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Research Article

A note on Hadwiger's conjecture for path-chromatic number

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Abstract

Let G be a graph and $(T, \mathcal{B})$ be a tree-decomposition of G, where $\mathcal{B} = (B_t)_{t \in V(T)}$. The *chromatic number* of $(T, \mathcal{B})$ is the maximum chromatic number among the subgraphs induced by B_t. The *tree-chromatic number* of G is the minimum chromatic number of the tree-decompositions of G. Huynh, Reed, Wood, and Yepremyan [*2019-20 MATRIX Annals*, Springer, Cham, 2021, 489–498] posed a Hadwiger-type conjecture for tree-chromatic number, and asked for a short proof that every K_6-minor-free graph has tree-chromatic number at most 5 even if it is allowed to use the Four Color Theorem. The present article answers this question by observing that, assuming Hadwiger's Conjecture for K_p-minor-free graphs, every K_{p+1}-minor-free graph has tree-chromatic number at most p. More precisely, only path-decompositions of graphs are considered.

Keywords: tree-decomposition; tree-chromatic number; path-chromatic number; Hadwiger's Conjecture.

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1. Introduction

In this paper, we only consider simple, undirected, and finite graphs. We use the terminology given in [3]. Given a graph G and $U \subseteq V(G)$, $G[U]$ denotes the subgraph induced by U. For a graph G and a vertex $v \in V(G)$, $N_G(v)$ is the neighborhood of v in G, and $N_G[v]$ is the closed neighborhood of v, i.e., $N_G[v] = N_G(v) \cup \{v\}$. For $U \subseteq V(G)$, we write $N_G(U) = \bigcup_{v \in U} N_G(v) \setminus U$ and $N_G[U] = \bigcup_{v \in U} N_G[v]$. Let G and H be graphs. We say that G contains an *H-minor* if a graph isomorphic to H can be obtained from a subgraph of G by repeating contractions of an edge. If G does not contain an H-minor, we call G *H-minor-free*.

A *tree-decomposition* of a graph G is a pair $(T, \mathcal{B})$, where T is a tree and $\mathcal{B} = (B_t)_{t \in V(T)}$ is a family of subsets of $V(G)$ indexed by the vertices of T satisfying the following conditions:

- For any $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in B_t$.
- For any $v \in V(G)$, the set $\{t \in V(T) \mid v \in B_t\}$ induces a nonempty subtree of T.

If T is a path, the tree-decomposition $(T, \mathcal{B})$ is called a *path-decomposition*.

In [8], Seymour introduced the tree-chromatic number and the path-chromatic number of a graph as follows. Let G be a graph and $(T, \mathcal{B})$ be a tree-decomposition of G, where $\mathcal{B} = (B_t)_{t \in V(T)}$. The *chromatic number* of $(T, \mathcal{B})$ is the maximum chromatic number of $G[B_t]$ among $t \in V(T)$. The *tree-chromatic number* (respectively, *path-chromatic number*) of G, denoted by $\mathsf{tree}\text{-}\chi(G)$ (respectively, $\mathsf{path}\text{-}\chi(G)$), is the minimum chromatic number of the tree-decompositions (respectively, path-decompositions) of G. By definition, any graph G satisfies $\mathsf{tree}\text{-}\chi(G) \le \mathsf{path}\text{-}\chi(G) \le \chi(G)$.

Hadwiger's Conjecture [4] is well-known in graph theory:

Conjecture 1.1. *For any integer $p > 0$, every K_{p+1}-minor-free graph G satisfies $\chi(G) \le p$.*

However, Conjecture 1.1 is extremely difficult and is proved only for $p \le 5$ (see [7]). Considering this situation, Huynh, Reed, Wood, and Yepremyan [6] posed Conjecture 1.1 for tree-chromatic number and path-chromatic number, respectively:

Conjecture 1.2. *For any integer $p > 0$, every K_{p+1}-minor-free graph G satisfies* $\mathsf{tree}\text{-}\chi(G) \le p$.

Conjecture 1.3. *For any integer $p > 0$, every K_{p+1}-minor-free graph G satisfies* $\mathsf{path}\text{-}\chi(G) \le p$.

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Note that if Conjecture 1.3 is true, then Conjecture 1.2 is true. In [6], Huynh et al. showed that Conjecture 1.2 is more tractable than Conjecture 1.1 by giving a proof of Conjecture 1.2 for $p = 4$ without using the Four Color Theorem [1, 2]. They also asked the following question in Section 3 of [6]:

Question 1.1. *Can we give a short proof that Conjecture 1.2 holds for $p = 5$ even if we are allowed to use the Four Color Theorem?*

We answer Question 1.1 by observing that the following theorem holds:

Theorem 1.1. *Let $p > 0$ be an integer. If Conjecture 1.1 holds for p, then Conjecture 1.3 holds for $p + 1$.*

Since Conjecture 1.1 holds for $p \leq 5$ (see [7]), Conjecture 1.3 holds for $p \leq 6$ by Theorem 1.1. In particular, Conjecture 1.1 for $p = 4$ is equivalent to the Four Color Theorem by the classical Wagner's Theorem [9], thus we give an answer to Question 1.1.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we prepare the following simple path-decomposition introduced by Huynh and Kim [5]. Let G be a graph of order n and given an enumeration $(v_1, \dots, v_n)$ of $V(G)$. For each i with $1 \leq i \leq n$, put

$$X_i = N_G[\{v_1, \dots, v_i\}] \setminus \{v_1, \dots, v_{i-1}\}.$$

It is easy to check that $(P_n, (X_i)_{1 \leq i \leq n})$ is a path-decomposition of G. Now, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. For an integer $p > 0$, let G be a K_{p+1}-minor-free graph of order n. We may assume that G is connected. First, we choose a vertex v_1 of G arbitrarily. Next, for each i with $1 \leq i \leq n$, we repeatedly choose a vertex v_i so that $G[\{v_1, \dots, v_i\}]$ is connected. For an enumeration $(v_1, \dots, v_n)$ of $V(G)$ obtained by this procedure, let $X_i = N_G[\{v_1, \dots, v_i\}] \setminus \{v_1, \dots, v_{i-1}\}$. Then, $(P_n, (X_i)_{1 \leq i \leq n})$ is a path-decomposition of G.

To complete the proof, we show that each $G[X_i]$ is p-colorable for $i = 1, \dots, n$. To this end, it suffices to show

$$\chi(G[X_i \setminus \{v_i\}]) \leq p - 1.$$

Let r be the largest integer such that $G[X_i \setminus \{v_i\}]$ contains a K_r-minor. By the definition of X_i, every vertex in $X_i \setminus \{v_i\}$ is adjacent to some vertex in $\{v_1, \dots, v_i\}$. Thus, by contracting $G[\{v_1, \dots, v_i\}]$ to a single vertex and $G[X_i \setminus \{v_i\}]$ to K_r, we find a K_{r+1}-minor in $G[X_i \cup \{v_1, \dots, v_{i-1}\}]$. Since $G[X_i \cup \{v_1, \dots, v_{i-1}\}]$ is K_{p+1}-minor-free, we have $r + 1 < p + 1$, i.e., $r < p$. This implies that $G[X_i \setminus \{v_i\}]$ is K_p-minor-free. Since we have assumed that Conjecture 1.1 holds for p, we obtain $\chi(G[X_i \setminus \{v_i\}]) \leq p - 1$. □

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