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Research Article

On the spectra and energy of the A_α matrix of digraphs

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Abstract

Let D be a digraph of order n with adjacency matrix $A(D)$. For $\alpha \in [0,1]$, the A_α-matrix of digraph D is defined as $A_\alpha(D) = \alpha\,\mathrm{Deg}(D) + (1-\alpha)A(D)$, where $\mathrm{Deg}(D) = \mathrm{diag}(d_1^+, d_2^+, \ldots, d_n^+)$ is the diagonal matrix of vertex outdegrees of D. The A_α-energy of D is defined as $E(A_\alpha(D)) = \sum_{i=1}^{n} |\mathrm{Re}(z_i)|$, where $z_1, z_2, \ldots, z_n$ are the eigenvalues of $A_\alpha(D)$, and $\mathrm{Re}(z_i)$ denotes the real part of z_i. In this paper, we determine the A_α-spectrum for several classes of digraphs. We also compute the A_α-energy for directed paths, directed cycles, bipartite digraphs, and k-regular digraphs. As a consequence, we obtain McClelland's inequality for the A_α-energy of k-regular digraphs.

Keywords: digraph; adjacency matrix; generalized adjacency matrix; A_α-spectrum; A_α-spectral radius; energy; A_α-energy.

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1. Introduction

A directed graph (or just digraph) D consists of two sets $\mathcal{V}$ and $\mathcal{A}$, where $\mathcal{V}$ is a non-empty finite set whose elements are called the vertices and $\mathcal{A}$ is a set of ordered pairs of the elements of $\mathcal{V}$ known as the arc set. A digraph D is strongly connected if for every pair of vertices $u, v \in \mathcal{V}(D)$, there are directed paths from u to v and from v to u. Throughout this paper, we consider finite, simple, connected digraphs. We follow [13] for terminology and notations.

In a digraph D, an arc from a vertex u to v is denoted by (u, v). In this case, we say that u is the tail and v is the head of the arc (u, v). Two vertices u and v of a digraph D are said to be adjacent if they are joined by an arc $(u, v) \in \mathcal{A}(D)$ or $(v, u) \in \mathcal{A}(D)$, and doubly adjacent if both (u, v) and (v, u) are in $\mathcal{A}(D)$. Let $N_D^-(u) = \{v \in \mathcal{V}(D) : (v, u) \in \mathcal{A}(D)\}$ and $N_D^+(u) = \{v \in \mathcal{V}(D) : (u, v) \in \mathcal{A}(D)\}$ denote the in-neighbours and out-neighbours of the vertex u, respectively. Let $d_u^- = |N_D^-(u)|$ and $d_u^+ = |N_D^+(u)|$, respectively, denote the indegree and outdegree of the vertex u in D. The minimum outdegree is denoted by δ^+, the maximum outdegree by Δ^+, and the minimum indegree by δ^-. A digraph is regular if each vertex has the same indegree and the same outdegree. Specifically, a digraph is k-regular if each vertex has indegree and outdegree k.

In a digraph, we denote a directed path by $\overrightarrow{P_n}$, with the arc set $\mathcal{A}(\overrightarrow{P_n}) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)\}$; a directed cycle by $\overrightarrow{C_n}$, with the arc set $\mathcal{A}(\overrightarrow{C_n}) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$; and the oriented complete bipartite graph by $\overrightarrow{K}_{r,s}$. Here, $\overrightarrow{K}_{r,s}$ consists of partite sets $\{u_1, u_2, \ldots, u_r\}$ and $\{v_1, v_2, \ldots, v_s\}$, with arcs of the form (u_i, v_j) for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$.

A digraph D is said to be discrete if it has no arcs. It is called symmetric if for any $(u, v) \in \mathcal{A}(D)$, also $(v, u) \in \mathcal{A}(D)$, where $u, v \in V(D)$. There is a one-to-one correspondence between the simple graphs and the symmetric digraphs given by $G \to \overleftrightarrow{G}$, where $\overleftrightarrow{G}$ has the same vertex set as the graph G, and each edge uv of G is replaced by a pair of symmetric arcs (u, v) and (v, u). Under this correspondence, a graph can be identified with a symmetric digraph.

The adjacency matrix of a digraph D is an $n \times n$ matrix $A(D) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in \mathcal{A}(D), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathrm{Deg}(D) = \mathrm{diag}(d_1^+, d_2^+, \ldots, d_n^+)$, where $d_i^+ = d_{v_i}^+$, be the diagonal matrix of vertex outdegrees of D. The matrices $L(D) = \mathrm{Deg}(D) - A(D)$ and $Q(D) = \mathrm{Deg}(D) + A(D)$ are called the Laplacian and the signless Laplacian matrices of the digraph D.

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Nikiforov [11] defined the generalized adjacency matrix $A_\alpha(G)$ for a graph G, as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$ for $\alpha \in [0,1]$. Liu et al. [10] defined the analogous generalized adjacency matrix $A_\alpha(D)$ for a digraph D as

$$A_\alpha(D) = \alpha \operatorname{Deg}(D) + (1-\alpha)A(D),$$

where α is any real number in $[0,1]$. It is clear that $A_\alpha(D) = A(D)$ if $\alpha = 0$, $2A_\alpha(D) = Q(D)$ if $\alpha = \frac{1}{2}$, and $A_\alpha(D) = \operatorname{Deg}(D)$ if $\alpha = 1$. From this, it follows that the matrix $A_\alpha(D)$ extends the spectral theory of both the adjacency matrix $A(D)$ and the signless Laplacian matrix $Q(D)$ of the digraph. In general, the matrix $A_\alpha(D)$ is not symmetric, so its eigenvalues can be complex numbers. The eigenvalues of $A_\alpha(D)$ are called the generalized adjacency eigenvalues of the digraph D. The set of distinct eigenvalues of $A_\alpha(D)$ together with their multiplicities is called the spectrum of $A_\alpha(D)$. If D is a digraph of order n with distinct generalized adjacency eigenvalues $z_1(A_\alpha(D)), z_2(A_\alpha(D)), \ldots, z_k(A_\alpha(D))$ and if their respective multiplicities are $m_1, m_2, \ldots, m_k$, we write the spectrum of $A_\alpha(D)$ as

$$\operatorname{Spec}(A_\alpha(D)) = \left\{ z_1^{(m_1)}, z_2^{(m_2)}, \ldots, z_k^{(m_k)} \right\}.$$

The eigenvalue of $A_\alpha(D)$ with the largest modulus is called the generalized adjacency spectral radius, or A_α-spectral radius of the digraph D, and is denoted by $z(A_\alpha(D))$. For recent work on A_α matrix of a digraph, see [3,4].

Gutman [8] defined the energy of a graph G as $E(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the adjacency eigenvalues of G. Abreu et al. [1] defined the signless Laplacian energy of a graph G as $QE(G) = \sum_{i=1}^{n} \left(q_i - \frac{2m}{n}\right)$, where $q_1 \geq q_2 \geq \cdots \geq q_n$ are the signless Laplacian eigenvalues of G and $\frac{2m}{n}$ is the average degree of G. For recent works on the energy, the Laplacian energy, and the signless Laplacian energy, we refer to [2,6,9,15]. Gou et al. [7] defined the generalized adjacency energy or A_α-energy of a graph G with order n and size m as the mean deviation of the A_α-eigenvalues of G, that is,

$$E(A_\alpha(G)) = \sum_{i=1}^{n} \left| p_i - \frac{2\alpha m}{n} \right|,$$

where p_i denotes the A_α-eigenvalues of G. From the definition, it is clear that $E(A_0(G)) = E(G)$ and $2E(A_{\frac{1}{2}}(G)) = QE(G)$. Therefore, the α-adjacency energy of a graph G merges the theories of adjacency energy and signless Laplacian energy.

The rest of the paper is organized as follows. In Section 2, we determine the A_α-spectrum for various types of digraphs, including $\overrightarrow{P_n}$, $\overrightarrow{C_n}$ and $\overrightarrow{K}_{r,s}$. In Section 3, we calculate the A_α-energy for several digraphs and derive a sharp upper bound for the A_α-energy of a k-regular digraph, expressed in terms of the number of arcs. This bound generalizes McClelland's inequality for the energy of digraphs, as given by Rada [16].

2. A_α-spectra of some families of digraphs

The following result gives the A_α-spectrum of the directed path $\overrightarrow{P_n}$.

Theorem 2.1. *The A_α-spectrum for the directed path $\overrightarrow{P_n}$ is given by*

$$\operatorname{Spec}(A_\alpha(\overrightarrow{P_n})) = \left\{ \alpha^{(n-1)}, 0^{(1)} \right\}.$$

Proof. Let $\overrightarrow{P_n}$ be a directed path with n vertices $v_1, v_2, \ldots, v_n$ and $n-1$ arcs given as (v_i, v_{i+1}) for $i = 1, 2, \ldots, n-1$. We assume that $n \geq 2$. For $\overrightarrow{P_n}$, the degree matrix $\operatorname{Deg}(\overrightarrow{P_n})$ is a diagonal matrix with entries equal to 1 for the vertices $v_1, v_2, \ldots, v_{n-1}$ and 0 for the vertex v_n, that is,

$$\operatorname{Deg}(\overrightarrow{P_n}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The adjacency matrix $A(\overrightarrow{P_n})$ has 1's on the first superdiagonal (corresponding to the arcs) and 0's elsewhere, that is,

$$A(\overrightarrow{P_n}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Therefore, the matrix $A_\alpha(\overrightarrow{P_n})$ is given by

$$A_\alpha(\overrightarrow{P_n}) = \alpha\,\mathrm{Deg}(\overrightarrow{P_n}) + (1-\alpha)A(\overrightarrow{P_n}),$$

$$= \alpha \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

$$= \begin{pmatrix} \alpha & 1-\alpha & 0 & \cdots & 0 \\ 0 & \alpha & 1-\alpha & \cdots & 0 \\ 0 & 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1-\alpha \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Therefore, the A_α-spectrum of the matrix $A_\alpha(\overrightarrow{P_n})$ is $\mathrm{Spec}(A_\alpha(\overrightarrow{P_n})) = \{\alpha^{(n-1)}, 0^{(1)}\}$. □

Next, we obtain the A_α-spectrum of the directed cycle $\overrightarrow{C_n}$.

Theorem 2.2. *The A_α-spectrum for the directed cycle $\overrightarrow{C_n}$ is given by*

$$\{\alpha + (1-\alpha)\omega^k : k = 0, 1, \ldots, n-1\},$$

where $\omega = e^{2\pi i/n}$ is a primitive n-th root of unity.

Proof. Let $\overrightarrow{C_n}$ be a directed cycle with n vertices, $v_1, v_2, \ldots, v_n$ and n arcs given as (v_i, v_{i+1}) for $i = 1, 2, \ldots, n-1$, with (v_n, v_1) completing the cycle. We assume that $n \geq 2$. The degree matrix $\mathrm{Deg}(\overrightarrow{C_n})$ is the identity matrix I_n because each vertex has outdegree 1, that is,

$$\mathrm{Deg}(\overrightarrow{C_n}) = I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The adjacency matrix $A(\overrightarrow{C_n})$ is the full-cycle permutation matrix of order n. The $(i, i+1)$-element of $A(\overrightarrow{C_n})$ is 1, $i = 1, 2, \ldots, n-1$, the $(n, 1)$-element of $A(\overrightarrow{C_n})$ is 1, and the remaining elements of $A(\overrightarrow{C_n})$ are zero.

$$A(\overrightarrow{C_n}) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Therefore, the generalized adjacency matrix $A_\alpha(\overrightarrow{C_n})$ is given by

$$A_\alpha(\overrightarrow{C_n}) = \alpha\,\mathrm{Deg}(\overrightarrow{C_n}) + (1-\alpha)A(\overrightarrow{C_n}) = \alpha I_n + (1-\alpha)A(\overrightarrow{C_n}).$$

Note that the eigenvalues of $A(\overrightarrow{C_n})$ are the n-th roots of unity, given by $\omega^k = e^{2\pi ik/n}, \quad k = 0, 1, \ldots, n-1$. Therefore, the eigenvalues of $A(\overrightarrow{C_n})$ are

$$z_k = \alpha \cdot 1 + (1-\alpha)\omega^k = \alpha + (1-\alpha)e^{2\pi ik/n}, \quad k = 0, 1, \ldots, n-1.$$

Thus, the A_α-spectrum of the directed cycle $\overrightarrow{C_n}$ is

$$\mathrm{Spec}(A_\alpha(\overrightarrow{P_n})) = \{\alpha + (1-\alpha)\omega^k : k = 0, 1, \ldots, n-1\},$$

where $\omega = e^{2\pi i/n}$ is a primitive n-th root of unity. □

We have the following observation.

Corollary 2.1. *For $\alpha = 0$, the A_α-spectrum of $\overrightarrow{C_n}$ reduces to the adjacency spectrum $\mathrm{Spec}(A(\overrightarrow{C_n})) = \{\omega^k : k=0, 1, \ldots, n-1\}$, as stated in* [5].

Next, we give the A_α-spectrum of the complete bipartite digraph $\overrightarrow{K}_{r,s}$.

Theorem 2.3. *The A_α-spectrum of the complete bipartite digraph $\overrightarrow{K}_{r,s}$ is given by $\left\{\alpha s^{(r)}, 0^{(s)}\right\}$.*

Proof. Let $\overrightarrow{K}_{r,s}$ be a complete bipartite digraph with partite sets $X = \{u_1, u_2, \ldots, u_r\}$ and $Y = \{v_1, v_2, \ldots, v_s\}$ and arcs (u_i, v_j), where $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$. The generalized adjacency matrix $A_\alpha(\overrightarrow{K}_{r,s})$ is given by

$$A_\alpha(\overrightarrow{K}_{r,s}) = \alpha\,\mathrm{Deg}(\overrightarrow{K}_{r,s}) + (1-\alpha)A(\overrightarrow{K}_{r,s}),$$

where $\mathrm{Deg}(\overrightarrow{K}_{r,s})$ is the outdegree diagonal matrix of order $(r+s)$ and $A(\overrightarrow{K}_{r,s})$ is the adjacency matrix of order $(r+s)$ of the digraph. We have

$$\begin{aligned}
A_\alpha(\overrightarrow{K}_{r,s}) &= \alpha\,\mathrm{Deg}(\overrightarrow{K}_{r,s}) + (1-\alpha)\,A(\overrightarrow{K}_{r,s}) \\
&= \alpha \begin{pmatrix} s & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \\
&= \alpha \begin{pmatrix} sI_r & 0_{r\times s} \\ 0_{s\times r} & 0_{s\times s} \end{pmatrix} + (1-\alpha) \begin{pmatrix} 0_{r\times r} & J_{r\times s} \\ 0_{s\times r} & 0_{s\times s} \end{pmatrix} \\
&= \begin{pmatrix} \alpha s I_r & (1-\alpha)J_{r\times s} \\ 0_{s\times r} & 0_{s\times s} \end{pmatrix}.
\end{aligned}$$

Therefore, the A_α-spectrum of $\overrightarrow{K}_{r,s}$ is $\mathrm{Spec}(A_\alpha(\overrightarrow{K}_{r,s})) = \left\{\alpha s^{(r)}, 0^{(s)}\right\}$. □

Remark 2.1. *When $r = 1$ and $s = n-1$, the complete bipartite digraph $\overrightarrow{K}_{1,n-1}$ is a directed star. The A_α-spectrum of $A_\alpha(\overrightarrow{K}_{1,n-1})$ can be directly calculated from the spectrum of $A_\alpha(\overrightarrow{K}_{r,s})$ by setting $r = 1$ and $s = n-1$ in the formula given in Theorem 2.3.*

3. A_α-Energy of digraphs

Gou et al. [7] defined the generalized adjacency energy or A_α-energy for graphs. We extend this concept to digraphs.

Definition 3.1. *Let D be a digraph with generalized adjacency eigenvalues $z_1, z_2, \ldots, z_n$. We define the A_α-energy $E(A_\alpha(D))$ of D as*

$$E(A_\alpha(D)) = \sum_{i=1}^{n} |\mathrm{Re}(z_i)|,$$

where $\mathrm{Re}(z_i)$ denotes the real part of the eigenvalue z_i.

Example 3.1. *Let $\overrightarrow{P_n}$ be the directed path. By Theorem 2.1, the A_α-spectrum of $A_\alpha(\overrightarrow{P_n})$ is $\{\alpha^{(n-1)}, 0\}$. Therefore, the A_α-energy is given as*

$$E(A_\alpha(\overrightarrow{P_n})) = \sum_{i=1}^{n} |\mathrm{Re}(z_i)| = |\alpha| + |\alpha| + \cdots + |\alpha| + |0| = (n-1)\alpha.$$

Remark 3.1. *For $\alpha = 0$, the A_α-energy of the directed path $\overrightarrow{P_n}$ reduces to the energy of the path as computed by Pena and Rada.* [12]

Example 3.2. *Let $\overrightarrow{K}_{r,s}$ be the complete bipartite digraph. By Theorem 2.3, A_α-spectrum of $A_\alpha(\overrightarrow{K}_{r,s})$ is given by $\{\alpha s^{(r)}, 0^{(s)}\}$. Therefore,*

$$E(A_\alpha(\overrightarrow{K}_{r,s})) = \sum_{i=1}^{n} |\mathrm{Re}(z_i)| = |\alpha s| + |\alpha s| + \cdots + |\alpha s| + |0| + |0| + \cdots + |0| = rs\alpha.$$

Remark 3.2. *The A_α-energy of the directed star $\overrightarrow{K}_{1,n-1}$ is $(n-1)\alpha$. Notably, this value coincides with the A_α-energy of the directed path $\overrightarrow{P_n}$, as established in Example 3.1.*

Now, we obtain the A_α-energy of the directed cycle $\overrightarrow{C_n}$.

Theorem 3.1. *The A_α-energy of the directed cycle $\overrightarrow{C_n}$ is given by*

$$\sum_{k=0}^{n-1}\left|\alpha+(1-\alpha)\cos\left(\frac{2\pi k}{n}\right)\right|.$$

Proof. By Theorem 2.2, the A_α-spectrum of $\overrightarrow{C_n}$ is given by

$$\left\{\alpha, \alpha+(1-\alpha)\omega, \alpha+(1-\alpha)\omega^2, \cdots, \alpha+(1-\alpha)\omega^{n-1}\right\},$$

where $\omega = e^{2\pi i/n}$ is a primitive n-th root of unity. We have

$$z_k = \alpha + (1-\alpha)e^{\frac{2\pi i k}{n}}, \quad k = 0, 1, \ldots, n-1.$$

The real part of z_k is then given by

$$\mathrm{Re}(z_k) = \alpha + (1-\alpha)\cos\left(\frac{2\pi k}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

Therefore,

$$E(A_\alpha(\overrightarrow{C_n})) = \sum_{i=1}^{n}|\mathrm{Re}(z_i)| = \sum_{k=0}^{n-1}\left|\alpha+(1-\alpha)\cos\left(\frac{2\pi k}{n}\right)\right|.$$

□

Corollary 3.1. *For $\alpha = 0$, the A_α-energy of the directed cycle $\overrightarrow{C_n}$ reduces to the energy calculated from the adjacency matrix of the directed cycle, as given by Pirzada and Bhat [14].*

Next, we determine the value of the A_α-energy of the directed cycle $\overrightarrow{C_n}$ for $\frac{1}{2} \le \alpha \le 1$.

Theorem 3.2. *If $\frac{1}{2} \le \alpha \le 1$, then the A_α-energy of the directed cycle $\overrightarrow{C_n}$ is $n\alpha$.*

Proof. From Theorem 3.1, we know that

$$E(A_\alpha(\overrightarrow{C_n})) = \sum_{k=0}^{n-1}\left|\alpha+(1-\alpha)\cos\left(\frac{2\pi k}{n}\right)\right|.$$

For $\frac{1}{2} \le \alpha \le 1$, we claim that $\alpha + (1-\alpha)\cos\left(\frac{2\pi k}{n}\right) \ge 0$. Since $-1 \le \cos\frac{2\pi k}{n} \le 1$, consider the case when $\cos\frac{2\pi k}{n} = -1$ so that $\alpha + (1-\alpha)\cos\left(\frac{2\pi k}{n}\right) = \alpha + (1-\alpha)(-1) = 2\alpha - 1$. This expression is non-negative if $\alpha \ge \frac{1}{2}$. Therefore, for $\frac{1}{2} \le \alpha \le 1$, the claimed inequality holds. Thus,

$$\begin{aligned} E(A_\alpha(\overrightarrow{C_n})) &= \sum_{k=0}^{n-1}\left|\alpha+(1-\alpha)\cos\left(\frac{2\pi k}{n}\right)\right| \\ &= \sum_{k=0}^{n-1}\left(\alpha+(1-\alpha)\cos\left(\frac{2\pi k}{n}\right)\right) \\ &= \sum_{k=0}^{n-1}\alpha + (1-\alpha)\sum_{k=0}^{n-1}\cos\left(\frac{2\pi k}{n}\right). \end{aligned}$$

Since, $\sum_{k=0}^{n-1}\cos(\frac{2\pi k}{n}) = 0$, it follows that $E(A_\alpha(\overrightarrow{C_n})) = n\alpha$. □

Remark 3.3. *We observe that the A_α-energy of the directed cycle $\overrightarrow{C_n}$ is an increasing function of α for $\frac{1}{2} \le \alpha \le 1$. This follows directly from the fact that $E(A_\alpha(\overrightarrow{C_n})) = n\alpha$, which is a linear function of α in this range.*

Next, we obtain an upper bound for the A_α-energy of the directed cycle for $0 \le \alpha < \frac{1}{2}$.

Theorem 3.3. *For $\alpha \ge 0$, the A_α-energy of the directed cycle $\overrightarrow{C_n}$ satisfies*

$$E(A_\alpha(\overrightarrow{C_n})) \le \begin{cases} n\alpha + 2\cot\frac{\pi}{n} & \text{if } n = 4m, \\ n\alpha + \csc\frac{\pi}{2n} & \text{if } n = 4m+1 \text{ or } n = 4m+3, \\ n\alpha + 2\csc\frac{\pi}{n} & \text{if } n = 4m+2. \end{cases}$$

Moreover, equality holds if and only if $\alpha = 0$.

Proof. By Theorem 3.1, we have

$$\begin{aligned} E(A_\alpha(\overrightarrow{C_n})) &= \sum_{k=0}^{n-1} \left| \alpha + (1-\alpha)\cos\left(\frac{2\pi k}{n}\right) \right| \\ &= \sum_{k=0}^{n-1} \left| \alpha + \cos\left(\frac{2\pi k}{n}\right) - \alpha\cos\left(\frac{2\pi k}{n}\right) \right| \\ &= \sum_{k=0}^{n-1} \left| \alpha\left(1-\cos\left(\frac{2\pi k}{n}\right)\right) + \cos\left(\frac{2\pi k}{n}\right) \right| \\ &\le \sum_{k=0}^{n-1} \left| \alpha\left(1-\cos\left(\frac{2\pi k}{n}\right)\right) \right| + \sum_{k=0}^{n-1} \left| \cos\left(\frac{2\pi k}{n}\right) \right|. \end{aligned} \tag{1}$$

Since $\alpha \ge 0$ and $(1-\cos\left(\frac{2\pi k}{n}\right)) \ge 0$, it follows that

$$\begin{aligned} E(A_\alpha(\overrightarrow{C_n})) &\le \sum_{k=0}^{n-1} \alpha\left(1-\cos\left(\frac{2\pi k}{n}\right)\right) + \sum_{k=0}^{n-1} \left| \cos\left(\frac{2\pi k}{n}\right) \right| \\ &= \sum_{k=0}^{n-1} \alpha - \alpha\sum_{k=0}^{n-1}\cos\left(\frac{2\pi k}{n}\right) + \sum_{k=0}^{n-1} \left| \cos\left(\frac{2\pi k}{n}\right) \right| \\ &= n\alpha + \sum_{k=0}^{n-1} \left| \cos\left(\frac{2\pi k}{n}\right) \right|, \end{aligned}$$

where the last equality follows from $\sum_{k=0}^{n-1}\cos\left(\frac{2\pi k}{n}\right) = 0$. In [14], Pirzada and Bhat have shown that

$$\sum_{k=0}^{n-1} \left| \cos\left(\frac{2\pi k}{n}\right) \right| = \begin{cases} 2\cot\frac{\pi}{n} & \text{if } n \equiv 0 \pmod 4, \\ \csc\frac{\pi}{2n} & \text{if } n \equiv 1 \pmod 4 \text{ or } n \equiv 3 \pmod 4, \\ 2\csc\frac{\pi}{n} & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Thus, it follows that

$$E(A_\alpha(\overrightarrow{C_n})) \le \begin{cases} n\alpha + 2\cot\frac{\pi}{n} & \text{if } n \equiv 0 \pmod 4, \\ n\alpha + \csc\frac{\pi}{2n} & \text{if } n \equiv 1 \pmod 4 \text{ or } n \equiv 3 \pmod 4, \\ n\alpha + 2\csc\frac{\pi}{n} & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

We now determine the equality case. Note that the inequality above becomes equality if and only if equality holds in each of the triangle inequalities (1). This happens if and only if the two terms inside each modulus are non-negative multiples of each other or one of them is zero, that is, $\alpha(1-\cos\left(\frac{2\pi k}{n}\right)) \cdot \cos\left(\frac{2\pi k}{n}\right) \ge 0$, for all $k = 0, 1, 2, \ldots, n-1$. Since $\alpha \ge 0$ and $1-\cos\left(\frac{2\pi k}{n}\right) \ge 0$, this condition reduces to $\cos\left(\frac{2\pi k}{n}\right) \ge 0$, for all $k = 0, 1, 2, \ldots, n-1$. However, this is not possible for all $k = 0, 1, 2, \ldots, n-1$. Therefore, the only possibility is $\alpha = 0$. This completes the proof. □

In [12], Pena and Rada defined the energy of a digraph as

$$E(D) = \sum_{i=1}^{n} \left| \mathrm{Re}(\lambda_i) \right|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of the digraph D and $\mathrm{Re}(\lambda_i)$ denotes the real part of λ_i.

In the following theorem, we establish an upper bound for the A_α-energy of a k-regular digraph D of order n, expressed in terms of α, k, n, and the energy of the digraph $E(D)$. Here, we note that a discrete digraph can be regarded as a regular digraph with degree of regularity equal to zero.

Theorem 3.4. *If D is a k-regular digraph of order n and $\alpha \in [0, 1]$, then*

$$E(A_\alpha(D)) \le \alpha kn + (1-\alpha)E(D). \tag{2}$$

Moreover, the equality holds if and only if D satisfies one of the following (a) D is a discrete digraph (b) $\alpha = 0$ or 1 (equality holds for all k-regular digraphs).

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the adjacency eigenvalues of D. If D is k-regular, then $\mathrm{Deg}(D) = kI_n$. The generalized adjacency matrix of D is given by $A_\alpha(D) = \alpha\,\mathrm{Deg}(D) + (1-\alpha)A(D)$, which simplifies to $A_\alpha(D) = \alpha k I_n + (1-\alpha)A(D)$. From this equality, it follows that the A_α-spectrum of D is $\{\alpha k + (1-\alpha)\lambda_1, \ldots, \alpha k + (1-\alpha)\lambda_n\}$. We know that $E(A_\alpha(D)) = \sum_{i=1}^{n} |\mathrm{Re}(z_i)|$, where $z_i = \alpha k + (1-\alpha)\lambda_i$. Therefore, we have

$$E(A_\alpha(D)) = \sum_{i=1}^{n} |\mathrm{Re}(\alpha k + (1-\alpha)\lambda_i)| = \sum_{i=1}^{n} \left(|\alpha k + (1-\alpha)\,\mathrm{Re}(\lambda_i)|\right).$$

Applying the triangle inequality and noting that αk is constant for all i, we obtain

$$E(A_\alpha(D)) \leq \sum_{i=1}^{n} |\alpha k| + \sum_{i=1}^{n} (1-\alpha)\,|\mathrm{Re}(\lambda_i)| = n\alpha k + (1-\alpha)E(D).$$

This proves (2). Assume that the equality holds in (2). Then equality holds in each triangle inequality, and the equality in the triangle inequality holds if and only if for each i the quantities αk and $(1-\alpha)\,\mathrm{Re}(\lambda_i)$ have the same sign or one of them is zero. Since $\alpha k \geq 0$, therefore $(1-\alpha)\,\mathrm{Re}(\lambda_i) \geq 0$. This implies that $\mathrm{Re}(\lambda_i) \geq 0$, for all $i = 1, 2, \ldots, n$. Since $\sum_{i=1}^{n} \mathrm{Re}(\lambda_i) = 0$, it follows that $\mathrm{Re}(\lambda_i) = 0$, for all $i = 1, 2, \ldots, n$. As $A(D)$ is a non-negative matrix, the spectral radius $\rho(D)$ is a non-negative eigenvalue of D. Moreover, $|\lambda_i| \leq \rho(D)$, for all $\lambda_i \in \mathrm{spec}(D)$. From the above, we conclude that $\lambda_i = 0$, for all $i = 1, 2, \ldots, n$, which implies that D is an acyclic digraph [Proposition 2.1, [16]]. Among acyclic digraphs, only regular digraph is a discrete digraph, which we consider as a zero regular digraph. Conversely, if $\alpha \in [0, 1]$ and D is a discrete digraph, then both sides of (2) are equal to zero. If $\alpha = 0$ then both sides of (2) are equal to $E(D)$. If $\alpha = 1$, then both sides of the (2) are equal to kn, so equality holds for all k regular digraphs. □

In [16], Rada extended McClelland's inequality to digraphs and derived a sharp upper bound for the energy of a digraph in terms of the number of arcs, which is given by

$$E(D) \leq \sqrt{\frac{n(a + c_2)}{2}}, \tag{3}$$

where c_2 denotes the number of closed walks of length 2. Equality holds if and only if $D \cong \bigoplus_{i=1}^{n/2} \overleftrightarrow{K_2}$, the direct sum of $\frac{n}{2}$ copies of the symmetric complete digraph on 2 vertices.

As an immediate consequence of Theorem 3.4 and inequality (3), we obtain the following upper bound for the A_α-energy of a k-regular digraph.

Corollary 3.2. *If D is a k-regular digraph with n vertices, a arcs and c_2 closed walks of length 2, then*

$$E(A_\alpha(D)) \leq \alpha k n + (1-\alpha)\sqrt{\frac{n(a + c_2)}{2}}.$$

Moreover, the equality holds if and only if (a) D is a discrete digraph (b) $\alpha = 0$ and $D \cong \bigoplus_{i=1}^{n/2} \overleftrightarrow{K_2}$ (c) $\alpha = 1$ and D is any k-regular digraph

Remark 3.4. *Setting $\alpha = 0$ in Corollary 3.2, we recover inequality (3) given by Rada for the energy of a digraph as stated in [16].*

4. Conclusion

By setting $\alpha = 0$ and $\alpha = \frac{1}{2}$ in the results derived in Sections 2 and 3, we recover the corresponding spectrum, energy, and bounds for the adjacency matrix $A(D)$ and the signless Laplacian matrix $Q(D)$, respectively. Most of the results of Sections 2 and 3 have already been studied in the context of the adjacency matrix $A(D)$ and/or the signless Laplacian matrix $Q(D)$. Therefore, our findings serve as a generalization of these results. Furthermore, by replacing D with $\overleftrightarrow{G}$, where $\overleftrightarrow{G}$ is the symmetric digraph corresponding to the underlying graph G, we obtain analogous results for graphs. While we have determined the value of the A_α-energy of the directed cycle $\overrightarrow{C_n}$ for $\frac{1}{2} \leq \alpha \leq 1$, the value of $E(A_\alpha(\overrightarrow{C_n}))$ for $0 < \alpha < \frac{1}{2}$ remains unknown.

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