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Research Article

Counting Boolean intervals in the weak Bruhat order of a finite Coxeter group

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Abstract

Given a finite Coxeter system (W, S), we enumerate Boolean intervals in its weak Bruhat order by expressing them in terms of independent sets in its Coxeter graph Γ_W. In particular, we show that the number of Boolean intervals of rank k is $i_k(\Gamma_W) \cdot |W| / 2^k$, where $i_k(\Gamma_W)$ is the number of independent sets of size k in Γ_W. Specializing to type A_n, we recover work of Tenner [*J. Combin.* **13** (2022) 135–165], as well as Elder, Harris, Kretschmann, and Martínez Mori [*J. Combin.* **16** (2025) 65–89]. We derive analogous specialized results for types C_n and D_n, together with related generating functions and new connections to known integer sequences.

Keywords: Coxeter groups; weak Bruhat order; Boolean intervals; Fibonacci numbers; independent sets.

2020 Mathematics Subject Classification: 05A05, 05A15, 05A19, 06A07, 20F55.

1. Introduction

Given a poset P, it is a common problem to determine if (and if so, how many times) a particular poset T appears as an interval (subposet) within P. For an introduction to this area of mathematics we point the interested reader to [11, Chapter 5]. In this work, we settle this when P is the weak order of a (finite) Coxeter group and T is a Boolean poset.

We begin by recalling some definitions. Let W be a Coxeter group with generating set S forming the Coxeter system (W, S). Let m be its associated Coxeter matrix, of dimensions $|S| \times |S|$, whose entries are defined by $m(s, s') = m(s', s)$ and $m(s, s') = 1$ if and only if $s = s'$. Associated to a Coxeter system is an undirected labeled graph known as the *Coxeter graph*.

Definition 1.1. *Given a Coxeter system (W, S), its Coxeter graph Γ_W has vertex set S and an edge between $s, s' \in S$ if and only if $m(s, s') \geq 3$. By convention, edges are labeled with their corresponding weight $m(s, s')$ only when $m(s, s') > 3$.*

A pair of generators $s, s' \in S$ commute if and only if they are not adjacent in Γ_W.

The *(right) weak order* of a Coxeter group W with generating set S is a poset whose element set is W and whose cover relations arise from the right-hand side application of a generator $s \in S$. A poset is said to be *Boolean* if it is isomorphic to the poset of subsets of a set I ordered by inclusion. If $|I| = k < \infty$, then a Boolean poset is ranked.

Let $\ell(\pi)$ denote the *length* of $\pi \in W$ (i.e., $\ell(\pi)$ is the smallest integer k such that π can be written as a product of k generators). For $\pi \in W$, let

$$\mathrm{Des}_W(\pi) = \{s \in S \mid \ell(\pi s) = \ell(\pi) - 1\}$$

denote the *(right) descent set* of π. We note the classical fact, see [1] for instance, that in the case of $W = A_{n-1} \cong \mathfrak{S}_n$ this definition is equivalent to the classical definition of descents in permutations where $\mathrm{Des}(\pi) = \{i \in [n-1] \mid \pi_i > \pi_{i+1}\}$. Additionally the *(right) ascent set* of $\pi \in W$ is defined as

$$\mathrm{Asc}_W(\pi) = \{s \in S \mid \ell(\pi s) = \ell(\pi) + 1\}.$$

We will omit the subscript for both the ascent and descent sets when it is clear from context. A generator $s \in S$ is said to be in the *support* of π, denoted $\mathrm{supp}(\pi)$, if it appears in a minimal decomposition of π. A result of Tenner [12, Corollary 4.4] states that, in the weak order of a (not necessarily finite) Coxeter group W, an interval $[\pi, \sigma]$ is Boolean if and only if $\pi^{-1}\sigma$ is a product of commuting generators. The following is a restatement of this result using the language of descent sets; this version plays a key role in our enumerative results.

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Theorem 1.1. *The interval $[\pi, \pi s_{i_1} s_{i_2} \dots s_{i_k}]$ is a Boolean interval in the weak order of W if and only if $J = \{s_{i_1}, \dots, s_{i_k}\}$ consists of commuting generators and $J \cap \mathrm{Des}(\pi) = \emptyset$.*

Recall that given a graph $G = (V, E)$, a subset $S \subseteq V$ of its vertices is an *independent set* if no two vertices $u, v \in S$ are adjacent in G. We use Theorem 1.1 to establish a bijection between Boolean intervals in the weak order of a (not necessarily finite) Coxeter group W and the collection of independent sets of an induced subgraph of its Coxeter graph Γ_W.

Corollary 1.1. *For any $\pi \in W$, the set of $\sigma \geq \pi$ such that $[\pi, \sigma]$ is a Boolean interval in the weak order of W is in bijection with the collection of independent sets in $\Gamma_W[S \setminus \mathrm{Des}(\pi)]$ (i.e., the induced subgraph of Γ_W obtained by deleting $\mathrm{Des}(\pi)$ from its vertex set).*

Let $i_k(G)$ denote the number of independent sets of size k in G. If W is finite, then the bijection in Corollary 1.1 implies our main enumerative result.

Theorem 1.2. *If W is finite, there are $i_k(\Gamma_W) \cdot |W| \,/\, 2^k$ Boolean intervals of rank k in the weak order of W.*

The remainder of this paper is organized as follows. In Section 2 we prove Theorem 1.2. In Section 3 we specialize this result to Coxeter groups of types A_n, C_n, and D_n. For Coxeter groups of type A_n, we recover results given in [12] and [4]. For Coxeter groups of types C_n and D_n, we prove that the number of Boolean intervals (of all ranks) with a fixed minimal element involves products of Fibonacci numbers and that, for type D_n, it also involves a Fibonacci-like sequence with initial values 1 and 4 [10, A000285]. For the type C_n result see Theorem 3.2 and for the type D_n result see Theorem 3.4. We also give formulas for the number of Boolean intervals of rank k in their weak orders; refer to Theorem 3.3 for the type C_n result and Theorem 3.5 for the type D_n result. In Theorem 1.1 we do not assume that W is finite, or even irreducible. Hence, in Section 4, we give formulas for the number of Boolean intervals with any given minimal element for all of the infinite families of affine irreducible Coxeter groups, whose counts also involve products of Fibonacci numbers.

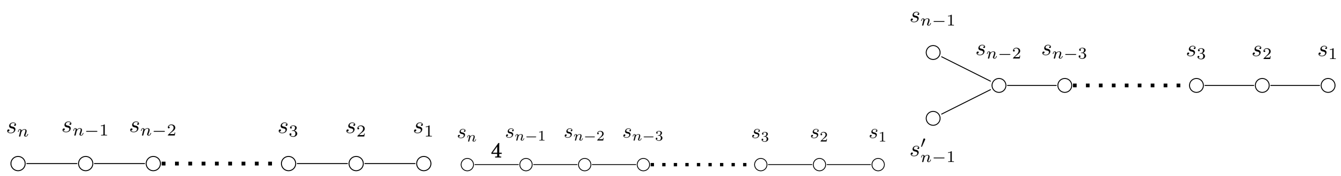
2. Results for Coxeter groups

We now turn to Theorem 1.2. Before proving the result, recall that if $J \subseteq S$, then W_J is the parabolic subgroup generated by J. Define $W^J = \{\pi \in W \,|\, \mathrm{Des}(\pi) \subseteq S \setminus J\}$. It is known that W^J is a system of minimal left coset representatives for W/W_J. If W is finite, this implies $|W^J| = |W|/|W_J|$ [1, Corollary 2.4.5]. Returning to our problem, for each subset $J \subseteq S$ of commuting generators, we count the elements $\pi \in W$ for which there is a Boolean interval $[\pi, \sigma]$ with $\mathrm{supp}(\pi^{-1}\sigma) = J$.

Proof of Theorem 1.2. Let $J = \{s_{i_1}, s_{i_1}, \dots, s_{i_k}\} \subseteq S$ consist of any k commuting generators and let $u = s_{i_1} s_{i_2} \cdots s_{i_k}$. By Theorem 1.1, for each element $\pi \in W$ with $\mathrm{Des}(\pi) \cap J = \emptyset$, there is a unique element $\sigma \in W$ with $\sigma \geq \pi$ and interval $[\pi, \sigma] \simeq [e, u]$. Note that W^J is the set of elements $\pi \in W$ for which $\mathrm{Des}(\pi) \cap J = \emptyset$, and $|W_J| = 2^k$ since the generators in J commute. Therefore, there are $|W^J| = |W|/|W_J| = |W|/2^k$ such intervals. Lastly, recall that J is any independent set of Γ_W of size k. In particular, this count is the same for any such independent set, and each distinct choice of independent set defines distinct intervals. Therefore, we conclude that there are $i_k(\Gamma_W) \cdot |W| \,/\, 2^k$ Boolean intervals of rank k. □

3. Specializing to Classical Coxeter Groups

We now specialize the results in Section 2 to the weak order of Coxeter groups of type A_n, C_n, and D_n. Throughout this section we refer to the Coxeter graphs in Figure 3.1.



(a) Γ_{A_n}. (b) Γ_{C_n}. (c) Γ_{D_n}.

Figure 3.1: Coxeter graphs of classical Coxeter groups.

3.1. Boolean intervals in the weak order of A_n

For $n \geq 1$, the Coxeter group of type A_n is described as follows. Let $S = \{s_1, s_2, \dots, s_n\}$ with $m(s_i, s_j) = 1$ if $i = j$, $m(s_i, s_j) = 3$ if $|i - j| = 1$, and $m(s_i, s_j) = 2$ if $|i - j| > 1$. Note that A_n is isomorphic to the symmetric group $\mathfrak{S}_{n+1}$. As illustrated in Figure 3.1a, this system has a path graph P_n on n vertices as its Coxeter graph Γ_{A_n}.

Define the Fibonacci numbers by $F_{n+2} = F_{n+1} + F_n$, with initial values $F_1 = F_2 = 1$ as in [10, A000045]. The number of independent sets of the path graph P_n on n vertices is known to be the Fibonacci number F_{n+2}. Moreover, a key insight into our results is that the number of independent sets of a graph is the product of the number of independent sets of its connected components. Therefore, the total number of Boolean intervals (of all possible ranks) above an element of A_n is a product of Fibonacci numbers. This specialization of Theorem 1.1 recovers [4, Theorem 1.1] (which generalizes [12, Proposition 5.9]). We state it below.

Theorem 3.1 (see Theorem 1.1 in [4])**.** *Let* $\pi = \pi_1\pi_2\cdots\pi_{n+1} \in A_n$ *be in one-line notation. Let* $\Gamma_{A_n}[S \setminus \mathrm{Des}(\pi)]$ *be the subgraph of* Γ_{A_n} *induced by deleting* $\mathrm{Des}(\pi)$ *from its vertex set. Partition the vertex set of* $\Gamma_{A_n}[S \setminus \mathrm{Des}(\pi)]$ *into connected components* $b_1, b_2, \ldots, b_k$*. Then, the number of Boolean intervals* $[\pi, \sigma]$ *in the weak order of the Coxeter group of type* A_n *with fixed minimal element* π *is* $\prod_{i=1}^{k} F_{|b_i|+2}$*, where* F_ℓ *is the* ℓ*th Fibonacci number and* $F_1 = F_2 = 1$*.*

The number of independent sets of size k of P_n is known to be $\binom{n+1-k}{k}$ [6, Proposition 1.1.iv]. Therefore, the specialization of Theorem 1.2 to A_n recovers [4, Theorem 1.3].

Corollary 3.1 (see Theorem 1.3 in [4])**.** *There are* $\frac{n!}{2^k}\binom{n-k}{k}$ *Boolean intervals of rank* k *in the weak order of the Coxeter group of type* A_{n-1}*.*

3.2. Boolean intervals in the weak order of C_n

For $n \geq 2$, the Coxeter group of type C_n is described as follows. Let $S = \{s_1, s_2, \ldots, s_n\}$ with $m(s_i, s_j) = 1$ if $i = j$, $m(s_i, s_j) = 3$ if $|i - j| = 1$ and $i, j < n$, $m(s_i, s_j) = 2$ if $|i - j| > 1$, and $m(s_i, s_j) = 4$ if $i + j = 2n - 1$. As illustrated in Figure 3.1b, this system has an underlying unlabeled path graph P_n on n vertices as its Coxeter graph Γ_{C_n}. Namely, the edge labels on the Coxeter graph do not matter, and this is why the Coxeter graph of A_n and C_n are both path graphs.

Now, consider the isomorphic representation of the Coxeter group of type C_n in $\mathfrak{S}_{2n}$ where, if $\pi = \pi_1\pi_2\cdots\pi_{2n} \in \mathfrak{S}_{2n}$, we impose the additional requirement that

$$\pi_i = k \text{ if and only if } \pi_{2n-i+1} = 2n - k + 1. \tag{1}$$

In this way, we treat the elements of C_n as *mirrored permutations* of $[2n]$ in one-line notation, as in [3]. The cover relations defining the weak order of C_n are as follows:

$$\sigma \lessdot \tau \text{ if and only if } \begin{cases} \sigma s_n = \tau & \text{when } n \in \mathrm{Des}(\tau) \\ \sigma s_i s_{2n-i+1} = \tau & \text{when } i \in \mathrm{Des}(\tau) \cap [n-1]. \end{cases} \tag{2}$$

Figure 3.2a depicts the Hasse diagram of the weak order of the Coxeter group of type C_3.

Consider the following illustrative example.

Example 3.1. *Consider the Boolean interval* B_3 *in the weak order of* $\mathfrak{S}_6$ *with minimal element* $\pi = 451623$ *and maximal element* $\sigma = 546132 = \pi s_1 s_3 s_5$*, depicted in Figure 3.3a. Note that both* π *and* σ *are elements of* C_3*, as they satisfy the condition given in* (3) *(i.e., they are mirrored permutations). Note that* $\mathrm{Des}(\sigma) = \{1, 3, 5\}$ *and consider the application of* s_1*,* s_3*, and* s_5*:*

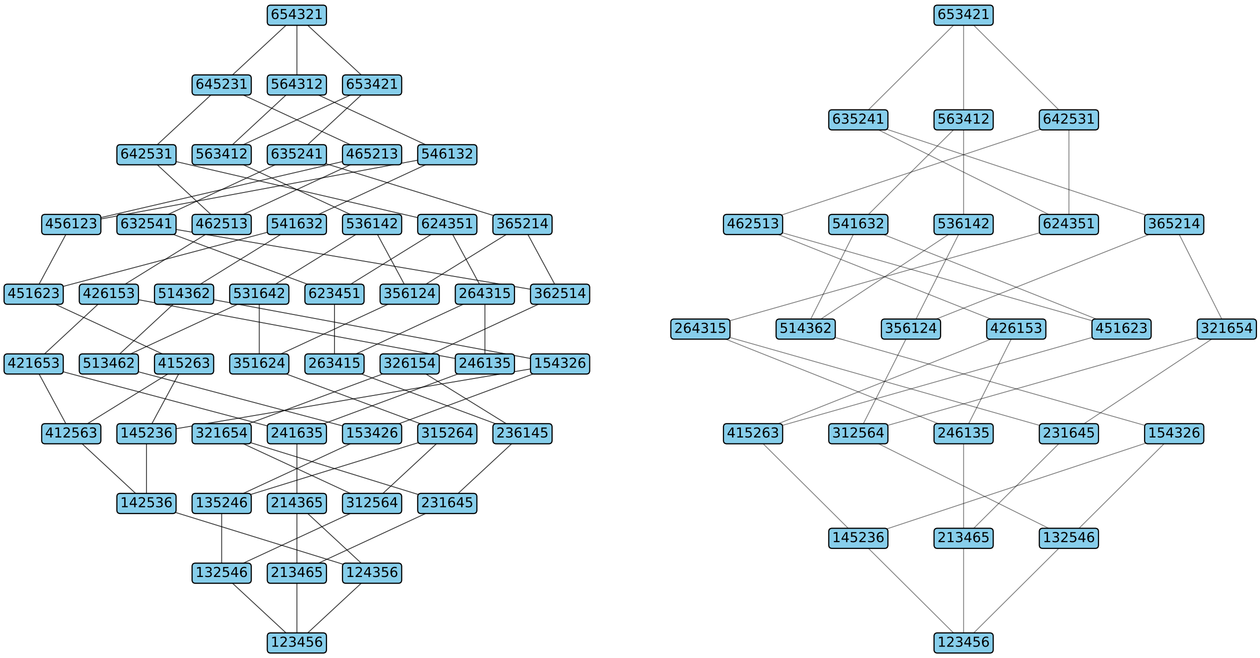
- *Applying* s_3 *to* π *and* σ*, respectively, gives another element of* C_3*. Hence,* π *is covered by* πs_3 *and* σs_3 *is covered by* σ*, and there are edges from* 451623 *to* 456123 *and from* 541632 *to* 546132 *in the weak order of* C_3*.*

- *Applying any single one of* s_1 *or* s_5 *to* π *yields permutations that are not in* C_3*, as they do not satisfy the condition given in* (3) *(i.e., they are not mirrored permutations). However, applying* $s_1 s_5$ *to* π *gives another element of* C_3*. Hence,* π *is covered by* $\pi s_1 s_5 = 541632$ *and there is an edge from* 451623 *to* 541632 *in the weak order of* C_3*.*

- *Applying any single one of* s_1 *or* s_5 *to* $\pi s_3 = 456123$ *yields permutations that are not in* C_3*, as they do not satisfy the condition given in* (3)*. However, applying* $s_1 s_5$ *to* πs_3 *gives another element of* C_3*. Hence,* πs_3 *is covered by* $\pi s_3 s_1 s_5 = 546132$ *and there is an edge from* 456123 *to* 546132 *in the weak order of* C_3*.*

The edges described in this example form a Boolean interval of rank two in the weak order of C_3 *with minimal element* 451623 *and maximal element* 546132*, as depicted in Figure 3.3b.*

The total number of Boolean intervals (of all possible ranks) above $\pi \in C_n$ is a product of Fibonacci numbers.



(a) C_3. **(b)** D_3.

Figure 3.2: Weak order of Coxeter groups C_3 and D_3 represented as mirrored permutations.

Theorem 3.2. *Let $\pi = \pi_1\pi_2\cdots\pi_{2n} \in \mathfrak{S}_{2n}$ satisfy $\pi_i = k$ if and only if $\pi_{2n-i+1} = 2n-k+1$ for all $i \in [n]$. Let $\Gamma_{C_n}[S \setminus \mathrm{Des}(\pi)]$ be the subgraph of Γ_{C_n} induced by deleting $\mathrm{Des}(\pi)$ from its vertex set. Partition the vertex set of $\Gamma_{C_n}[S\setminus\mathrm{Des}(\pi)]$ into connected components $b_1, b_2, \ldots, b_k$. Then, the number of Boolean intervals $[\pi, \sigma]$ in the weak order of the Coxeter group of type C_n with fixed minimal element π is $\prod_{i=1}^{k} F_{|b_i|+2}$, where F_ℓ is the ℓth Fibonacci number and $F_1 = F_2 = 1$.*

Proof. Condition (1) implies that, for $\tau \in C_n$ with associated mirrored permutation π and $1 \le i \le n$, $s_i \in \mathrm{Des}(\tau)$ if and only if $i \in \mathrm{Des}(\pi)$. Therefore, the elements of $\Gamma_{C_n}[S \setminus \mathrm{Des}(\tau)]$ correspond to $i \in \{1, 2, \ldots, n\}$ such that $s_i \in \mathrm{Asc}_{C_n}(\pi)$. This means that the connected components of $\Gamma_{C_n}[S \setminus \mathrm{Des}(\tau)]$ are exactly the maximal blocks of consecutive entries, call them $b_1, b_2, \ldots, b_k$. Since each b_i corresponds to a path graph of size $|b_i|$, by Theorem 1.1, we have that the number of Boolean intervals above τ (equivalently above π) is $\prod_{i=1}^{k} F_{|b_i|+2}$, as desired. □

Example 3.2. *Let $n = 9$ and suppose $\pi \in \mathfrak{S}_{18}$ satisfies $\mathrm{Asc}_{C_n}(\pi) = \{s_1, s_3, s_4, s_9\}$. Then, there are $F_3 \cdot F_4 \cdot F_3 = 12$ Boolean intervals in the weak order of C_n with minimal element π. For an explicit example, consider the following permutation in $\mathfrak{S}_{18}$:*

$$\pi = 3(17)47(18)(14)(11)96(13)(10)851(12)(15)2(16),$$

which is written in one-line notation and we have placed parenthesis around numbers with two digits. Note that the cover relations are determined by left multiplying π by s_1s_{17}, s_3s_{15}, s_4s_{14}, and s_9. There is 1 Boolean interval of rank 0, there are 4 Boolean intervals of rank 1, 5 Boolean intervals of rank 2, and 2 Boolean intervals of rank 3. This gives a total of 12 Boolean intervals with minimal element π, as expected.

Since Γ_{C_n} is a path graph P_n on n vertices, the next result follows directly from Theorem 1.2 and [6, Proposition 1.1.iv].

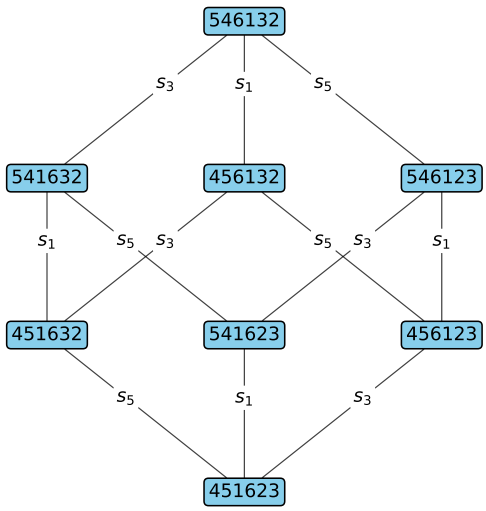
Theorem 3.3. *There are $2^{n-k}n!\binom{n+1-k}{k}$ Boolean intervals of rank k in the weak order of the Coxeter group of type C_n.*

Setting $k = 1$ in Theorem 3.3 gives the number of edges in the weak order of C_n, namely the number of Boolean intervals of rank one, denoted B_1. This case corresponds to [10, A014479].

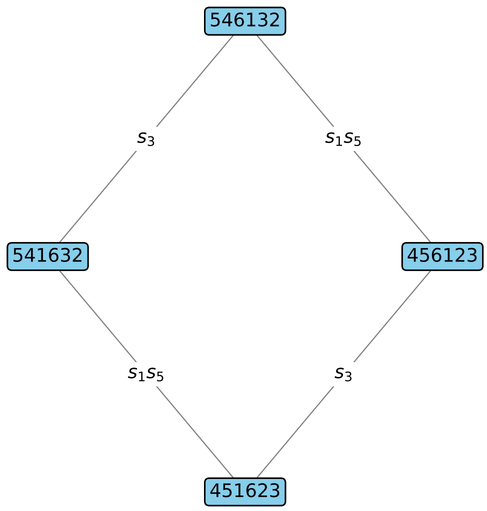
Corollary 3.2. *The number of edges in the weak order of C_n is $\frac{n}{2}2^n n!$.*

Setting $k = 2$ in Theorem 3.3 gives the sequence for the number of Boolean intervals of rank 2 in the weak order of C_n, which for $n \ge 2$ begins with $0, 12, 288, 5760, 115200, 2419200, 54190080, 1300561920, 33443020800, \ldots$. We added this sequence to the Online Encyclopedia of Integer Sequences (OEIS) [10, A380082].

We conclude this subsection with the following exponential generating function for the number of Boolean intervals of rank k in the weak order of the Coxeter groups of type C_n.



(a) A Boolean interval of rank three in the weak order of $\mathfrak{S}_6$ with two edges, corresponding to s_3, which are kept in the weak order of C_3.



(b) A Boolean interval of rank two in weak order of C_3.

Figure 3.3: Examples of Boolean intervals in the weak order of C_3.

Corollary 3.3. *Let $f_C(n,k)$ count the Boolean intervals of rank k in C_n. Then* $\displaystyle\sum_{n\geq 0}\sum_{k\geq 0} f_C(n,k)q^k\frac{x^n}{n!} = \frac{(2+q)x+2qx^2}{1-2x-2qx^2}+1.$

Proof. If $n \geq 3$, then $f_C(n,k)$ satisfies the following recurrence: $f_C(n,k) = 2nf_C(n-1,k) + 2n(n-1)f_C(n-2,k-1)$ depending on whether s_1 is supported in the interval. So, using standard techniques of exponential generating functions which can be found in [13] for example, the desired generating function will be rational with denominator $1-2x-2qx^2$. Then note in C_1 there are 2 Boolean intervals of rank 0 and 1 Boolean interval of rank 1. In C_2 the number of Boolean intervals is 8 of rank 0, and 8 of rank 1 as there are 2 intervals above the minimal element and 1 above every element that is not maximal. Applying the recurrence, we find that if b is the coefficient of $\frac{x^2}{2}$, then $b+8+4q$ will correspond to $\sum_{i=0}^{2} f_C(2,i)q^i = 8+8q$. So, the coefficient of $\frac{x^2}{2}$ is $4q$. Therefore the numerator is $(2+q)x+2qx^2$, leading to the generating function $\frac{(2+q)x+2qx^2}{1-2x-2qx^2}$. To account for $n=0$ we add 1. □

Corollary 3.4. *Let $f_C(n)$ count the Boolean intervals in C_n. Then,* $\displaystyle\sum_{n\geq 0} f_C(n)\frac{x^n}{n!} = \frac{3x+2x^2}{1-2x-2x^2}+1.$

3.3. Boolean intervals in the weak order of D_n

We now consider the Coxeter group of type D_n. Let $S = \{s_{n-1}, s'_{n-1}, s_{n-2}, \dots, s_1\}$ with $m(s_i,s_j) = 1$ if $s_i = s_j$, $m(s_i,s_j) = 3$ if $|i-j| = 1$, and $m(s_i,s_j) = 2$ if $|i-j| > 1$, and that $m(s_{n-1}, s'_{n-1}) = m(s'_{n-1}, s_{n-1}) = 2$. In this case, Γ_{D_n} is the graph on n vertices illustrated in Figure 3.1c.

We again return to the perspective of mirrored permutations. Consider the isomorphic representation of the Coxeter group of type $D_n \subset C_n \subset A_{2n-1}$ where, if $\pi = \pi_1\pi_2\dots\pi_n\pi_{n+1}\pi_{n+2}\dots\pi_{2n} \in \mathfrak{S}_{2n}$ we impose the additional requirements that

$$\pi_i = k \text{ if and only if } \pi_{2n-i+1} = 2n-k+1. \tag{3}$$

and $\{\pi_1, \pi_2, \dots, \pi_n\}$ always contains an even number of elements from the set $\{n+1, n+2, \dots, 2n\}$. In this way, we treat the elements of D_n as mirrored permutations of $[2n]$ in one-line notation, as in [3]. The cover relations defining the weak order of D_n are as follows:

$$\sigma \lessdot \tau \text{ if and only if } \begin{cases} \sigma(s_n s_{n-1} s_{n+1} s_n) = \tau & \text{when } n \in \mathrm{Des}(\tau) \\ \sigma s_i s_{2n-i+1} = \tau & \text{when } i \in \mathrm{Des}(\tau) \cap [n-1]. \end{cases} \tag{4}$$

Figure 3.2b depicts the Hasse diagram of the weak order of the Coxeter group of type D_3. One can compare Figures 3.2a and 3.2b to verify that $D_n \subset C_n$.

We begin by giving a recursive formula for the number of independent sets of Γ_{D_n}.

Lemma 3.1. *Let $(d_n)_{n\geq 1}$ be the sequence given by $d_n = d_{n-1} + d_{n-2}$ with $d_1 = 1, d_2 = 4$. Then, for $n \geq 1$, d_n is the number of independent sets of Γ_{D_n}.*

Proof. For $n = 1, 2, 3$, note that Γ_{D_1} is the empty graph, which has 1 independent set, Γ_{D_2} is two disjoint vertices which has 4 independent sets, and $\Gamma_{D_3} = P_3$ which has 5 independent sets.

The initial conditions are such that d_4 and d_5 correspond to where Γ_{D_n} has 4 and 5 vertices, which is a star with 3 leaves having 9 independent sets and adjoining a new vertex connected to one of the leaves, which has 14 independent sets. In general, the number of independent sets of Γ_{D_n} for $n \geq 6$ can be enumerated via the case where the leaf connected to a vertex of degree-2 is in the set, and alternatively where the leaf connected to a vertex of degree-2 is not in the set. If the leaf connected to a vertex of degree-2 is in the set, then the remaining elements are an independent set of the graph obtained by deleting the leaf and its neighbor, which is $\Gamma_{D_{n-2}}$. On the other hand, if the leaf is connected to a vertex of degree-2 is not in the set, then the independent set is of the graph obtained by just deleting the leaf, which is $\Gamma_{D_{n-1}}$. Adding those cases yields the desired recurrence $d_n = d_{n-1} + d_{n-2}$. □

The sequence d_n of Lemma 3.1 appears in [10, A000285], which already mentions the enumeration of independent sets in this family of graphs (unfortunately, no attribution is available). The same sequence has been considered before in the context of collapse free Hecke algebras in D_n [7].

Theorem 3.4. *Let $\pi \in D_n$. Let $\Gamma_{D_n}[S \setminus \mathrm{Des}(\pi)]$ be the subgraph of Γ_{D_n} induced by deleting the elements in $\mathrm{Des}(\pi)$ from its vertex set. Partition the vertex set of $\Gamma_{D_n}[S \setminus \mathrm{Des}(\pi)]$ into connected components $b_1, b_2, \ldots, b_k$, and let b_0 be the (possibly empty) connected component containing a vertex of degree-3. Then, the number of Boolean intervals $[\pi, \sigma]$ in the weak order of the Coxeter group of type D_n with fixed minimal element π is $d_{|b_0|} \cdot \prod_{i=1}^{k} F_{|b_i|+2}$, where we define $d_0 = 1$, F_ℓ is the ℓth Fibonacci number, and $F_1 = F_2 = 1$.*

Proof. Note that $\Gamma_{D_n}[S \setminus \mathrm{Des}(\pi)]$ has a single, possibly empty, component containing a vertex of degree-3, and that any other connected component is a path graph. Then, by Theorem 1.1, together with Lemma 3.1, and the fact that the number of independent sets of a path graph on k vertices is F_{k+2}, the result follows. □

To count Boolean intervals of a fixed rank, it suffices to count independent sets of Γ_{D_n} of size k for all $n \geq 4$. We give this count next.

Lemma 3.2. *For $n \geq 4$, the number of independent sets of Γ_{D_n} of size k is $i_k(\Gamma_{D_n}) = \binom{n-k}{k-2} + \binom{n-k-1}{k-1} + \binom{n-k}{k}$.*

Proof. An independent set of Γ_{D_n} of size k must contain some subset of the 2 leaves adjacent to the degree-3 vertex. If it contains both, then the remaining independent set is an independent set of size $k-2$ on the graph obtained by deleting these leaves and the degree-3 vertex. This deletion gives a path graph on $n-3$ vertices, so by [6, Proposition 1.1.iv], the number of independent sets of size $k-2$ is $\binom{n-k}{k-2}$. If only one of these vertices is in the independent set, then the remaining elements are again an independent set on the path graph obtained by deleting the degree-3 vertex and the adjacent leaves. As there are two choices for which of these vertices is to be in the independent set, there are $2\binom{n-k-1}{k-1}$ such independent sets. Finally, if neither of these vertices are in the independent set, then the independent set is just an independent set of size k on the graph obtained by deleting these leaves, which is a path graph on $n-2$ vertices. In this case, the number of such independent sets is [6, Proposition 1.1.iv] as $\binom{n-k-1}{k}$. The claim then follows from the binomial identity $\binom{n-k-1}{k-1} + \binom{n-k-1}{k} = \binom{n-k}{k}$. □

Specializing Theorem 1.2 to the Coxeter group of type D_n and using Lemma 3.2 yields the following result.

Theorem 3.5. *There are $2^{n-k-1}n!\left(\binom{n-k}{k-2} + \binom{n-k-1}{k-1} + \binom{n-k}{k}\right)$ Boolean intervals of rank k in the weak order of the Coxeter group of type D_n.*

Setting $k = 1$ in Theorem 3.5 gives the number of edges in the weak order of D_n, namely the number of Boolean intervals of rank one, denoted B_1. This case corresponds to [10, A019999], which has been studied in the context of local bisection refinement for n-simplicial grids generated by reflection [9].

Corollary 3.5. *The number of edges in the weak order of D_n is $2^{n-2}n!n$.*

Setting $k = 2$ in Theorem 3.5 gives the sequence for the number of Boolean intervals of rank 2 in the weak order of D_n, which for $n \geq 1$ begins with $0, 1, 6, 144, 2880, 57600, 1209600, 27095040, 650280960, 16721510400, 459841536000, \ldots$. This sequence does not appear in the OEIS [10].

We conclude this section with an exponential generating function for the number of Boolean intervals of rank k in the weak order of D_n.

Corollary 3.6. *Let $f_D(n,k)$ count Boolean intervals of rank k in D_n. Then,*

$$\sum_{n\geq 0}\sum_{k\geq 0} f_D(n,k)q^k\frac{x^n}{n!} = \frac{1}{2}\left(\frac{2x+(4q+q^2)x^2}{1-2x-2qx^2}\right)+1.$$

Proof. If $n \geq 3$, then $f_D(n,k)$ satisfies the following recurrence: $f_D(n,k) = 2nf_D(n-1,k) + 2n(n-1)f_D(n-2,k-1)$ depending on whether s_1 is supported in the interval. Using standard techniques of exponential generating functions which can be found in [13] for example, the desired generating function will be rational with denominator $1-2x-2qx^2$. Then note that in D_1 there is a single Boolean interval of rank 0. In D_2 the number of Boolean intervals of rank 4 is 0, of rank 1 is 4, and of rank 2 is 1. Applying the recurrence, we have that if b is the coefficient of $\frac{x^2}{2}$, then $b+4$ will correspond to $\sum_{i=0}^{2} f_D(2,i)q^i = 4+4q+q^2$. So the coefficient of $\frac{x^2}{2}$ is $4q+q^2$. Consequently the numerator is $x+\frac{(4q+q^2)x^2}{2}$, leading to the generating function

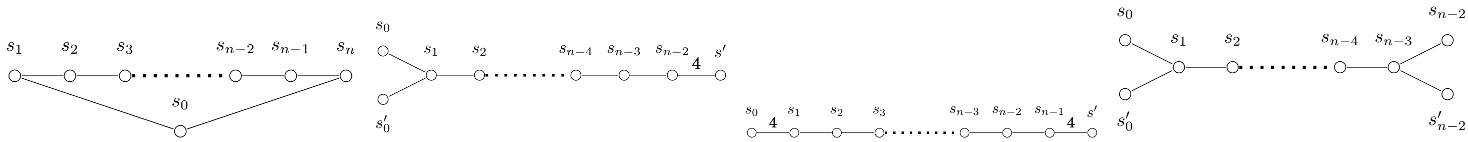
$$\frac{x+(4q+q^2)\frac{x^2}{2}}{1-2x-2qx^2} = \frac{1}{2}\left(\frac{2x+(4q+q^2)x^2}{1-2x-2qx^2}\right).$$

To account for $n=0$ we add 1. □

Corollary 3.7. *Let $f_D(n)$ count Boolean intervals in D_n. Then,* $\sum_{n\geq 0} f_D(n)\frac{x^n}{n!} = \frac{1}{2}\left(\frac{2x+5x^2}{1-2x-2x^2}\right)+1.$

4. Specializing to affine Coxeter groups

In this section, we briefly consider the enumeration of Boolean intervals in the weak order of the Coxeter groups $\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n$, and $\widetilde{D}_n$. As Theorem 1.2 only applies in the case where W is finite, we will only be able to apply Theorem 1.1 to obtain results for the number of Boolean intervals above a given element. One thing we mention with regards to the affine types is that in type $\widetilde{X}_n$, a set of generators S is such that $|S| = n+1$, differing from the finite case where in X_n, $|S| = n$. We use the definitions of these groups via their diagrams from [8, Section 2.5]. This differs from previous sections where we additionally provided combinatorial embeddings of the Coxeter groups as subgroups of a permutation group, but as the arguments for these groups will not differ we omit them. Again in contrast to previous sections we do not include any further discussion about generating functions counting Boolean intervals by rank in the affine irreducible types as in these cases there are an infinite number of Boolean intervals of a given rank. Throughout this section we reference the Coxeter graphs in Figure 4.1.



(a) $\widetilde{A}_n$. **(b)** $\widetilde{B}_n$. **(c)** $\widetilde{C}_n$. **(d)** $\widetilde{D}_n$.

Figure 4.1: Summary of Coxeter graphs of affine Coxeter groups.

We begin with type $\widetilde{A}_n$, for $n \geq 2$. In type $\widetilde{A}_n$, the relations are exactly those of A_n, except that there is a new generator s_0 with $m(s_0,s_1) = m(s_0,s_n) = 3$ as well. This means that $\Gamma_{\widetilde{A}_n}$ is a cycle graph with $n+1$ vertices.

Corollary 4.1. *Suppose $\pi \in \widetilde{A}_n$. If $\pi \neq e$, let $b_1, b_2, \dots, b_r$ be the connected components of $\Gamma_{\widetilde{A}_n}[S \setminus \mathrm{Des}(\pi)]$. Then the number of Boolean intervals of the form $[\pi,\sigma]$ with fixed minimal element π is counted by $\prod_{i=1}^{r} F_{|b_i|+2}$. If $\pi = e$ (the identity), then the number of Boolean intervals of the form $[\pi,\sigma]$ is the $(n+1)$th Lucas number, see* [2, p .46].

Proof. If $\pi \neq e$, then $\mathrm{Des}(\pi) \neq \emptyset$. Consequently $\Gamma_{\widetilde{A}_n}[S \setminus \mathrm{Des}(\pi)]$ is a disjoint union of r path graphs, with the sizes of the connected components $b_1, b_2, \dots, b_r$. Then by Theorem 1.1 and the fact that the number of independent sets of a path graph on k vertices is F_{k+2} the claim follows.

Instead if $\pi = e$, then by Theorem 1.1 the number of Boolean intervals above v is the number of independent sets of a cycle on $n+1$ vertices, which is the $(n+1)$th Lucas number [10, A000032]. □

For type $\widetilde{B}_n$, as $\Gamma_{\widetilde{B}_n}$ is the same graph when edge labels are omitted as $\Gamma_{D_{n+1}}$, we have the following result via a proof identical to that of Theorem 3.5.

Corollary 4.2. *Suppose $\pi \in \widetilde{B}_n$. Let $b_0, b_1, b_2, \ldots, b_r$ be the connected components of $\Gamma_{\widetilde{B}_n}[S \setminus \mathrm{Des}(\pi)]$ with b_0 the, potentially empty, connected component containing a degree-3 vertex. Then the number of Boolean intervals of the form $[\pi, \sigma]$ with fixed minimal element π is counted by $d_{|b_0|} \prod_{i=1}^{r} F_{|b_i|+2}$, with $d_0 = 1$.*

For Type $\widetilde{C}_n$, as $\Gamma_{\widetilde{C}_n}$ is a path graph on $n+1$ vertices, we have the following result as a consequence of Theorem 1.1 and the fact that every induced subgraph of a path is a path.

Corollary 4.3. *Suppose $\pi \in \widetilde{C}_n$. Let $b_1, b_2, \ldots, b_r$ be the connected components of $\Gamma_{\widetilde{C}_n}[S \setminus \mathrm{Des}(\pi)]$. Then the number of Boolean intervals of the form $[\pi, \sigma]$ with fixed minimal element π is counted by $\prod_{i=1}^{r} F_{|b_i|+2}$.*

For type $\widetilde{D}_n$, we have the following consequence of Theorem 1.1.

Corollary 4.4. *Suppose $\pi \in \widetilde{D}_n$. If $\pi \neq e$, let $b_0, b_0', b_1, b_2, \ldots, b_r$ be the connected components of $\Gamma_{\widetilde{D}_n}[S \setminus \mathrm{Des}(\pi)]$, with b_0, b_0' the, potentially empty, connected components containing a vertex of degree-3. Then the number of Boolean intervals of the form $[\pi, \sigma]$ with fixed minimal element π is counted by $d_{|b_0|} d_{|b_0'|} \prod_{i=1}^{r} F_{|b_i|+2}$, where $d_0 = 1$. If $\pi = e$ (the identity), then the number of Boolean intervals of the form $[\pi, \sigma]$ is $d_n + 2d_{n-2}$.*

Proof. When $\pi \neq e$, so $\mathrm{Des}(\pi) \neq \emptyset$, the first statement follows immediately from Theorem 1.1 together with the fact that any induced subgraph obtained by deleting at least one vertex of $\Gamma_{\widetilde{D}_n}$ will have as its connected components path graphs, together with at most two components that are copies of the Coxeter diagram for type D_k for some smaller k's.

For the second statement, the number of Boolean intervals above e is just the number of independent sets of any size of $\Gamma_{\widetilde{D}_n}$. If an independent set contains s_{n-2}, then the remaining elements are an independent set of the graph $\Gamma_{D_{n-2}}$ together with an isolated vertex s'_{n-2}, as s_{n-3} cannot be in the set, of which there are $2d_{n-2}$ such independent sets by Lemma 3.1. In the case where s_{n-2} is not in the set, the independent set is an independent set of the graph Γ_{D_n} of which there are d_n independent sets, again by Lemma 3.1. □

5. Future work

In [4] the authors provided a parking function interpretation for the Boolean intervals of rank k in the weak order of the Coxeter group of type A_n. We wonder if there is such a direct interpretation for Boolean intervals in other weak orders of classical Coxeter groups, which does not utilize the embedding of them into a larger permutation group. Moreover, such a combinatorial interpretation in terms of parking functions for the affine cases remains an open problem worthy of investigation. We also ask whether other combinatorial families of objects arise in characterizing and enumerating Boolean intervals in other well-known posets, including the strong Bruhat order of Coxeter groups and the Tamari lattice. For the reader interested in the latter, we point to the work of Fishel who enumerates chains in the Tamari lattice [5].

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