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Research Article

Pattern avoidance in flattened derangements

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Abstract

To flatten a permutation π, expressed in standard cycle form, is to remove the parentheses enclosing the cycles and consider the resulting permutation π' in the one-line notation. Then π is said to avoid a pattern τ in the flattened sense if π' avoids τ in the usual sense. In this paper, we consider the problem of avoidance of one or more classical patterns of length three in the flattened sense by derangements, which extends earlier results on flattened permutations and other structures. We establish explicit formulas enumerating each corresponding avoidance class of derangements according to the number of cycles. As a consequence of our results, we obtain the equivalences $213 \approx 312$ and $231 \approx 321$ for derangements in the flattened sense. To establish the generating function formula in the case of the pattern 321, we make use of the kernel method and Lagrange inversion.

Keywords: flattened permutation; derangement; pattern avoidance; generating function.

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1. Introduction

A permutation $\pi \in \mathcal{S}_n$ is said to be in standard cycle form if its (disjoint) cycles are written from left to right in increasing order of smallest elements, with the smallest element first in each cycle. Let flat(π) denote the permutation in one-line notation obtained from π by erasing each pair of parentheses enclosing the cycles of π and considering the resulting sequence, which is referred to as the flattening of π. For example, if $\pi \in \mathcal{S}_8$ has disjoint cycles (24), (315) and (876), then its standard cycle form is $(153)(24)(687)$ and flat$(\pi) = 15324687$. The notion of a flattened permutation was originally introduced by Carlitz [5] in his definition of a certain kind of inversion statistic on $\mathcal{S}_n$; see also [15], where further properties of this statistic are studied. The idea of flattening was later extended independently by Callan [4] to finite set partitions, where here the brackets enclosing the blocks of a partition of $[n] = \{1, \ldots, n\}$, expressed in standard form, are erased and the resulting sequence is considered. Analogous notions of flattening have since been defined on other discrete structures, such as Catalan words [1], Stirling permutations [3] and parking functions [7].

Let $\rho = \rho_1 \cdots \rho_n$ and $\tau = \tau_1 \cdots \tau_m$ be positive integral sequences such that the distinct letters of τ comprise $[\ell]$ for some $\ell \geq 1$. Then ρ is said to *contain* the pattern τ in the classical sense if there exists a subsequence of ρ isomorphic to τ. That is, there exist indices $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $\rho_{i_j}\, x\, \rho_{i_k}$ if and only if $\tau_j\, x\, \tau_k$ for all $j, k \in [m]$ and $x \in \{<, >, =\}$. Otherwise, ρ is said to *avoid* τ. In analogy with the definition given for flattened set partitions [4], we say that ρ contains or avoids τ *in the flattened sense* if $\rho' =$ flat(ρ) contains or avoids τ in the usual sense. For example, let $\rho = 641935872 \in \mathcal{S}_9$ in the one-line notation whose standard cycle form is given by $(1653)(249)(78)$. Then ρ contains 312 in the flattened sense, as $\rho' = 165324978$ is seen to contain 312 in the usual sense (as witnessed by each of the four subsequences 634, 624, 534, 524 of ρ'). On the other hand, ρ itself contains several occurrences of 231, whereas ρ' avoids 231. In general, the containment (or avoidance) of a pattern τ by a permutation ρ in one sense is not related to its containment (or avoidance) in the other.

Recall that a *derangement* is a permutation of $[n]$ without fixed points, i.e., each of its cycles has length at least two. Let $\mathcal{D}(n)$ denote the set of all derangements of $[n]$. Here, we consider the problem of avoiding a pattern of length three by members of $\mathcal{D}(n)$ in the flattened sense. This extends prior work concerning the pattern avoidance problem in the flattened sense as well as related statistics (such as number of subwords or runs) on other discrete structures, including permutations [10, 12], set partitions [4, 11], Catalan words [1], Stirling permutations [3] and parking functions [7]. By contrast, in [13], the set of distinct permutations that arise as flattened partitions of $[n]$ are considered (instead of the original partitions themselves), and the statistic recording the number of increasing runs on this set is studied.

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Our results also extend work related to the avoidance of classical patterns by derangements in the usual sense. Robertson et al. in [14] enumerated the members of $\mathcal{D}(n)$ that avoid a single pattern of length three when represented in the one-line notation, which was extended to the avoidance any subset of $\mathcal{S}_3$ in [9]. Among the results, it was shown in [14] that members of $\mathcal{D}(n)$ avoiding 321 in the usual sense are enumerated by the n-th Fine number (see also [6]). By contrast, we find here that the generating function enumerating members of $\mathcal{D}(n)$ that avoid 321 in the flattened sense satisfies a third, instead of a second, order algebraic equation.

Given a set S of patterns, let $\mathcal{D}_S(n)$ denote the subset of $\mathcal{D}(n)$ whose members avoid each pattern in S in the flattened sense and let $d_S(n) = |\mathcal{D}_S(n)|$. Let $d_S(n;y)$ denote the distribution of the statistic on $\mathcal{D}_S(n)$ recording the number of cycles (marked by y). Note $d_S(n;1) = d_S(n)$, by the definitions. When S is a single pattern τ, we will write τ in place of S in the preceding quantities.

The organization of this paper is as follows. In the next section, we find explicit expressions and/or generating function formulas for $d_\tau(n;y)$, and in particular $d_\tau(n)$ (upon setting $y = 1$), for each pattern τ of length three. In particular, it is shown the Wilf-equivalence $213 \approx 312$ for derangements in the flattened sense via a bijective proof. The case of avoiding 321 is apparently more difficult and here we make use of an auxiliary array and the kernel method, along with Lagrange inversion, to establish a formula for the (ordinary) generating function of $d_{321}(n;y)$. Next, we employ a system of intertwined linear recurrences to prove the result for $d_{231}(n;y)$ and find $231 \approx 321$, with both this equivalence and the preceding one respecting the number of cycles. In the third section, we briefly describe the main results concerning the avoidance of two or more patterns in $\mathcal{S}_3$, where here the generating function for $d_S(n;y)$ works out to be rational in each case of $S \subseteq \mathcal{S}_3$ for which $|S| \geq 2$. Several entries from the OEIS [16] arise as enumerators of $\mathcal{D}_S(n)$ when $|S| \geq 2$, and hence we obtain new combinatorial interpretations for these sequences in terms of pattern-avoiding derangements.

2. Avoiding a single pattern of length three

Let $f_n(y) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} y^i$ for $n \geq 0$ denote the n-th Fibonacci polynomial. Our first result concerns the patterns 123 and 132.

Proposition 2.1. *The distribution for the statistic tracking the number of cycles (marked by y) on $\mathcal{D}_{123}(n)$ and $\mathcal{D}_{132}(n)$ is given by y and $yf_{n-2}(y)$, respectively, for all $n \geq 2$.*

Proof. Let $\pi \in \mathcal{D}(n)$, where $n \geq 2$. In order for $\pi' = \text{flat}(\pi)$ to avoid 123, we must have $\pi' = 1n(n-1)\cdots 2$, since the first cycle in standard form always starts with 1. The only member of $\mathcal{D}(n)$ whose flattened form is as given is $\pi = (1n(n-1)\cdots 2)$, which yields the first assertion. On the other hand, we have that π avoids 132 if and only if $\pi' = 12\cdots n$. Recall that $f_n(y)$ gives the distribution for the number of dominos on the set $\mathcal{F}_n$ consisting of the (linear) square-and-domino tilings of length n (see, e.g., [2, Chapter 3]). Let $\mathcal{F}'_n$ denote the subset of $\mathcal{F}_n$ whose members start with a domino. Then members of $\mathcal{D}_{132}(n)$ are in one-to-one correspondence with the tilings in $\mathcal{F}'_n$ by putting a domino for the first two letters of each cycle (going from left to right) and a square for each additional letter occurring within a cycle. Thus, the statistic on $\mathcal{D}_{132}(n)$ tracking the number of cycles corresponds to the parameter on $\mathcal{F}'_n$ recording the number of dominos, with the latter distribution being given by $yf_{n-2}(y)$ for all $n \geq 2$, which implies the second statement. □

For the sake of brevity, let us denote $d_{312}(n;y)$ by $a_n = a_n(y)$ for $n \geq 2$ and let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the n-th Catalan number. Then the sequence a_n is given recursively as follows.

Lemma 2.1. *If $n \geq 3$, then*

$$a_n = a_{n-1} + yC_{n-2} + \sum_{i=0}^{n-4} \left(yC_i + (1+y)C_{i+1}\right) a_{n-i-2}, \qquad n \geq 3, \tag{1}$$

with $a_2 = y$.

Proof. Let $\pi \in \mathcal{D}_{312}(n)$, where we may assume $n \geq 4$, as (1) is seen to hold for $n = 3$ since $a_3 = 2y$. Note first that any letter occurring to the left of 2 within $\pi' = \text{flat}(\pi)$ must be less than any letter to the right of 2 in order to avoid 312. Thus, we may decompose π' as $1\alpha 2\beta$, where α consists of all letters in $[3, i+2]$ for some $i \geq 0$. Note that there are C_i ways in which to order the letters within α as they must avoid 312. We now consider several cases based on the position of 2 within π and let C denote the first cycle of π. If 2 starts a cycle of π, then 1α comprises C and is of length $i+1$ for some $i \geq 1$. Independent of the choice of α, there are a_{n-i-1} possibilities concerning the arrangement of the elements of $[i+3, n] \cup \{2\}$, which comprise the remaining cycles of π. Considering all possible $1 \leq i \leq n-3$ then yields a contribution of $y\sum_{i=1}^{n-3} C_i a_{n-i-1}$ towards the overall weight a_n, where the initial factor of y accounts for the cycle 1α.

If 2 does not start a cycle and is not the last letter of C, then $0 \leq i \leq n-3$ and we get a contribution of $\sum_{i=0}^{n-3} C_i a_{n-i-1}$ in this case. Note that no extra y factor is required here, upon treating the terminal section 2γ of C, where γ is nonempty by assumption, as the first cycle in the derangement of $[i+3,n]\cup\{2\}$ accounted for by a_{n-i-1}. Finally, if 2 is the final letter of C and π does not consist of a single cycle, then we must have $0\le i\le n-4$ in this case since then at least two elements must occur to the right of 2 in π'. This yields $y\sum_{i=0}^{n-4}C_ia_{n-i-2}$ additional possibilities, where the factor of y again accounts for the first cycle of π. Finally, if π consists of a single cycle ending in 2, then there are yC_{n-2} possibilities. Combining all of the prior cases yields $a_n=y\sum_{i=1}^{n-3}C_ia_{n-i-1}+\sum_{i=0}^{n-3}C_ia_{n-i-1}+y\sum_{i=0}^{n-4}C_ia_{n-i-2}+yC_{n-2}$, which may be rewritten as (1). □

Let $A(x,y)=\sum_{n\ge2}a_n(y)x^n$ and $C(x)=\sum_{n\ge0}C_nx^n=\frac{1-\sqrt{1-4x}}{2x}$, which will often be denoted by C.

Theorem 2.1. *The generating function enumerating the members of either $\mathcal{D}_{213}(n)$ or $\mathcal{D}_{312}(n)$ for $n\ge2$ according to the number of cycles is given by*

$$A(x,y)=\frac{x^2yC(x)}{1+xy-x(1+y+xy)C(x)}. \tag{2}$$

Moreover, the number of members of $\mathcal{D}_{213}(n)$ or $\mathcal{D}_{312}(n)$ with exactly m cycles is given by

$$\sum_{j=0}^{m-1}\frac{2m+j}{n+j}\binom{m-1}{j}\binom{2n-2m+j-1}{n-2m},\qquad 1\le m\le\lfloor n/2\rfloor.$$

Proof. Multiplying both sides of (1) by x^n, and summing over all $n\ge3$, we obtain $A(x,y)-x^2y=xA(x,y)+x^2y(C-1)$ $+x^2yCA(x,y)+x(1+y)(C-1)A(x,y)$, which yields (2). By the fact $xC^2=C-1$, we have

$$A(x,y)=\frac{x^2yC^2}{1-y(C-x-1)}=\frac{x^2yC^2}{1-x^2yC^2(C+1)}=\sum_{m\ge1}C^{2m}(C+1)^{m-1}x^{2m}y^m.$$

Hence, if $n\ge2$ and $1\le m\le\lfloor n/2\rfloor$, we then get

$$[x^ny^m]A(x,y)=\sum_{j=0}^{m-1}\binom{m-1}{j}[x^{n-2m}]C^{2m+j}=\sum_{j=0}^{m-1}\frac{2m+j}{n+j}\binom{m-1}{j}\binom{2n-2m+j-1}{n-2m},$$

as desired, where we have applied [18, Equation 2.5.16] in the second equality.

To complete the proof, we define a bijection $f=f_n$ inductively as follows for all $n\ge2$ between $\mathcal{D}_{312}(n)$ and $\mathcal{D}_{213}(n)$, which preserves the number of cycles. We may take f_n to be the identity if $n=2,3$, so assume $n\ge4$. Let $\pi\in\mathcal{D}_{312}(n)$ and first suppose 2 does not lie in the same cycle as 1, with (1α) denoting the first cycle of π. Then π avoiding 312 implies α consists of all the letters in $[3,i]$ for some $3\le i\le n-1$. Note α itself is a 312-avoiding permutation in the usual sense on its alphabet. Let g be any of the known bijections between 312- and 213-avoiding permutations of the same length and we apply g to α. We then add $n-i$ to each letter in $g(\alpha)$ to obtain a 213-avoiding permutation of the set $[n-i+3,n]$, which we denote by γ. Next, we apply f_{n-i+1} to the partial derangement $\pi-(1\alpha)$, and express the resulting derangement, which we denote by ρ, in standard cycle form using the letters in $[2,n-i+2]$ instead of $[i+1,n]\cup\{2\}$. Then let $f(\pi)=(1\gamma)\rho$, which is seen to belong to $\mathcal{D}_{213}(n)$ and has the same number of cycles as π. We define f similarly if 2 occurs in the first cycle, where 2 is to maintain its relative position within the first cycle of $f(\pi)$. One may verify that f yields the desired bijection between $\mathcal{D}_{312}(n)$ and $\mathcal{D}_{213}(n)$. □

2.1. The case 321

Given $n\ge2$ and $2\le i\le n$, let $\mathcal{D}_{321}(n,i)$ denote the subset of $\mathcal{D}_{321}(n)$ whose members end in i. Let us denote the distribution $d_{321}(n;y)$ by $d_n=d_n(y)$ and let $d_{n,i}=d_{n,i}(y)$ be the restriction of d_n to $\mathcal{D}_{321}(n,i)$ for $2\le i\le n$. Note $d_n=\sum_{i=2}^n d_{n,i}$, by the definitions.

In order to write a recurrence for $d_{n,i}$, it is convenient to consider its restriction to a particular subset of $\mathcal{D}_{321}(n,i)$ as follows. Given $n\ge3$ and $2\le i\le n$, let $w_{n,i}=w_{n,i}(y)$ denote the restriction of $d_{n,i}$ to those members of $\mathcal{D}_{321}(n,i)$ whose final cycle has length at least three, the subset of which we denote by $\mathcal{W}(n,i)$, and let $w_n=\sum_{i=2}^n w_{n,i}$. For example, when $n=4$, we have $d_{4,2}=y$, $d_{4,3}=d_{4,4}=2y+y^2$ and $w_{4,2}=y$, $w_{4,3}=w_{4,4}=2y$. Put zero for $d_{n,i}$ and $w_{n,i}$ in all cases where the enumerated set of derangements is empty.

The $d_{n,i}$ and $w_{n,i}$ are given recursively as follows.

Lemma 2.2. *If $n\ge3$, then*

$$d_{n,i}=d_{n-1,i}+\sum_{j=2}^{i-1}(d_{n-1,j}+yw_{n-1,j}),\qquad 2\le i\le n-1, \tag{3}$$

$$w_{n,i}=\sum_{j=2}^{i}d_{n-1,j},\qquad 2\le i\le n-1, \tag{4}$$

with $d_{n,n}=d_{n-1}+yw_{n-1}$ and $w_{n,n}=d_{n-1}$ for $n\ge3$ and $d_2=d_{2,2}=y$ and $w_2=w_{2,2}=0$.

Proof. The initial conditions for $n=2$ are clear, so assume $n\geq 3$. To show (3), let $\pi\in\mathcal{D}_{321}(n,i)$ where $i\in[2,n-1]$ and consider the penultimate letter j in flat(π). If $j=n$, then the final cycle of π must be of length at least three, and hence deleting n in this case results in a member of $\mathcal{D}_{321}(n-1,i)$. Note $i<j<n$ is not possible due to the avoidance of 321, so assume $i\geq 3$ with $2\leq j\leq i-1$. If j does not start a cycle, then we may delete i (as it is preceded by j with $j<i$), resulting in a member of $\cup_{j=2}^{i-1}\mathcal{D}_{321}(n-1,j)$ and thus yielding a contribution of $\sum_{j=2}^{i-1}d_{n-1,j}$ towards the weight. Otherwise, j and i comprise a 2-cycle, in which case we delete i and add j to the end of the penultimate cycle of π to obtain a new derangement π^*. Note that $\pi^*\in\mathcal{W}(n-1,j)$, since the final cycle of π^* has length at least three. Thus, we obtain a contribution of $y\sum_{j=2}^{i-1}w_{n-1,j}$, where the factor of y accounts for the terminal cycle (ji) which was deleted from π in forming π^*. Combining each of the prior cases then implies (3). Note that deleting i from $\pi\in\mathcal{W}(n,i)$ where $i\in[2,n-1]$ results in an arbitrary member of $\cup_{j=2}^{i-1}\mathcal{D}_{321}(n-1,j)$ when $j<i$, whereas deleting n from π yields a member of $\mathcal{D}_{321}(n-1,i)$ when $j=n$. Combining these two cases implies (4). Finally, the formulas for $d_{n,n}$ and $w_{n,n}$ follow by similar arguments and the definitions of d_n and w_n. $\square$

Define the generating functions $D(x,y;v)=\sum_{n\geq 2}\left(\sum_{i=2}^{n}d_{n,i}v^{i-2}\right)x^n$ and $W(x,y;v)=\sum_{n\geq 3}\left(\sum_{i=2}^{n}w_{n,i}v^{i-2}\right)x^n$. Then $D(x,y;v)$ satisfies the following functional equation.

Lemma 2.3. *We have*

$$\frac{K(v)}{v(1-v)^2}D(x/v,y;v)=\frac{x^2y}{v^2}+\frac{x(v-xy-1)}{v(1-v)^2}D(x,y;1)-\frac{xy}{v(1-v)}W(x,y;1), \tag{5}$$

where $K(v)=v^3-2v^2+(x+1)v-x^2y-x$.

Proof. First note

$$\sum_{n\geq 4}\sum_{i=3}^{n-1}\sum_{j=2}^{i-1}d_{n-1,j}x^nv^{i-2}=\sum_{n\geq 4}x^n\sum_{j=2}^{n-2}d_{n-1,j}\sum_{i=j+1}^{n-1}v^{i-2}=\sum_{n\geq 3}x^n\sum_{j=2}^{n-1}d_{n-1,j}\frac{v^{j-1}-v^{n-2}}{1-v}=\frac{xv}{1-v}D(x,y;v)-\frac{x}{v(1-v)}D(xv,y;1)$$

and, similarly,

$$y\sum_{n\geq 4}\sum_{i=3}^{n-1}\sum_{j=2}^{i-1}w_{n-1,j}x^nv^{i-2}=\frac{xyv}{1-v}W(x,y;v)-\frac{xy}{v(1-v)}W(xv,y;1).$$

By the formula for $d_{n,n}$, we also have

$$\sum_{n\geq 3}d_{n,n}x^nv^{n-2}=\sum_{n\geq 3}x^nv^{n-2}\sum_{i=2}^{n}d_{n-1,i}+y\sum_{n\geq 4}x^nv^{n-2}\sum_{i=2}^{n-1}w_{n-1,i}=\frac{x}{v}D(xv,y;1)+\frac{xy}{v}W(xv,y;1).$$

Thus, multiplying both sides of (3) by x^nv^{i-2}, summing over all $n\geq 3$ and $2\leq i\leq n-1$, and taking into account the contribution from the $d_{n,n}$ terms yields

$$\begin{aligned}D(x,y;v)&=x^2y+xD(x,y;v)+\frac{xv}{1-v}D(x,y;v)-\frac{x}{v(1-v)}D(xv,y;1)+\frac{xyv}{1-v}W(x,y;v)-\frac{xy}{v(1-v)}W(xv,y;1)\\&\quad+\frac{x}{v}D(xv,y;1)+\frac{xy}{v}W(xv,y;1)\\&=x^2y+\frac{x}{1-v}\left(D(x,y;v)-D(xv,y;1)\right)+\frac{xy}{1-v}\left(vW(x,y;v)-W(xv,y;1)\right).\end{aligned} \tag{6}$$

By (4) and the formula for $w_{n,n}$, we also have

$$\begin{aligned}W(x,y;v)&=\sum_{n\geq 3}\sum_{i=2}^{n-1}w_{n,i}x^nv^{i-2}+\sum_{n\geq 3}w_{n,n}x^nv^{n-2}=\sum_{n\geq 3}\sum_{i=2}^{n-1}\sum_{j=2}^{i}d_{n-1,j}x^nv^{i-2}+\sum_{n\geq 3}\sum_{j=2}^{n-1}d_{n-1,j}x^nv^{n-2}\\&=\sum_{n\geq 3}\sum_{j=2}^{n-1}\sum_{i=j}^{n-1}d_{n-1,j}x^nv^{i-2}+\sum_{n\geq 2}\sum_{j=2}^{n}d_{n,j}x^{n+1}v^{n-1}\\&=\frac{x}{v(1-v)}\left(vD(x,y;v)-D(xv,y;1)\right)+\frac{x}{v}D(xv,y;1)=\frac{x}{1-v}\left(D(x,y;v)-D(xv,y;1)\right).\end{aligned} \tag{7}$$

Substituting (7) into (6) yields, after some algebra,

$$\left(1-\frac{x}{1-v}-\frac{x^2yv}{(1-v)^2}\right)D(x,y;v)=x^2y-\frac{x-xv+x^2yv}{(1-v)^2}D(xv,y;1)-\frac{xy}{1-v}W(xv,y;1),$$

which may be rewritten to give (5), upon replacing x with x/v. $\square$

Note that the kernel equation $K(v)=0$ in (5) has three solutions v_0,v_1,v_2, where

$$\begin{aligned} v_0 &= x+(1+y)x^2+\cdots,\\ v_1 &= 1+\frac{\sqrt{1+4y}-1}{2}x+\frac{1+3y-(1+y)\sqrt{1+4y}}{2\sqrt{1+4y}}x^2+\cdots,\\ v_2 &= 1-\frac{\sqrt{1+4y}+1}{2}x-\frac{1+3y+(1+y)\sqrt{1+4y}}{2\sqrt{1+4y}}x^2+\cdots. \end{aligned}$$

Taking $v=v_1$ or $v=v_2$ in (5), and solving the resulting system in $D(x,y;1)$ and $W(x,y;1)$, we obtain

$$D(x,y;1)=\frac{-(v_1v_2-v_1-v_2+1)}{v_1v_2} \text{ and } W(x,y;1)=\frac{-(v_1v_2xy-v_1v_2-xy+v_1+v_2-1)}{v_1v_2y}.$$

By the kernel equation, we have $v_0v_1v_2=x(1+xy)$ and $v_0+v_1+v_2=2$. Hence, rewriting the formulas above for $D(x,y;1)$ and $W(x,y;1)$ in terms of v_0 yields the following result.

Lemma 2.4. *We have*

$$D(x,y;1)=\frac{v_0(1-v_0)}{x(1+xy)}-1 \text{ and } W(x,y;1)=\frac{v_0(xy-1+v_0)}{xy(1+xy)}+\frac{1-xy}{y}, \tag{8}$$

where $v_0=x+(1+y)x^2+\cdots$ *is a solution of* $K(v)=v^3-2v^2+(x+1)v-x^2y-x=0$.

Theorem 2.2. *The generating function* $D(x,y;1)$ *is given explicitly by*

$$D(x,y;1)=\frac{xy}{1+xy}\sum_{n\geq1}\sum_{j=0}^{n}\frac{1}{n}\binom{n}{j}\binom{2n+j}{n-1}x^{n+j}y^j. \tag{9}$$

Moreover, the number of members of $\mathcal{D}_{321}(a+b)$ *with exactly* a *cycles is given by* $\sum_{j=1}^{a}\frac{(-1)^{a-j}}{b}\binom{b}{j-1}\binom{2b+j-1}{b-1}$ *for* $1\leq a\leq b$.

Proof. Let $f(z)=z^3+xz^2-x^2yz+x^4y^2$ and let p denote a root to the kernel equation $K(v)=0$. Note that

$$\begin{aligned} -f(p(1-p)-x) &= p^6-3p^5+(2x+3)p^4-(4x+1)p^3-x(xy-x-2)p^2+x^2(y-1)p-x^3y(1+xy)\\ &= (p^3-2p^2+(x+1)p-x(1+xy))(p^3-p^2+xp+x^2y)=K(p)(p^3-p^2+xp+x^2y)=0, \end{aligned}$$

and hence the roots r to $f(z)=0$ are given by $r=v_i(1-v_i)-x$ for $0\leq i\leq 2$. Let $s=\frac{r}{xy}$, and note $z=s$ is a root of $z=x+z^2+yz^3$. Then $h=s-x$ satisfies $h=(h+x)^2(1+y(h+x))$, and applying the Lagrange inversion formula (see, e.g., [17, Section 5.4] or [18, Section 5.1]), we have $h=s-x=\sum_{n\geq1}\sum_{j=0}^{n}\frac{1}{n}\binom{n}{j}\binom{2n+j}{n-1}x^{n+j+1}y^j$. Note that $xy(h+x)=v_i(1-v_i)-x$ for some $0\leq i\leq 2$, and upon observing that $v_i(1-v_i)-x$ has x coefficient zero only for $i=0$, we must have $xy(h+x)=v_0(1-v_0)-x$. By (8), we then get

$$D(x,y;1)=\frac{v_0(1-v_0)-x-x^2y}{x(1+xy)}=\frac{yh}{1+xy}=\frac{xy}{1+xy}\sum_{n\geq1}\sum_{j=0}^{n}\frac{1}{n}\binom{n}{j}\binom{2n+j}{n-1}x^{n+j}y^j,$$

as desired. Note that the coefficient of $x^{c+1}y^{d+1}$ where $c>d\geq0$ in $D(x,y;1)$ is given by $\sum_{j=0}^{d}\frac{(-1)^{d-j}}{c-d}\binom{c-d}{j}\binom{2c-2d+j}{c-d-1}$, which is zero if $c\leq 2d$, by [8, Identity 5.25], as it should be on combinatorial grounds. Thus, the only other possible nonzero coefficients of $D(x,y;1)$ are those of the form $x^{a+b}y^a$ for some $1\leq a\leq b$, in which case we get

$$[x^{a+b}y^a]D(x,y;1)=\sum_{j=1}^{a}\frac{(-1)^{a-j}}{b}\binom{b}{j-1}\binom{2b+j-1}{b-1},$$

which yields the second statement and completes the proof. □

By the second statement in Theorem 2.2, we have

$$d_{321}(n)=\sum_{m=1}^{\lfloor n/2\rfloor}\sum_{j=1}^{m}\frac{(-1)^{m-j}}{n-m}\binom{n-m}{j-1}\binom{2n-2m+j-1}{n-m-1}, \qquad n\geq2,$$

with this sequence apparently not occurring in [16]. By (5) and (8), we have

$$D(x,y;v)=\frac{xy(\hat{v}_0^2+(v-2)\hat{v}_0+x^2yv+x)}{(1+xyv)(1-x-(2-x+x^2y)v+v^2)}, \tag{10}$$

where $\hat{v}_0$ denotes v_0 with x replaced by xv. Using (10) and the kernel equation, we obtain the next result for general v.

Theorem 2.3. *The generating function $g(v) = D(x, y; v)$ satisfies*

$$x^4y^3 + x^2y^2(3x^2yv - xv + 3x - 1)g(v) + xy(3x^2yv - 2xv + 2v + 3x - 2)(xyv + 1)g^2(v)$$
$$- ((1 - v)^2 - x(1 - v + xyv))(xyv + 1)^2g^3(v) = 0.$$

Moreover, expressing the solution to the preceding cubic equation trigonometrically, we obtain

$$D(x, y; v) = \frac{xy\left[2\sqrt{Q}\cos\left(\frac{1}{3}\arccos\left(\frac{R}{2Q\sqrt{Q}}\right)\right) + 3x^2yv - 2xv + 2v + 3x - 2\right]}{3(1 + xyv)(1 - x - (2 - x + x^2y)v + v^2)}, \quad |v| < 1,$$

where $Q = 1 - 2(2x + 1)v - (6x^2y - x^2 - 7x - 1)v^2 + 3x(3xy - 1)v^3$ *and* $R = 2 - 6(2x + 1)v - 3(6x^2y - 5x^2 - 11x - 2)v^2 + (36x^3y + 2x^3 + 45x^2y - 33x^2 - 30x - 2)v^3 + 9x(3x^3y^2 - 3x^2y - 6xy + 2x + 1)v^4 + 27x^2yv^5$.

To establish a comparable trigonometric formula for $D(x, y; v)$ when $v = 1$, we first observe

$$v_0(1 - v_0) = \frac{2x}{3}\left(1 - \sqrt{3y + 1}\cos\left(\frac{1}{3}\arccos\left(\frac{27xy^2 + 9y + 2}{2(3y + 1)^{3/2}}\right) - \frac{2\pi}{3}\right)\right). \tag{11}$$

Note that (11) can be shown by demonstrating that the right-hand side quantity satisfies the same minimal equation, namely, $p^3 - 2xp^2 + x^2(1 - y)p + x^3y(1 + xy) = 0$, as the $v_i(1 - v_i)$, and hence it must equal $v_i(1 - v_i)$ for some $0 \le i \le 2$. Since only $v_0(1 - v_0)$ has leading coefficient x in its Taylor expansion like the right-hand side, equality (11) follows. Using (8) and (11), we then obtain the following formula for $D(x, y; 1)$.

Corollary 2.1. *The generating function $g = D(x, y; 1)$ counting the members of $\mathcal{D}_{321}(n)$ for $n \ge 2$ according to the number of cycles is given by*

$$D(x, y; 1) = \frac{2\sqrt{3y + 1}\cos\left(\frac{1}{3}\arccos\left(\frac{27xy^2+9y+2}{2(3y+1)^{3/2}}\right) + \frac{\pi}{3}\right) - 3xy - 1}{3(xy + 1)}$$

and satisfies the algebraic equation $x^2y^2 + y(3x^2y + 2x - 1)g + (xy + 1)(3xy + 1)g^2 + (xy + 1)^2g^3 = 0$.

In particular, the generating function $f = D(x, 1; 1)$ of the sequence $d_{321}(n)$ for $n \ge 2$ is given by

$$D(x, 1; 1) = \frac{4\cos\left(\frac{1}{3}\arccos\left(\frac{27x+11}{16}\right) + \frac{\pi}{3}\right) - 3x - 1}{3(x + 1)},$$

with $x^2 + (x + 1)(3x - 1)f + (x + 1)(3x + 1)f^2 + (x + 1)^2f^3 = 0$.

2.2. The case 231

In this subsection, we show that 231 is Wilf-equivalent to 321 for derangements in the flattened sense. Let $b_n = b_n(y)$ denote the distribution $d_{231}(n; y)$. To assist in finding a formula for the generating function of b_n, we consider its restriction $v_n = v_n(y)$ to those members of $\mathcal{D}_{231}(n)$ whose final cycle has length at least three, the subset of which we will denote by $\mathcal{V}(n)$. The sequences b_n and v_n satisfy the following system of intertwined recurrences.

Lemma 2.5. *If $n \ge 4$, then*

$$b_n = 2b_{n-1} + y(v_{n-1} + b_{n-2}) + (1/y)\sum_{i=2}^{n-2}(b_i + yv_i)(b_{n-i} + yb_{n-i-1}), \qquad n \ge 4, \tag{12}$$

$$v_n = b_{n-1} + v_{n-1} + b_{n-2} + 2yv_{n-2} + (1/y)\sum_{i=2}^{n-3}(b_i + yv_i)(v_{n-i} + yv_{n-i-1}), \qquad n \ge 4, \tag{13}$$

with $b_1 = v_1 = v_2 = 0$, $b_2 = y$ and $b_3 = v_3 = 2y$.

Proof. We may assume $n \ge 4$, the initial conditions being clear. To show (12), let $\pi \in \mathcal{D}_{231}(n)$ and we consider the position of n within $\pi' = \text{flat}(\pi)$. Note that any letter occurring to the left of n within π' is smaller than any occurring to the right in order to avoid 231, and hence those to the left comprise the set $[i]$ for some $i \in [n - 1]$. If n occurs as the second letter of π', then the weight of such π is seen to be $yb_{n-2} + b_{n-1}$, upon considering whether or not the 2-cycle $(1n)$ occurs. If n is the last letter of π', then the weight is $yv_{n-1} + b_{n-1}$, upon considering whether or not n belongs to a 2-cycle, since in cases when it does, we may add i occurring in the terminal cycle (in) to the end of the penultimate cycle of π and delete (in), which accounts for the extra factor of y.

So assume that the set of letters occurring to the left of n within π' is $[i]$ for some $2 \le i \le n-2$ and we consider the following four subcases:

(a) n is second in its cycle, but not last,

(b) n occurs in a 2-cycle,

(c) n is neither second nor last in its cycle,

(d) n is last in its cycle, but does not occur in a 2-cycle.

Note that $i \ge 2$ implies n cannot appear within the first cycle of π in case (a) and that the 2-cycle $(1n)$ cannot occur in (b), with these excluded cases already having been treated above. Likewise, n cannot occur as the final element of π' in either (b) or (d). One then obtains the following respective contributions to the overall weight from the four cases: (a) $\sum_{i=3}^{n-2} v_i b_{n-i}$, (b) $y\sum_{i=3}^{n-3} v_i b_{n-i-1}$, (c) $(1/y)\sum_{i=2}^{n-2} b_i b_{n-i}$, (d) $\sum_{i=2}^{n-3} b_i b_{n-i-1}$.

To show (a), note that if the cycle C containing n starts j, n for some $j \in [2, i]$, then appending j to the end of the cycle preceding C results in a member of $\mathcal{V}(i)$, with all elements of π to the right of and including n within π' being enumerated by b_{n-i} (where here we treat n as the "smallest" element, with its cycle comprising $C - \{j\}$). Note $i \ge 3$ in this case since n being second in its cycle implies at least three letters must occur to the left of n in π'. Considering all possible $3 \le i \le n-2$ then gives a contribution of $\sum_{i=3}^{n-2} v_i b_{n-i}$. If (b) holds, then we delete the 2-cycle (jn) and append j to the cycle preceding C. Note that $3 \le i \le n-3$ in (b) since n is neither second nor last in π'. Considering all possible i then yields $y\sum_{i=3}^{n-3} v_i b_{n-i-1}$, where the extra factor of y accounts for the deleted cycle (jn). Comparable arguments may be given for the contributions from (c) and (d), where in these cases no letter needs to be moved forward to the cycle preceding C as n here is not the second letter of C. Note that the summation formulas in cases (a) and (b) above may be started at $i = 2$ since $v_2 = 0$, while those in (b) and (d) may go up to $i = n-2$ as $b_1 = 0$. Thus, the formulas from (a) and (b) combine to give $\sum_{i=2}^{n-2} v_i(b_{n-i} + yb_{n-i-1})$, whereas those from (c) and (d) give $(1/y)\sum_{i=2}^{n-2} b_i(b_{n-i} + yb_{n-i-1})$. Combining cases (a)–(d) then yields $(1/y)\sum_{i=2}^{n-2}(b_i + yv_i)(b_{n-i} + yb_{n-i-1})$. Taking this expression together with the cases above where n was either second or last in π' implies (12).

A comparable argument applies to (13). Let $\pi \in \mathcal{V}(n)$, where $n \ge 4$. Note first that π for which n occurs second in π' contribute $yv_{n-2}+v_{n-1}$ towards the overall weight v_n, upon considering whether or not the 2-cycle $(1n)$ occurs, whereas π for which n is last in π' contribute b_{n-1}. So assume n is neither second nor last within π'. We obtain from the cases (a)–(d) above respective contributions towards the weight of (a) $yv_{n-2} + \sum_{i=3}^{n-3} v_i v_{n-i}$, (b) $y\sum_{i=3}^{n-4} v_i v_{n-i-1}$, (c) $b_{n-2} + (1/y)\sum_{i=2}^{n-3} b_i v_{n-i}$ and (d) $\sum_{i=2}^{n-4} b_i v_{n-i-1}$. Note that the initial yv_{n-2} term in (a) accounts for the case in which the final cycle of π is $(in(n-1))$ for some $i \in [2, n-2]$, whereas the b_{n-2} term in (c) accounts for π in which the last cycle has the form $(j\alpha n(n-1))$, where α is nonempty. Thus (a)–(d), taken together, yields

$$b_{n-2} + yv_{n-2} + (1/y)\sum_{i=2}^{n-3}(b_i + yv_i)(v_{n-i} + yv_{n-i-1}),$$

and combining with the first two cases above gives (13). □

Theorem 2.4. *The patterns 231 and 321 are equivalent in the flattened sense on derangements, with this equivalence respecting the number of cycles.*

Proof. Let $B(x,y) = \sum_{n\ge2} b_n(y)x^n$ and $V(x,y) = \sum_{n\ge3} v_n(y)x^n$, which we will denote by B and V, respectively. Multiplying both sides of (12) and (13) by x^n, and summing over $n \ge 4$, we obtain

$$B = x^2y + 2xB + xy(xB + V) + \frac{1+xy}{y}B(B + yV), \tag{14}$$

$$V = x^3y + x(1+x)B + x(1+2xy)V + \frac{1+xy}{y}V(B + yV). \tag{15}$$

Multiplying both sides of (14) by V and both sides of (15) by B, and comparing the resulting equations, gives

$$xyV + yV^2 = x^2yB + (1+x)B^2 - (1-xy)BV. \tag{16}$$

Solving for V in terms of B in (14), and substituting into (16), we obtain after several algebraic steps,

$$x^2y^2 - y(1 - 2x - 3x^2y)B + (1+xy)(1+3xy)B^2 + (1+xy)^2B^3 = 0.$$

A comparison with Corollary 2.1 now shows $B(x,y) = D(x,y;1)$, which implies the result. □

We were unable to find a direct bijective proof of the preceding theorem, which we leave as an open question.

3. Avoiding two or more patterns of length three

In this section, we state without proof the results concerning the avoidance of two or more patterns of length three. Our work is shortened somewhat in this regard by noting that if $S \subseteq \mathcal{S}_3$ contains 123 or 132, where $|S| \geq 2$, then $d_S(n;y)$ may easily be obtained from the proof of Proposition 2.1. In cases when S contains 123, then $d_S(n;y)$ is trivial, either being 0 or y for all $n \geq 4$ depending on whether or not S also contains 132 or 321. On the other hand, if S contains 132, but not 123, then $d_S(n;y) = d_{132}(n;y) = yf_{n-2}(y)$ for $n \geq 2$. Thus, we may restrict attention to S not containing 123 or 132.

Our first result features several pattern pairs for which $d_S(n;y)$ satisfies a recurrence of second order.

Theorem 3.1. *If $n \geq 1$, then $d_S(n;y) = j_n$ and $d_{\{231,321\}}(n;y) = k_n$, where $S = \{213, 231\}$, $\{231, 312\}$ or $\{312, 321\}$, j_n is given recursively by $j_n = 2j_{n-1} + 2yj_{n-2}$ for $n \geq 3$, with $j_1 = 0$, $j_2 = y$, and k_n is given by $k_n = 2k_{n-1} + 3yk_{n-2}$ for $n \geq 3$, with the same initial conditions.*

We have that j_n and k_n for $n \geq 1$ reduce when $y = 1$ to the sequences $A002605[n-1]$ and $A015518[n-1]$, respectively, in [16]. Note that the formulas in Theorem 3.1 may be obtained directly by arguing that $d_S(n;y)$ satisfies the stated two-term recurrence in each case, upon considering the relative position of certain elements (such as 2 or n) within a pattern-restricted derangement expressed in standard cycle form.

There are the following generating function formulas in the cases when $S = \{213, 312\}$ or $\{213, 321\}$.

Theorem 3.2. *We have*

$$\sum_{n\geq 2} d_{\{213,312\}}(n;y)x^n = \frac{x^2y(1-x)}{(1-2x)(1-x-x^2y)}, \tag{17}$$

$$\sum_{n\geq 2} d_{\{213,321\}}(n;y)x^n = \frac{x^2y(1-x+x^2+x^3y)}{(1-x)(1-x-x^2y)^2}. \tag{18}$$

Upon taking $y = 1$ in (17), we have $d_{\{213,312\}}(n) = 2^{n-1} - F_n$, where $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ denotes the n-th Fibonacci number with $F_0 = 0$, $F_1 = 1$. Hence, $d_{\{213,312\}}(n)$ coincides with $A027934[n-1]$ for all $n \geq 1$. Note that it is also possible to show this result directly by defining a bijection between the members of $\mathcal{D}_{\{213,312\}}(n)$ and compositions of n containing at least one even part. Further, from the form of the generating functions in (17) and (18), it is seen that the sequences $d_{\{213,312\}}(n;y)$ and $d_{\{213,321\}}(n;y)$ satisfy third and fifth order linear recurrences, respectively.

We have the following result for avoiding more than two patterns of length three, where $f_n(y) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} y^k$ for $n \geq 0$, with $f_{-1}(y) = 0$.

Theorem 3.3. *If $n \geq 2$, then $d_T(n;y) = f_n(y) - 1$, $d_{\{213,231,321\}}(n;y) = \frac{(n-1)yf_{n-1}(y)+2ny^2f_{n-2}(y)}{1+4y}$, $d_{\{231,312,321\}}(n;y) = \ell_n$ and $d_{\{213,231,312,321\}}(n;y) = y(f_{n-2}(y) + f_{n-3}(y))$, where $T = \{213, 231, 312\}$ or $\{213, 312, 321\}$ and ℓ_n denotes the sequence defined recursively by $\ell_n = \ell_{n-1} + (2y+1)\ell_{n-2} + y\ell_{n-3}$ for $n \geq 4$, with $\ell_1 = 0$, $\ell_2 = y$, $\ell_3 = 2y$.*

Let $L_n = L_{n-1} + L_{n-2}$ and $p_n = 2p_{n-1} + p_{n-2}$ for $n \geq 2$ denote the n-th Lucas and n-th Pell number, respectively, where $L_0 = 2$, $L_1 = 1$, $p_0 = 0$, $p_1 = 1$. Taking $y = 1$ in Theorem 3.3 implies $d_T(n) = F_{n+1} - 1$, $d_{\{213,231,321\}}(n) = \frac{nL_n - F_n}{5}$, $d_{\{231,312,321\}}(n) = p_{n-1}$ and $d_{\{213,231,312,321\}}(n) = F_n$, where T is as above. Note that $F_{n+1} - 1$ and $\frac{nL_n-F_n}{5}$ occur respectively as $A000071[n-1]$ and $A001629[n]$ in [16], and hence one obtains new combinatorial interpretations for these sequences like others in this section in terms of flattened derangements.

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