreasing edges are of length t, the edges $(t+2,2)$ and $(t+1,1)$ must be part of $P_{t+2\to 1}$, as these are the only ways to return to vertices 1 and 2. Therefore, $P_{t+2\to 1}$ can be expressed as $(t+2,2)\cup P_{2\to t+1}\cup(t+1,1)$.

In $T_{t+2}\langle 1,3,7;t\rangle$, the only allowed increasing edges have lengths 1, 3, and 7. For the subpath $P_{2\to t+1}$, we can use one single edge or five consecutive edges of length 1, because otherwise, the other path $P_{1\to t+2}$, would not be able to complete the traversal as there are only jumps of 3 and 7. Also we cannot use exclusively consecutive edges of length 3 or 7 either, as there is no jump of 2 in $P_{1\to t+2}$, so between jumps of length 3 or 7, we need edges of length 1 or 5. Furthermore, the edges $(2,3)$ and $(t,t+1)$ must be part of path $P_{2\to t+1}$.

For $P_{2\to t+1}$, to reach vertex $t+1$ from vertex 2, while maintaining valid transitions, we have the following options:
$P_{2\to t} = (2,3,4,5,6,7)\cup P_{7\to t}\cup(t,t+1)$,
$P_{2\to t} = (2,3)\cup(3,6)\cup P_{6\to t}\cup(t,t+1)$,
$P_{2\to t} = (2,3)\cup(3,10)\cup P_{10\to t}\cup(t,t+1))$.
For each $P_{r\to t}$ with $r\in\{6,7,10\}$, we again have the same options. Since, the total distance from vertex r to vertex t is $t-r$, for $r\in\{6,7,10\}$. Clearly, the jumps of length 3 or 7 contribute $0 mod 4$ to the total distance. For the total distance $t-r$ to be achievable, we must have $t-r\equiv 0 \pmod 4$, which means $t\equiv 1,2 \pmod 4$ as $r\in\{6,7,10\}$. However, our assumption is that $t\equiv 0\pmod 4$, which contradicts. Hence, the graph cannot be Hamiltonian under the given conditions. □

In the next two theorems, we will investigate the hamiltonicity of the graph $T_n\langle 1,3,7;t\rangle$, where $t\geq 16$ is an even integer, for the cases when $n\equiv 0\pmod 4$ and $n\equiv 3\pmod 4$. The latter case specifically considers the condition $t\equiv 0\pmod 4$; thus, the case $t\equiv\ 2\pmod 4$ is not investigated here, although it has been verified for small values and is stated as a conjecture. The cases $n\equiv 1\pmod 4$ and $n\equiv 2\pmod 4$ are also stated as conjectures, but have been verified for small values of n using Algorithm 1.

Theorem 5.2. *Let t be an even integer with $t\geq 16$. Then, the graph $T_n\langle 1,3,7;t\rangle$ is Hamiltonian for all $n\equiv\ 0\pmod 4$.*

Proof. We prove this theorem by explicitly constructing a Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ that contains the edge $(n-2,n-1)$ for $n<2t+1$. This allows us to apply Remark 1, which ensures the extension of this Hamiltonian cycle to larger graphs of the form $T_{n+(t-1)}\langle 1,3,7;t\rangle$. Let $n<2t+1$ and $n\equiv 0\ \pmod 4$. Since n is a multiple of 4 and $n-t-2\geq 0$, we analyze the subcases based on $n-t-2$.
(i) $n-t-2=0$. A Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ is constructed as $(1,4)\cup A_4(4)\cup A_8(4)\cup\cdots\cup A_{n-4=t-2}(4)\cup(n,n-t=2)\cup A_2(4)\cup A_6(4)\cup\cdots\cup A_{t-4}(4)\cup(n-2=t,n-1=t+1,1)$. See Figure 5.1 for reference. Applying Remark 2.1, we extend

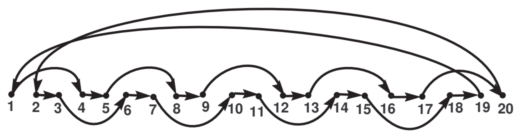

Figure 5.1: A Hamiltonian cycle in $T_{20}\langle 1,3,7;18\rangle$.

this cycle to $T_{n+t-1}\langle 1,3,7;t\rangle$, and so on.
(ii) For $n-t-2>0$. A Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ is $(1,2,3,\ldots,n-t-2)\cup A_{n-t-2}(4)\cup A_{n-t+2}(4)\cup\cdots\cup A_{t-2}(4)\cup(t+2,t+3,\ldots,n-2,n-1,n,n-t)\cup A_{n-t}(4)\cup A_{n-t+4}(4)\cup\cdots\cup A_{t-4}(4)\cup(t,t+1,1)$, see Figure 5.2 for reference. Since

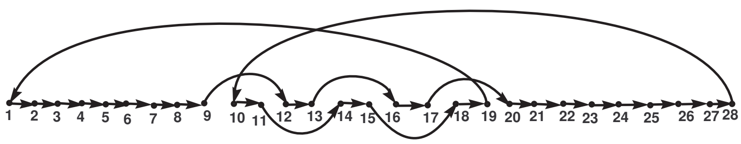

Figure 5.2: A Hamiltonian cycle in $T_{28}\langle 1,3,7;18\rangle$.

this cycle contains $(n-2,n-1)$, we apply Remark 2.1. This completes the proof. □

Theorem 5.3. *Let $t\equiv 0\pmod 4$ with $t\geq 16$. Then, the graph $T_n\langle 1,3,7;t\rangle$ is Hamiltonian for all $n\equiv\ 3\pmod 4$.*

Proof. We prove this theorem by explicitly constructing a Hamiltonian cycle in $T_n\langle 1,3,7;t\rangle$ that contains the edge $(n-2,n-1)$ for $n<2t+1$. This allows us to apply Remark 2.1, which ensures the extension of this Hamiltonian cycle to larger graphs of the form $T_{n+(t-1)}\langle 1,3,7;t\rangle$. Since n is odd, so is $2t-n$. we consider the following cases:

If $2t-n\equiv 3,5\pmod 8$, then a Hamiltonian cycle in $T_{n\neq t+3}\langle 1,3,7;t\rangle$ is $(1,4)\cup A_4(4)\cup A_8(4)\cup\cdots\cup A_{n-t-3}(4)\cup(n-t+1,n-t+4)\cup A_{n-t+4}(8)\cup A_{n-t+12}(8)\cup\cdots\cup A_{t-9}(8)\cup(t-1,t,t+3,t+4,\ldots,n-2,n-1,n,n-t,n-t+7)\cup A_{n-t+7}(8)\cup A_{n-t+15}(8)\cup\cdots\cup A_{t-6}(8)\cup(t+2,2)\cup A_2(4)\cup A_6(4)\cup\cdots\cup A_{n-t-5}(4)\cup(n-t-1,n-t+2)\cup B_{n-t+2}(8)\cup B_{n-t+10}(8)\cup\cdots\cup B_{t-11}(8)\cup(t-3,t-2,t+1,1)$, see Figure 5.3. Since $(n-2,n-1)$ is present, Remark 2.1 applies. And a Hamiltonian cycle in $T_{t+3}\langle 1,3,7;t\rangle$ is $(1,4,7)\cup A_7(8)\cup A_{15}(8)\cup\cdots\cup A_{n-12}(8)\cup(n-4,n-3,n,n-t=3,10)\cup A_{10}(8)\cup A_{18}(8)\cup\cdots\cup A_{n-9}(8)\cup(n-1,n-t-1=2,5)\cup B_5(8)\cup B_{13}(8)\cup\cdots\cup B_{t-7}(8)\cup(t+1,1)$, see Figure 5.4.

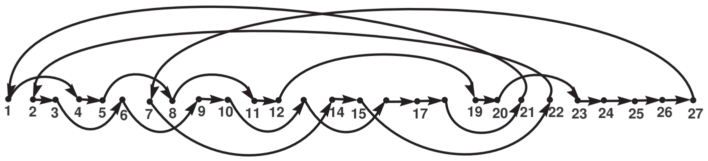

Figure 5.3: A Hamiltonian cycle in $T_{27}\langle 1,3,7;20\rangle$.

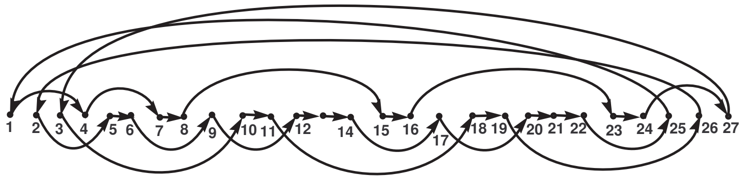

Figure 5.4: A Hamiltonian cycle in $T_{23}\langle 1,3,7;20\rangle$.

If $2t-n\equiv 1,7\,(\mathrm{mod}\,8)$, then a Hamiltonian cycle in $T_{n\neq t+3}\langle 1,3,7;t\rangle$ is $(1,4)\cup A_4(4)\cup A_8(4)\cup\cdots\cup A_{n-t-3}(4)\cup(n-t+1,n-t+4)\cup A_{n-t+4}(8)\cup A_{n-t+12}(8)\cup\cdots\cup A_{t-5}(8)\cup(t+3,t+4,\ldots,n-2,n-1,n,n-t,n-t+7)\cup A_{n-t+7}(8)\cup A_{n-t+15}(8)\cup\cdots\cup A_{t-10}(8)\cup(t-2,t-1,t+2,2)\cup A_2(4)\cup A_6(4)\cup\cdots\cup A_{n-t-5}(4)\cup(n-t-1,n-t+2)\cup B_{n-t+2}(8)\cup B_{n-t+10}(8)\cup\cdots\cup B_{t-11}(8)\cup(t-3,t-2,t+1,1)$, see Figure 5.5. Since $(n-2,n-1)$ is present, Remark 2.1 applies. And a Hamiltonian cycle in $T_{t+3}\langle 1,3,7;t\rangle$ is $(1,4,7)\cup A_7(8)\cup A_{15}(8)\cup\cdots\cup A_{n-8}(8)\cup(n,n-t=3)\cup A_3(8)\cup A_{11}(8)\cup\cdots\cup A_{n-13}(8)\cup(n-5,n-4,n-1,n-t-1=2,5)\cup B_5(8)\cup B_{13}(8)\cup\cdots\cup B_{t-10}(8)\cup(n-2=t+1,1)$, see Figure 5.6.

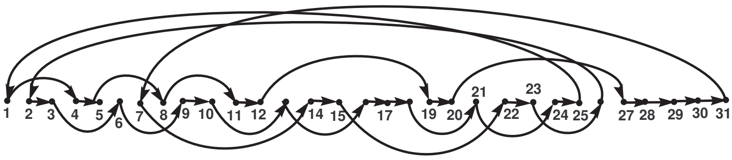

Figure 5.5: A Hamiltonian cycle in $T_{31}\langle 1,3,7;24\rangle$.

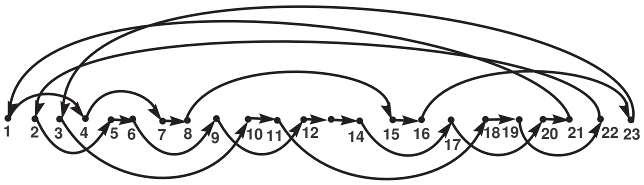

Figure 5.6: A Hamiltonian cycle in $T_{23}\langle 1,3,7;20\rangle$.

Since for $n=t+3$, the Hamiltonian cycle does not contain the edge $(n-2,n-1)$, we then consider $n=2t+1$. Since t is even, n is odd, and $n-t$ is also odd. We analyze three cases based on $n-t$ modulo 3. If $n-t\equiv 0\pmod 3$, then a Hamiltonian cycle in $T_{n=2t+1}\langle 1,3,7;t\rangle$ is $(1,4,7,\ldots,t+2,2,5,8,\ldots,t-3,t+4,t+7)\cup A_{t+7}(4)\cup A_{t+11}(4)\cup\cdots\cup A_{n-6}(4)\cup(n-2,n-1,n-t-1=t,t+3,3,6,9,\ldots,t-2,t+5)\cup A_{t+5}(4)\cup A_{t+9}(4)\cup\cdots\cup A_{n-4}(4)\cup(n,n-t=t+1,1)$, see Figure 5.7.

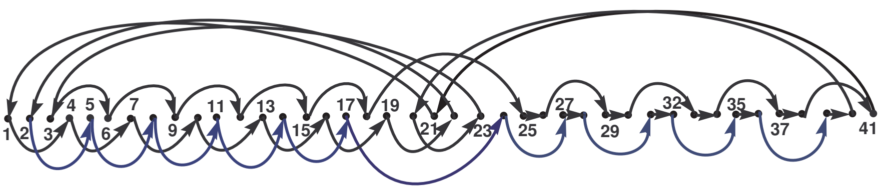

Figure 5.7: A Hamiltonian cycle in $T_{41}\langle 1,3,7;20\rangle$.

If $n-t\equiv 1\pmod 3$, then a Hamiltonian cycle in $T_{n=2t+1}\langle 1,3,7;t\rangle$ is $(1,8,11,14,\ldots,t+2,2,5,6,9,12,\ldots,t-3,t+4,t+7)\cup A_{t+7}(4)\cup A_{t+11}(4)\cup\cdots\cup A_{n-6}(4)\cup(n-2,n-1,n-t-1=t,t+3,3,4,7,10,\ldots,t-2,t+5)\cup A_{t+5}(4)\cup A_{t+9}(4)\cup\cdots\cup A_{n-4}(4)\cup(n,n-t=t+1,1)$, see Figure 5.8.

If $n-t\equiv 2\pmod 3$, then a Hamiltonian cycle in $T_{n=2t+1}\langle 1,3,7;t\rangle$ is $(1,8,9,12,15,\ldots,t+2,2,5,6,7,10,13,\ldots,t-3,t+4,t+7)\cup A_{t+7}(4)\cup A_{t+11}(4)\cup\cdots\cup A_{n-6}(4)\cup(n-2,n-1,n-t-1=t,t+3,3,4,11,14,17,\ldots,t-2,t+5)\cup A_{t+5}(4)\cup A_{t+9}(4)\cup\cdots\cup A_{n-4}(4)\cup(n,n-t=t+1,1)$, see Figure 5.9. Since $(n-2,n-1)$ is included in all three cases, Remark 1 ensures the cycle extends to $T_{2t+1+(t-1)}\langle 1,3,7;t\rangle$, and so on. This completes the proof. □

Conjecture 5.1. *Let t be an even integer such that $t\geq 16$.*

(i). The graph $T_n\langle 1,3,7;t\rangle$ is Hamiltonian for all $n\equiv 1,\,2\,(\mathrm{mod}\,4)$.

(ii). Let $t\equiv 2\,(\mathrm{mod}\,4)$. Then, the graph $T_n\langle 1,3,7;t\rangle$ is Hamiltonian for all $n\equiv 3\,(\mathrm{mod}\,4)$.

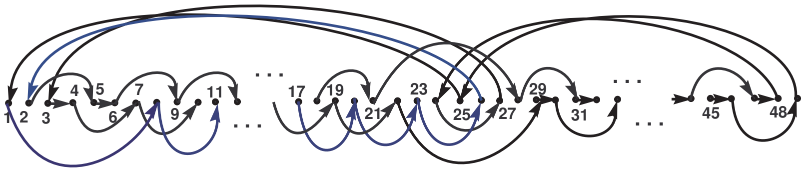

Figure 5.8: A Hamiltonian cycle in $T_{49}\langle 1, 3, 7; 24\rangle$

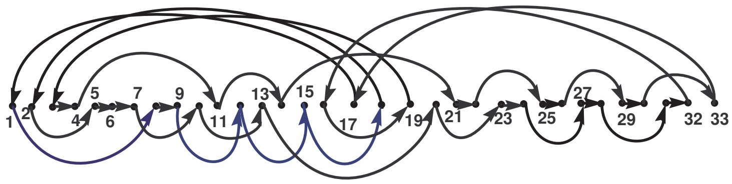

Figure 5.9: A Hamiltonian cycle in $T_{33}\langle 1, 3, 7; 16\rangle$

If both parts of Conjecture 5.1 are resolved affirmatively and Theorems 5.1–5.3 are applied, then the following result may be obtained:
Let t be an even integer with $t \geq 16$. Then, the graph $T_n\langle 1, 3, 7; t\rangle$ is Hamiltonian if and only if $n \neq t + 2$ when $t \equiv 0 \pmod 4$.

6. Concluding remarks

The Hamiltonicity of Toeplitz graphs $T_n\langle 1, 3, s_3; t\rangle$ has been extensively studied in the literature for $s_3 \in \{4, 5\}$. In this paper, we address the remaining cases where $s_3 \in \{6, 7\}$. Our investigation is complete for $s_3 = 6$. For $s_3 = 7$, most of the cases have been examined. However, some configurations remain unresolved. For these, we propose conjectures based on empirical observations. Due to the lack of an apparent pattern for constructing Hamiltonian cycles in the unresolved cases, a generalization remains challenging. Nevertheless, we have verified several small cases using Algorithm 1 and found consistent results that support our conjectures.

As a natural continuation of this work, future research should aim to resolve the conjectures proposed for $s_3 = 7$. Moreover, it would be worthwhile to explore Toeplitz graphs of the form $T_n\langle 1, 3, 8, s_4, s_5, \ldots, s_p; t_1, t_2, \ldots, b_q\rangle$ and extend the analysis to $s_3 \geq 8$ to investigate whether a broader structural pattern governs Hamiltonicity of these graphs.

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