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Research Article

On a class of graphs with equal domination and certified domination numbers

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Abstract

A graph G is called a $\mathcal{DD}_2$-graph if its vertex set V_G can be partitioned into a dominating set D and a 2-dominating set $V_G - D$. The domination number, $\gamma(G)$, is the minimum cardinality of a dominating set. The certified domination number, $\gamma_{\rm cer}(G)$, is the minimum size of a dominating set D such that every vertex in D has either zero or at least two neighbors in $V_G - D$. In this paper, the minimal $\mathcal{DD}_2$-graphs G for which $\gamma(G) = \gamma_{\rm cer}(G)$ are characterized, and it is shown that they are the cycle C_4, the stars $K_{1,n}$ ($n \geq 2$), and the subdivisions of the connected corona graphs $\operatorname{cor}(K_1)$ and $\operatorname{cor}(K_2)$. A characterization is also provided for bipartite graphs in which the smaller partite set serves simultaneously as a γ-set and a $\gamma_{\rm cer}$-set. A conjecture concerning trees and several open problems are presented in conclusion.

Keywords: dominating set; certified domination; $\mathcal{DD}_2$-graph; minimal $\mathcal{DD}_2$-graph; bipartite graph.

2020 Mathematics Subject Classification: 05C69.

1. Introduction

We investigate the conditions under which the domination number $\gamma(G)$ and the certified domination number $\gamma_{\rm cer}(G)$ of a graph G are equal, focusing on the class of $\mathcal{DD}_2$-graphs. We generally follow the notation of [1]. For a graph $G = (V_G, E_G)$, a set $D \subseteq V_G$ is a *dominating set* if every vertex in $V_G - D$ is adjacent to a vertex in D. The minimum cardinality of such a set is the *domination number* $\gamma(G)$. A set D is *2-dominating* if every vertex in $V_G - D$ has at least two neighbors in D. A *certified dominating set* is a dominating set D where every vertex in D has zero or at least two neighbors in $V_G - D$; the minimum cardinality is the *certified domination number* $\gamma_{\rm cer}(G)$ (see [3]). A leaf is a vertex of degree 1; its neighbor is a support vertex. The *corona graph* $\operatorname{cor}(H)$ of a multigraph H is obtained from H by adding at least one leaf adjacent to every vertex of H. The *subdivision graph* $S(H)$ of a multigraph H is the graph obtained from H by adding a new vertex n_e to each edge $e = uv$ of H. A graph G is a *$\mathcal{DD}_2$-graph* if its vertex set can be partitioned into a dominating set D and a 2-dominating set $V_G - D$ (see [4]). Such a partition is a *$\mathcal{DD}_2$-pair*. As noted in [3,6], if $(D, V_G - D)$ is a $\mathcal{DD}_2$-pair, then D is a certified dominating set. A graph G is a *minimal $\mathcal{DD}_2$-graph* if no proper spanning subgraph of G is a $\mathcal{DD}_2$-graph. Miotk, Topp, and Żyliński [6] characterized connected minimal $\mathcal{DD}_2$-graphs as being the stars $K_{1,n}$ ($n \geq 2$), the cycle C_4, and the subdivision graphs $S(\operatorname{cor}(H))$ of a connected corona multigraph H. In this paper, we build on this foundation to characterize the minimal $\mathcal{DD}_2$-graphs G for which $\gamma(G) = \gamma_{\rm cer}(G)$. We also provide a structural characterization for bipartite graphs where the smaller partite set is both a minimum dominating and minimum certified dominating set.

2. Equality condition in minimal $\mathcal{DD}_2$-graphs

In this section, we focus on the connection between the domination number $\gamma(G)$ and the certified domination number $\gamma_{\rm cer}(G)$, with a particular emphasis on the class of $\mathcal{DD}_2$-graphs. We first establish a necessary condition for the equality

$$\gamma(G) = \gamma_{\rm cer}(G)$$

and then provide a complete characterization of the connected minimal $\mathcal{DD}_2$-graphs for which this equality holds. Let D be a dominating set of G. A vertex $v \in D$ is *shadowed* if $N_G(v) \subseteq D$. In [3], it was stated that every support vertex should be included in every $\gamma_{\rm cer}$-set. We start by demonstrating that the equality of the two domination numbers indicates that the graph contains a particular partition that is associated with the $\mathcal{DD}_2$ property.

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Theorem 2.1. *Let G be a graph without isolated vertex. The following statements are equivalent:*

(i). $\gamma(G) = \gamma_{\text{cer}}(G)$.

(ii). *There exists a γ-set D, such that the set $V_G - D$ is 2-dominating.*

Proof. To prove the equivalence, we first assume that $\gamma(G) = \gamma_{\text{cer}}(G)$. Let D_{cer} be a γ_{cer}-set, which must also be a γ-set according to our assumption. For any vertex $w \in D_{\text{cer}}$, the certified property ensures that w has zero or at least two neighbors in $V_G - D_{\text{cer}}$. If any $w \in D_{\text{cer}}$ had zero neighbors in $V_G - D_{\text{cer}}$ (that is, w were shadowed by $N_G(w) \subseteq D_{\text{cer}}$), then $D' = D_{\text{cer}} - \{w\}$ would form a dominating set. Any vertex in $V_G - D_{\text{cer}}$ is dominated by $D_{\text{cer}} - \{w\}$ and since it is not adjacent to w, it must be dominated by $D_{\text{cer}} - \{w\}$, vertices in $D_{\text{cer}} - \{w\}$ would dominate themselves, and w would be dominated by its neighbors in $D_{\text{cer}} - \{w\}$. This would give $|D'| = \gamma(G) - 1$, contradicting the minimality of $\gamma(G)$. Therefore, every vertex in D_{cer} must have at least two neighbors in $V_G - D_{\text{cer}}$, which makes $V_G - D_{\text{cer}}$ a 2-dominating set, thereby proving $(i) \Rightarrow (ii)$.

Conversely, if there exists a γ-set D such that $V_G - D$ is 2-dominating, then every vertex in D has at least two neighbors outside D, making D a certified dominating set. This gives $\gamma_{\text{cer}}(G) \le |D| = \gamma(G)$, and since $\gamma(G) \le \gamma_{\text{cer}}(G)$ always holds by definition, we must have $\gamma(G) = \gamma_{\text{cer}}(G)$, proving $(ii) \Rightarrow (i)$. □

By Theorem 2.1, if $\gamma(G) = \gamma_{\text{cer}}(G)$ (and G has no isolated vertex), then G is a $\mathcal{DD}_2$-graph. However, the converse is not true. For example, the graph G in Figure 2.1 is a $\mathcal{DD}_2$-graph, as the pair $(D = \{v_1, v_3, v_5, v_7, v_9\}, D_2 = \{v_0, v_2, v_4, v_6, v_8, v_{10}\})$ is a $\mathcal{DD}_2$-pair. For this graph, a careful analysis reveals that $\gamma(G) = 4$ (with a minimum dominating set $\{v_0, v_2, v_5, v_9\}$), while $\gamma_{cer}(G) = 5$. Hence, this example demonstrates that a graph can be a $\mathcal{DD}_2$-graph while having $\gamma(G) < \gamma_{\text{cer}}(G)$, proving that the converse of Theorem 2.1 does not hold in general.

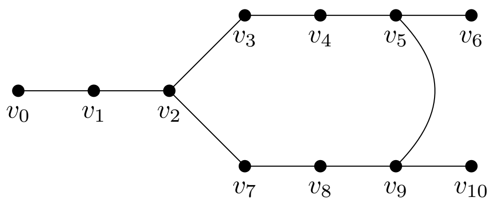

Figure 2.1: An example of a $\mathcal{DD}_2$-graph G. Here, $\gamma(G) = 4$ and $\gamma_{cer}(G) = 5$.

This motivates the description of minimal $\mathcal{DD}_2$-graphs G for which $\gamma(G) = \gamma_{\text{cer}}(G)$. A connected graph G is a *minimal $\mathcal{DD}_2$-graph* if G is a $\mathcal{DD}_2$-graph and no proper spanning subgraph of G is a $\mathcal{DD}_2$-graph. In [6], connected minimal $\mathcal{DD}_2$-graphs G were described as $K_{1,n} (n \ge 2)$, C_4, or $G = S(\text{cor}(H))$ for some connected H. While having $\gamma(G) = \gamma_{\text{cer}}(G)$ implies that G is a $\mathcal{DD}_2$ graph (if G contains no isolated vertices), the converse is not true. We now study the structure of *minimal* $\mathcal{DD}_2$-graphs with this equality. The following lemma considers the case $G = S(\text{cor}(H))$ and determines the structure of the γ_{cer}-set assuming equality.

Lemma 2.1. *Let G be a minimal $\mathcal{DD}_2$-graph such that $G = S(\text{cor}(H))$ for some connected graph H. If $\gamma(G) = \gamma_{\text{cer}}(G)$, then the set $D_{sub} = \{n_e : e \in E_{\text{cor}(H)}\}$ is a γ_{cer}-set of G.*

Proof. Assume $G = S(\text{cor}(H))$ is a connected minimal $\mathcal{DD}_2$-graph and that $\gamma(G) = \gamma_{\text{cer}}(G)$. Let $V_{\text{cor}(H)}$ be the vertex set of the corona graph $\text{cor}(H)$, and let $E_{\text{cor}(H)}$ be its edge set. The vertex set of G is $V_G = V_{\text{cor}(H)} \cup \{n_e \mid e \in E_{\text{cor}(H)}\}$, where n_e is the subdivision vertex corresponding to the edge $e \in E_{\text{cor}(H)}$. Consider the set $D_{sub} = \{n_e \mid e \in E_{\text{cor}(H)}\}$. We first demonstrate that D_{sub} is a certified dominating set of G. To establish *dominating property*, consider any vertex $w \in V_G \setminus D_{sub}$. This implies $w \in V_{\text{cor}(H)}$. Since H is connected and $\text{cor}(H)$ involves adding leaves to each vertex, the graph $\text{cor}(H)$ is connected and contains no isolated vertices. Thus, w must be an endpoint of at least one edge $e = wu$ in $\text{cor}(H)$. In the subdivision graph $G = S(\text{cor}(H))$, the vertex w is adjacent to the subdivision vertex n_e. As $n_e \in D_{sub}$, every vertex $w \in V_G \setminus D_{sub}$ is dominated by D_{sub}. Hence, D_{sub} is a dominating set of G. Next, we verify *certified property*. Let s be an arbitrary vertex in D_{sub}. Then $s = n_e$ for some edge $e = uv$ belonging to $E_{\text{cor}(H)}$. In the graph G, the neighbors of the subdivision vertex $s = n_e$ are precisely the endpoints of the original edge e, namely u and v. Both u and v are elements of $V_{\text{cor}(H)}$, which is equal to $V_G \setminus D_{sub}$. Therefore, the set of neighbors of s that lie outside D_{sub} is exactly $N_G(s) \cap (V_G \setminus D_{sub}) = \{u, v\}$. Since e is an edge, its endpoints u and v are distinct ($u \neq v$). Consequently, $|N_G(s) \cap (V_G \setminus D_{sub})| = 2$. Since this holds for every $s \in D_{sub}$, the set D_{sub} satisfies the condition for being a certified dominating set.

Now, consider the cardinality of D_{sub}. By definition, $|D_{sub}| = |E_{\mathrm{cor}(H)}|$. In [6], it was established that for a connected graph H, the domination number of $G = S(\mathrm{cor}(H))$ is precisely $\gamma(G) = |E_{\mathrm{cor}(H)}|$. We are given the initial assumption that $\gamma(G) = \gamma_{\mathrm{cer}}(G)$. Combining this with the known domination number, we obtain $\gamma_{\mathrm{cer}}(G) = \gamma(G) = |E_{\mathrm{cor}(H)}|$. Since D_{sub} is a certified dominating set and its cardinality $|D_{sub}|$ is equal to the certified domination number $\gamma_{\mathrm{cer}}(G)$, it follows by definition that D_{sub} is a certified dominating set of minimum cardinality. Therefore, D_{sub} is a γ_{cer}-set of G. □

Lemma 2.1 immediately results in the uniqueness of the minimum certified dominating set in particular graphs.

Corollary 2.1. *Let G be a minimal $\mathcal{DD}_2$-graph such that $G = S(\mathrm{cor}(H))$. If $\gamma(G) = \gamma_{\mathrm{cer}}(G)$, then the set*

$$D_{sub} = \{n_e : e \in E_{\mathrm{cor}(H)}\}$$

is the unique γ_{cer}-set of G.

Lemma 2.1 identifies the structure of the set γ_{cer} when $G = S(\mathrm{cor}(H))$ and $\gamma(G) = \gamma_{\mathrm{cer}}(G)$. We now establish a constraint on the base graph H that is necessary for this equality to hold. Specifically, we show that if H contains a vertex of degree two or more, the equality cannot be satisfied.

Lemma 2.2. *Let G be a minimal $\mathcal{DD}_2$-graph $G = S(\mathrm{cor}(H))$. If H has maximum degree $\Delta(H) \geq 2$, then $\gamma(G) < \gamma_{\mathrm{cer}}(G)$.*

Proof. Suppose $G = S(\mathrm{cor}(H))$ with $\Delta(H) \geq 2$, and assume that $\gamma(G) = \gamma_{\mathrm{cer}}(G)$. By Lemma 2.1, the set

$$D_{sub} = \{n_e : e \in E_{\mathrm{cor}(H)}\}$$

is a γ_{cer}-set, which implies $\gamma_{\mathrm{cer}}(G) = |D_{sub}| = |E_{\mathrm{cor}(H)}|$, and consequently, $\gamma(G) = |E_{\mathrm{cor}(H)}|$. Now, let $v \in V_H$ with degree $d_H(v) = k \geq 2$ (such a vertex exists since $\Delta(H) \geq 2$), and let $e_{vu_1}, \ldots, e_{vu_k}$ be the edges of H incident to v. Consider the set $D' = (D_{sub} - \{n_{e_{vu_1}}, \ldots, n_{e_{vu_k}}\}) \cup \{v\}$, which has cardinality $|D'| = |E_{\mathrm{cor}(H)}| - k + 1 < |E_{\mathrm{cor}(H)}|$ since $k \geq 2$. We claim that D' is a dominating set. The vertex v dominates itself, the vertices $u_1, \ldots, u_k$ and the subdivision vertices $n_{e_{vu_i}}$; any vertex $w \in V_H \setminus N_H[v]$ is dominated by some $n_{we'} \in D'$; any leaf attached to v is dominated by the corresponding subdivision vertex in D'; any leaf attached to $u \neq v$ is similarly dominated; and any subdivision vertex not incident to v remains in D'. Therefore, D' is a dominating set with $|D'| < |E_{\mathrm{cor}(H)}|$, which implies $\gamma(G) \leq |D'| < |E_{\mathrm{cor}(H)}|$. This contradicts our earlier conclusion that $\gamma(G) = |E_{\mathrm{cor}(H)}|$, proving that our initial assumption must be false, and thus $\gamma(G) < \gamma_{\mathrm{cer}}(G)$. □

Combining the known characterization of connected minimal $\mathcal{DD}_2$-graphs with the necessary condition derived in Lemma 2.2, we arrive at the main result of this section. This theorem fully characterizes the connected minimal $\mathcal{DD}_2$-graphs for which the domination number equals the certified domination number.

Theorem 2.2. *Let G be a connected minimal $\mathcal{DD}_2$-graph. Then $\gamma(G) = \gamma_{\mathrm{cer}}(G)$ if and only if G is the cycle C_4, a star $K_{1,n}$ $(n \geq 2)$ or a subdivision of $\mathrm{cor}(K_1)$ or $\mathrm{cor}(K_2)$.*

Proof. For a cycle C_4, direct verification shows that any minimum dominating set of size 2 (consisting of two non-adjacent vertices) is also certified, as each vertex in the dominating set has exactly two neighbors outside the set. For a star $K_{1,n}$ with $n \geq 2$, the center vertex forms a minimum dominating set of size 1, and this vertex has $n \geq 2$ neighbors outside the set, satisfying the certified property. For $G = S(\mathrm{cor}(K_1))$, which is a subdivision of a single vertex with at least one pendant edge, the unique subdivision vertex forms both a minimum dominating set and a minimum certified dominating set. Similarly, for $G = S(\mathrm{cor}(K_2))$, which is a subdivision of two vertices connected by an edge and each having at least one pendant edge, the two subdivision vertices form a minimum dominating set and a minimum certified dominating set. Conversely, assume that G is a connected minimal $\mathcal{DD}_2$-graph with $\gamma(G) = \gamma_{\mathrm{cer}}(G)$. By the characterization theorem in [6], we know that G must be C_4, a star $K_{1,n}$ with $n \geq 2$, or $G = S(\mathrm{cor}(H))$ for some connected graph H. If $G = C_4$ or $G = K_{1,n}$, we are done. If $G = S(\mathrm{cor}(H))$, then by Lemma 2.2, the equality $\gamma(G) = \gamma_{\mathrm{cer}}(G)$ implies $\Delta(H) < 2$, which means that the maximum degree of any vertex in H is at most 1. Since H is connected, there are only two possibilities: either $H = K_1$ (a single vertex) or $H = K_2$ (two vertices connected by an edge). Therefore, G must be $S(\mathrm{cor}(K_1))$ or $S(\mathrm{cor}(K_2))$, which completes the proof. □

Figure 2.2 presents examples of minimal $\mathcal{DD}_2$-graphs G for which $\gamma(G) = \gamma_{\mathrm{cer}}(G)$.

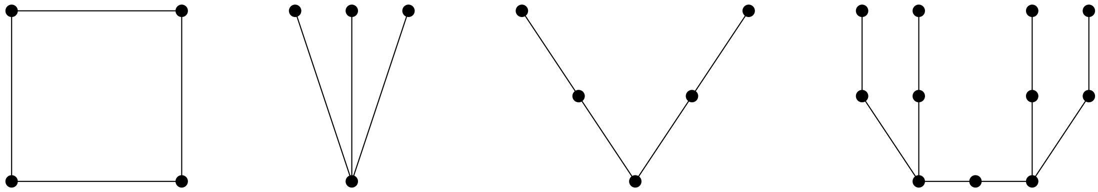

Figure 2.2: Examples of minimal $\mathcal{DD}_2$-graphs G for which $\gamma(G) = \gamma_{\rm cer}(G)$.

3. Certified domination relative to partite sets in bipartite graphs

This section focuses on connected bipartite graphs $G = ((A, B); E_G)$, where A and B are the partite sets with $1 \leq |A| \leq |B|$. Our primary goal is to characterize the graphs within this class for which the smaller partite set A is simultaneously a minimum dominating set (γ-set) and a minimum certified dominating set ($\gamma_{\rm cer}$-set). This condition implies the equality

$$\gamma(G) = \gamma_{\rm cer}(G) = |A|.$$

We build on existing characterizations of bipartite graphs where $\gamma(G) = |A|$ (see, e.g., [5,7]) and relate this specific condition to the properties of certified domination and $\mathcal{DD}_2$-graphs. Recall that a vertex $v \in D$ is *shadowed* with respect to D if $N_G(v) \subseteq D$. In [2] it was shown that the shadowed vertices, if they exist, must be leaves or weak support vertices. We will also utilize the structural characterization of the $\mathcal{DD}_2$-graphs of [6].

We begin by recalling the characterization of graphs possessing the $\mathcal{DD}_2$ property, established in [6], which connects this property to certified dominating sets without shadowed vertices and the neighborhood of weak support vertices.

Lemma 3.1 (see [6])**.** *Let G be a graph with no isolated vertex. Then the following three properties are equivalent:*

(i). *G has a certified dominating set with no shadowed vertex.*

(ii). *G is a $\mathcal{DD}_2$-graph.*

(iii). *$N_G(s) - (L_G \cup S_G) \neq \emptyset$ for every weak support s of G.*

The specific structure of bipartite graphs imposes a strong constraint on shadowed vertices within a partite set. The following simple lemma clarifies this point, showing that if the set A forms a certified dominating set, it inherently satisfies one of the key conditions related to $\mathcal{DD}_2$-graphs.

Lemma 3.2. *Let $G = ((A, B); E_G)$ be a connected bipartite graph with $1 \leq |A| \leq |B|$. If the set A is the minimal certified dominating set, then A does not contain a shadowed vertex.*

Proof. A vertex $u \in A$ is shadowed with respect to A if $N_G(u) \subseteq A$. However, by the definition of a bipartite graph, if $u \in A$, then all its neighbors must be in B, i.e., $N_G(u) \subseteq B$. Therefore, $N_G(u) \cap A = \emptyset$ (unless u is isolated, which is excluded). Consequently, no vertex in A can be shadowed with respect to A in a non-trivial bipartite graph. □

By combining the irrelevance of shadowing vertices in A (Lemma 3.2) with known properties of $\gamma_{\rm cer}$-sets (that is, any $\gamma_{\rm cer}$-set must contain all the support vertices), along with the description of $\mathcal{DD}_2$-graphs (Lemma 3.1), we obtain straightforward necessary conditions for A being a $\gamma_{\rm cer}$-set.

Corollary 3.1. *Let $G = ((A, B); E_G)$ be a connected bipartite graph with $1 \leq |A| \leq |B|$. If the set A is a $\gamma_{\rm cer}$-set of G, then G has the following two properties.*

(i). *$S_G \subseteq A$ or equivalently $S_G \cap B = \emptyset$;*

(ii). *G is a $\mathcal{DD}_2$-graph.*

We are now ready to state the main result of this section. The following theorem gives a complete characterization of connected bipartite graphs (of order at least three) in which the smaller partite set A functions as both a γ-set and a $\gamma_{\rm cer}$-set. The proof is structurally similar to that given for the characterization in [5] where $\gamma(G) = |A|$.

Theorem 3.1. *Let $G = ((A, B); E_G)$ be a connected bipartite graph with an order of at least three such that $1 \leq |A| \leq |B|$. Then, the following statements are equivalent.*

(i). *The set A is both a γ-set and a γ_{cer}-set of G.*

(ii). *The graph G has the following three properties.*

- *a) $S_G \cap B = \emptyset$;*
- *b) G is a $\mathcal{DD}_2$-graph;*
- *c) If $x, y \in A - S_G$ and $d_G(x, y) = 2$, then there exist at least two vertices $\overline{x}, \overline{y}$, such that $N_G(\overline{x}) = N_G(\overline{y}) = \{x, y\}$.*

Proof. Assume that the set A is both a γ-set and a γ_{cer}-set of G. Hence, $\gamma(G) = |A| = \gamma_{\text{cer}}(G)$, and by Corollary 3.1 we see that G is a $\mathcal{DD}_2$-graph and $S_G \cap B = \emptyset$. This proves properties $a)$ and $b)$. Now, let $x, y \in A - S_G$ for which $d_G(x, y) = 2$. Let $z \in N_G(x) \cap N_G(y)$. If there exists exactly one vertex z, then the set $A - \{x, y\} \cup \{z\}$ would be a smaller dominating set, which is a contradiction with equality $\gamma(G) = |A| = \gamma_{\text{cer}}(G)$. Therefore, it follows that there exists a vertex $\overline{x} \in B - \{z\}$ such that $N_G(\overline{x}) \subseteq \{x, y\}$. Since $x, y \notin S_G$ and $\overline{x}$ is not a support vertex, it follows that $N_G(\overline{x}) = \{x, y\}$ (otherwise if $d_G(\overline{x}) > 2$, then the set $A - N_G(\overline{x}) \cup \{\overline{x}\}$ would be a smaller dominating set, which contradicts the equality $\gamma(G) = |A| = \gamma_{\text{cer}}(G)$). Hence, there exists $\overline{y} \in B$ for which $N_G(\overline{y}) = \{x, y\}$. This proves property $c)$.

Now, assume that the graph G has the three properties $a)$, $b)$, and $c)$. First, it suffices to show that the set A is a γ-set of G. We claim that $\gamma(G) = |A|$. Since the set A is a dominating set, we see that $\gamma(G) \leq |A|$. Now, it suffices to show that $\gamma(G) \geq |A|$. Suppose, for contradiction, that $\gamma(G) < |A|$). Let D be a γ-set of G, such that $|D \cap A|$ is as large as possible (if $D \subset A$, then the set D would not be dominating in G). Since $|A - D| > |B \cap D| \geq 1$ and from the fact that every vertex in $A - D$ has at least one neighbor in $B \cap D$ we see that there exists two vertices $x, y \in A$ with common neighbor $z \in B$ (otherwise, we obtain a larger $A \cap D$, a contradiction). Since $S_G \subseteq A \cap D$ and $x, y \notin L_G$ we have $x, y \in A - S_G$. Then, by property $c)$ it follows that there exists vertices $\overline{x}, \overline{y} \in B$, such that $N_G(\overline{x}) = N_G(\overline{y}) = \{x, y\}$. Then the set $D' = D - \{\overline{x}, \overline{y}\} \cup \{x, y\}$ is the dominating set, a contradiction with maximality of $|D \cap A|$. Hence, $\gamma(G) = |A|$ and the set A is a γ-set of G. Now we show that the set A is also a certified dominating set. Since G is a connected graph on at least three vertices and from $a)$, it follows that every support vertex has at least two neighbors in the set B. From $c)$ we have it that for every two non-support vertices x, y belong to A there exists at least two neighbors in the set B. Since $\gamma(G) = |A|$ and for every vertex $v \in A$, we have it that $|N_G(v) \cap (V_G - A)| \geq 2$. This implies that the set A is also a certified dominating set. Consequently, A is both a γ-set and a γ_{cer}-set of G. □

Figure 3.1 shows an example of a bipartite graph G with partition $A = \{v_1, v_3, v_5, v_7, v_9\}$ and $B = \{v_0, v_2, v_4, v_6, v_8, v_{10}\}$. The set $D = \{v_1, v_3, v_5, v_7, v_9\}$ forms a certified dominating set because each vertex in D has exactly two neighbors outside D, which satisfies the certified property. Careful examination shows that the set $\{v_1, v_2, v_5, v_9\}$ forms a γ-set of G, and hence, $\gamma(G) = 4$. However, no certified dominating set smaller than D exists because any attempt to use fewer vertices would result in at least one vertex having exactly one neighbor outside the set, violating the certified condition. Thus, $\gamma_{\text{cer}}(G) = |D| = 5 = |A|$, demonstrating that

$$\gamma(G) < \gamma_{\text{cer}}(G) = |A|.$$

This example demonstrates that a graph can be a $\mathcal{DD}_2$-graph while having $\gamma(G) < \gamma_{\text{cer}}(G)$; thus, the converse of Theorem 2.1 does not hold in general.

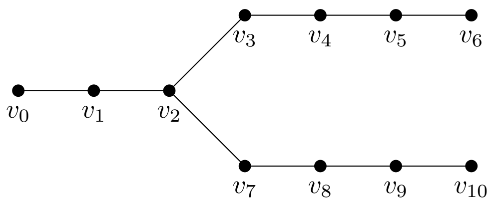

Figure 3.1: A bipartite $\mathcal{DD}_2$-graph $G = ((A, B); E_G)$ where the smaller partite set $A = \{v_1, v_3, v_5, v_7, v_9\}$ is a γ_{cer}-set, but not γ-set, demonstrating that $\gamma(G) < \gamma_{\text{cer}}(G) = |A|$.

4. Concluding remarks and open problems

In [6], it was observed that a tree T is a $\mathcal{DD}_2$-graph if and only if T can be decomposed into a forest F where each component is a star $K_{1,n} (n \geq 2)$ or the subdivision graph of a corona tree. By Theorem 2.2, it follows that if this decomposition contains a component of the form $S(\mathrm{cor}(H))$, where H is not K_1 or K_2, then $\gamma(T) < \gamma_{\mathrm{cer}}(T)$. Hence, we have the following conjecture.

Conjecture 4.1. *Let T be a tree. Then $\gamma(T) = \gamma_{\mathrm{cer}}(T)$ if and only if in the structural decomposition of T related to $\mathcal{DD}_2$-graphs described in [6], every component is a star $K_{1,n} (n \geq 2)$ or is a subdivision of $\mathrm{cor}(K_1)$ or $\mathrm{cor}(K_2)$.*

We end this paper with the following open problems.

Problem 4.1. *Determine the class of $\mathcal{DD}_2$-graphs G for which $\gamma(G) = \gamma_{\mathrm{cer}}(G)$.*

Problem 4.2. *Determine the class of bipartite $\mathcal{DD}_2$-graphs $G = ((A, B); E_G)$ for which $\gamma_{\mathrm{cer}}(G) = |A|$, where $1 \leq |A| \leq |B|$.*

Problem 4.3. *Determine an algorithm to check the equality $\gamma_{\mathrm{cer}}(G) = |A|$ in bipartite graphs $G = ((A, B); E_G)$, where $1 \leq |A| \leq |B|$.*

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