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Research Article

Integrals of hyperbolic tangent function

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Abstract

By means of the contour integration method, we evaluate, in closed form, a class of definite integrals involving hyperbolic tangent function.

Keywords: Cauchy residue theorem; contour integration; formal power series.

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1. Introduction and motivation

There exist numerous interesting and challenging integrals in the mathematical literature (see for example [1, 5, 8]). However, the following improper integrals do not seem to have been examined previously

$$J(m,n) := \int_0^\infty \frac{\tanh^m(z)}{z^n} dz, \tag{1}$$

where $m, n \in \mathbb{N}$ subject to $m \geq n \geq 2$ and $m \equiv_2 n$. Here and forth $m \equiv_2 n$ stands for that m is congruent to n modulo 2. When $m = n = 2$, the corresponding integral was proposed recently as a monthly problem by Holland [4], which has been the primary motivation for the authors to carry on the present research.

By employing the contour integration method, we shall establish, in this paper, the explicit formula for $J(m,n)$ when m and n are considered as two integer parameters. The final result states that the integral value $J(m,n)$ is expressed as a finitely linear combination of $\frac{\zeta(m+n-2k+1)}{\pi^{m+n-2k}}$ for $1 \leq k \leq \lceil \frac{m}{2} \rceil$, where $\lceil x \rceil$ denotes the smallest integer not less than x. As a showcase, the initial four examples are highlighted as follows:

$$\begin{aligned}
J(2,2) &= \frac{14\zeta(3)}{\pi^2},\\
J(3,3) &= -\frac{7\zeta(3)}{\pi^2} + \frac{186\zeta(5)}{\pi^4},\\
J(4,2) &= \frac{56\zeta(3)}{3\pi^2} - \frac{124\zeta(5)}{\pi^4},\\
J(4,4) &= -\frac{496\zeta(5)}{3\pi^4} + \frac{2540\zeta(7)}{\pi^6};
\end{aligned}$$

where the first one resolves the monthly problem proposed in [4]. Further concrete formulae will be presented at the end of the paper.

2. The contour integral

Rewriting the integrand in terms of the exponential function

$$T_{m,n}(z) := \frac{\tanh^m(z)}{z^n} = \frac{1}{z^n}\left(\frac{e^z - e^{-z}}{e^z + e^{-z}}\right)^m = \frac{1}{z^n}\left(\frac{e^{2z} - 1}{e^{2z} + 1}\right)^m, \tag{2}$$

we can express, by symmetry, the integral as

$$2J(m,n) = 2\int_0^\infty \frac{1}{z^n}\left(\frac{e^{2z} - 1}{e^{2z} + 1}\right)^m dz = \int_{-\infty}^\infty \frac{1}{z^n}\left(\frac{e^{2z} - 1}{e^{2z} + 1}\right)^m dz.$$

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Observe that $T_{m,n}(z)$ is a meromorphic function whose poles are determined by

$$1+e^{2z}=0.$$

We find that all the poles of $T_{m,n}(z)$ are given by

$$z_k=\left(k-\frac{1}{2}\right)\pi i, \quad \text{where} \qquad k\in\mathbb{Z}.$$

In order to evaluate the integral $J(m,n)$, consider the anti-clockwise contour $\mathcal{L}$, consisting of two segments $[-\pi M,-\varepsilon]$ and $[\varepsilon,\pi M]$ along the real axis plus two semi-circles $\mathcal{C}(\varepsilon)$ and $\mathcal{C}(\pi M)$ over the upper half-plane centered at the origin with radii ε and πM (where $M\in\mathbb{N}$), respectively. Hence, all the poles of $T_{m,n}(z)$ inside the contour $\mathcal{L}$ are of order m and given explicitly by

$$z_k=\left(k-\frac{1}{2}\right)\pi i, \quad \text{where} \quad k\in\mathbb{N} \;\text{ with }\; 1\le k\le M.$$

Then according to the residue theorem (see [2] and [7, §3.3]), we have the following equality

$$\int_{\mathcal{L}} T_{m,n}(z)dz=2\pi i\sum_{k=1}^{M}\operatorname*{Res}_{z=z_k}T_{m,n}(z)=\int_{\mathcal{C}(\varepsilon)}T_{m,n}(z)dz+\int_{\mathcal{C}(\pi M)}T_{m,n}(z)dz+\int_{-\pi M}^{-\varepsilon}T_{m,n}(z)dz+\int_{\varepsilon}^{\pi M}T_{m,n}(z)dz.$$

As we shall show in the next two subsections that

$$\lim_{\varepsilon\to 0}\int_{\mathcal{C}(\varepsilon)}T_{m,n}(z)dz=\lim_{M\to\infty}\int_{\mathcal{C}(\pi M)}T_{m,n}(z)dz=0,$$

the limiting case of the above equation as $\varepsilon\to 0$ and $M\to\infty$ will reduce to the identity given in the following lemma:

Lemma 2.1. *For two integers $m,n\in\mathbb{N}$ subject to conditions $m\ge n\ge 2$ and $m\equiv_2 n$, the following integral-sum formula holds:*

$$J(m,n)=\int_0^{\infty}\frac{\tanh^m(z)}{z^n}dz=\pi i\sum_{k=1}^{\infty}\operatorname*{Res}_{z=z_k}T_{m,n}(z).$$

3. The integral along $\mathcal{C}(\varepsilon)$

Keeping in mind $m\ge n$, we have

$$\lim_{z\to 0}T_{m,n}(z)=\lim_{z\to 0}\frac{1}{z^n}\left(\frac{e^{2z}-1}{e^{2z}+1}\right)^m=\begin{cases}1, & m=n,\\ 0. & m>n.\end{cases}$$

Therefore, for any ε with $|\varepsilon|$ being sufficiently small, the function $T_{m,n}(z)$ is bounded on C_ε, which implies that

$$\left|\int_{\mathcal{C}(\varepsilon)}T_{m,n}(z)dz\right|=\mathcal{O}(1)\times\left|\int_{\mathcal{C}(\varepsilon)}dz\right|=\mathcal{O}(\varepsilon\pi),$$

and consequently, the following limiting relation

$$\lim_{\varepsilon\to 0}\int_{\mathcal{C}(\varepsilon)}T_{m,n}(z)dz=0.$$

4. The integral along $\mathcal{C}(\pi M)$

Writing $z=\pi Me^{i\theta}$ on the semi-circle $\mathcal{C}(\pi M)$, we can explicitly express the modulus

$$\left|1+e^{2z}\right|^2=\left|1+e^{2\pi M(\cos\theta+i\sin\theta)}\right|^2=1+e^{4\pi M\cos\theta}+2e^{2\pi M\cos\theta}\cos(2\pi M\sin\theta).$$

Observe that the squared modulus results in 4 for $\theta=\frac{\pi}{2}$, and is equal to or greater than $(1-e^{2\pi M\cos\theta})^2$ when $\theta\neq\frac{\pi}{2}$. Then we can choose a small $\delta>0$ such that

$$\left|1+e^{2z}\right|^2\ge\begin{cases}3, & \frac{\pi}{2}-\delta\le\theta\le\frac{\pi}{2}+\delta;\\ (1-e^{-2\pi M\sin\delta})^2, & 0\le\theta<\frac{\pi}{2}-\delta \text{ and } \frac{\pi}{2}+\delta<\theta\le\pi.\end{cases}$$

Therefore $(1+e^{2z})^{-1}$ is bounded on $\mathcal{C}(\pi M)$, which implies further that the function below is bounded too on $\mathcal{C}(\pi M)$

$$\left(\frac{e^{2z}-1}{e^{2z}+1}\right)^m=\left(1-\frac{2}{e^{2z}+1}\right)^m=\mathcal{O}(1).$$

Under the change of variable $z=\pi Me^{i\theta}$, we estimate the integral

$$\Big|\int_{\mathcal{C}(\pi M)}T_{m,n}(z)dz\Big|=\mathcal{O}(1)\times\Big|\int_0^\pi\frac{ie^{(1-n)i\theta}}{(\pi M)^{n-1}}d\theta\Big|=\mathcal{O}\Big(\frac{\pi^{2-n}}{M^{n-1}}\Big).$$

Hence, we have shown that for $m\geq n\geq 2$:

$$\lim_{M\to\infty}\int_{\mathcal{C}(\pi M)}T_{m,n}(z)dz=0.$$

5. Computing residues

Finally, we have to determine the residues of $T_{m,n}(z)$ at z_k explicitly. When $m\geq n\geq 2$ are small integers, it is not difficult to do this. However, it becomes quite a tough task if m and n are considered as integer parameters.

Instead of calculating higher derivatives, we shall determine the residues by extracting coefficients from formal power series (cf. [3, §3.2], [6, §4.5], and [9, §2.1]). For the sake of brevity, denote by $[x^n]\phi(x)$ the coefficient of x^n in the formal power series $\phi(x)$. Keeping in mind that

$$e^{2z_k}=(-1)^{2k-1}=-1,$$

we can write

$$\begin{aligned}\operatorname*{Res}_{z=z_k}T_{m,n}(z)&=\big[(z-z_k)^{-1}\big]\frac{1}{z^n}\left(\frac{e^{2z}-1}{e^{2z}+1}\right)^m\\&=\big[(z-z_k)^{-1}\big]\frac{1}{z^n}\left(\frac{e^{2z}+e^{2z_k}}{e^{2z}-e^{2z_k}}\right)^m\\&=\big[(z-z_k)^{-1}\big]\frac{1}{z^n}\left(\frac{e^{z-z_k}+e^{z_k-z}}{e^{z-z_k}-e^{z_k-z}}\right)^m.\end{aligned}$$

By making the change of variables $y=z-z_k$, we have further

$$\begin{aligned}\operatorname*{Res}_{z=z_k}T_{m,n}(z)&=\big[y^{-1}\big]\frac{1}{(y+z_k)^n}\left(\frac{e^y+e^{-y}}{e^y-e^{-y}}\right)^m\\&=\big[y^{-1}\big]\frac{(e^y+e^{-y})^m}{(2y)^m(y+z_k)^n}\left(\frac{e^y-e^{-y}}{2y}\right)^{-m}.\end{aligned}$$

Then the above residue can be expressed as

$$\begin{aligned}\operatorname*{Res}_{z=z_k}T_{m,n}(z)&=\big[y^{m-1}\big]\frac{\mathrm{U}(y)\mathrm{V}(y)}{(y+z_k)^n}\\&=\sum_{\substack{1\leq j\leq m\\ j\equiv_2 m}}(-1)^{j-1}\binom{n+j-2}{j-1}\big[y^{m-j}\big]\frac{\mathrm{U}(y)\mathrm{V}(y)}{z_k^{n+j-1}},\end{aligned}\tag{3}$$

where $\mathrm{U}(y)$ and $\mathrm{V}(y)$ are two even functions given by

$$\mathrm{U}(y)=\frac{(e^y+e^{-y})^m}{2^m}\quad\text{and}\quad\mathrm{V}(y)=\left\{1-\left(1-\frac{e^y-e^{-y}}{2y}\right)\right\}^{-m}.$$

In order to extract the coefficient of y^{m-j} in (3), it is sufficient to expand $\mathrm{U}(y)$ and $\mathrm{V}(y)$ to Maclaurin polynomials up to order m.

First, by means of the binomial theorem, $\mathrm{U}(y)$ can be written as

$$\mathrm{U}(y)=\left(\frac{e^y+e^{-y}}{2}\right)^m=\sum_{\lambda=0}^m\binom{m}{\lambda}\frac{e^{(m-2\lambda)y}}{2^m}.\tag{4}$$

Then by observing the initial terms of the Maclaurin series

$$\frac{e^y-e^{-y}}{2y}=1+\frac{y^2}{6}+\frac{y^4}{120}+\mathcal{O}(y^6),$$

we can expand $\mathrm{V}(y)$ in succession as follows:

$$\begin{aligned}\mathrm{V}(y)&=\mathcal{O}(y^m)+\sum_{\ell=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^\ell\binom{-m}{\ell}\Big(1-\frac{e^y-e^{-y}}{2y}\Big)^\ell\\&=\mathcal{O}(y^m)+\sum_{\ell=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^\ell\binom{-m}{\ell}\sum_{\mu=0}^{\ell}(-1)^\mu\binom{\ell}{\mu}\Big(\frac{e^y-e^{-y}}{2y}\Big)^\mu.\end{aligned}$$

Let S stand for the above double sum. According to the binomial relations

$$\binom{-m}{\ell}\binom{\ell}{\mu}=\binom{-m}{\mu}\binom{-m-\mu}{\ell-\mu},$$

$$\sum_{\ell=\mu}^{\lfloor\frac{m}{2}\rfloor}(-1)^{\ell-\mu}\binom{-m-\mu}{\ell-\mu}=(-1)^{\lfloor\frac{m}{2}\rfloor-\mu}\binom{-m-\mu-1}{\lfloor\frac{m}{2}\rfloor-\mu};$$

we can manipulate S in the following manner:

$$\begin{aligned}\mathrm{S}&=\sum_{0\le\mu\le\ell\le\lfloor\frac{m}{2}\rfloor}(-1)^{\ell-\mu}\binom{-m}{\ell}\binom{\ell}{\mu}\Big(\frac{e^y-e^{-y}}{2y}\Big)^\mu\\&=\sum_{0\le\mu\le\ell\le\lfloor\frac{m}{2}\rfloor}(-1)^{\ell-\mu}\binom{-m}{\mu}\binom{-m-\mu}{\ell-\mu}\Big(\frac{e^y-e^{-y}}{2y}\Big)^\mu\\&=\sum_{\mu=0}^{\lfloor\frac{m}{2}\rfloor}\binom{-m}{\mu}\Big(\frac{e^y-e^{-y}}{2y}\Big)^\mu\sum_{\ell=\mu}^{\lfloor\frac{m}{2}\rfloor}(-1)^{\ell-\mu}\binom{-m-\mu}{\ell-\mu}\\&=\sum_{\mu=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^{\lfloor\frac{m}{2}\rfloor-\mu}\binom{-m}{\mu}\binom{-m-\mu-1}{\lfloor\frac{m}{2}\rfloor-\mu}\Big(\frac{e^y-e^{-y}}{2y}\Big)^\mu.\end{aligned}$$

Applying further the binomial theorem to the rightmost function and then making substitutions, we derive the following double sum expansion

$$\mathrm{V}(y)=\mathcal{O}(y^m)+\sum_{0\le\nu\le\mu\le\lfloor\frac{m}{2}\rfloor}(-1)^{\lfloor\frac{m}{2}\rfloor-\mu-\nu}\binom{-m-\mu-1}{\lfloor\frac{m}{2}\rfloor-\mu}\binom{-m}{\mu}\binom{\mu}{\nu}\frac{e^{(\mu-2\nu)y}}{(2y)^\mu}.\tag{5}$$

Now, substituting (4) and (5) into (3), we can further reformulate the residue as

$$\begin{aligned}\operatorname*{Res}_{z=z_k}T_{m,n}(z)=&\sum_{\substack{1\le j\le m\\ j\,\equiv_2\,m}}\sum_{\lambda=0}^{m}(-1)^{j-1}\binom{n+j-2}{j-1}\binom{m}{\lambda}\sum_{0\le\nu\le\mu\le\lfloor\frac{m}{2}\rfloor}\binom{-m-\mu-1}{\lfloor\frac{m}{2}\rfloor-\mu}\\&\times\frac{(-1)^{\lfloor\frac{m}{2}\rfloor-\mu-\nu}}{2^{m+\mu}z_k^{n+j-1}}\binom{-m}{\mu}\binom{\mu}{\nu}[y^{m-j}]\,\frac{e^{(m+\mu-2\nu-2\lambda)y}}{y^\mu}\\=&\sum_{\substack{1\le j\le m\\ j\,\equiv_2\,m}}\sum_{\lambda=0}^{m}(-1)^{j-1}\binom{n+j-2}{j-1}\binom{m}{\lambda}\sum_{0\le\nu\le\mu\le\lfloor\frac{m}{2}\rfloor}\binom{-m-\mu-1}{\lfloor\frac{m}{2}\rfloor-\mu}\\&\times\frac{(-1)^{\lfloor\frac{m}{2}\rfloor-\mu-\nu}}{z_k^{n+j-1}}\binom{-m}{\mu}\binom{\mu}{\nu}\frac{(m+\mu-2\nu-2\lambda)^{m+\mu-j}}{2^{m+\mu}(m+\mu-j)!}.\end{aligned}$$

Taking into account the binomial equality

$$\binom{-m}{\mu}\binom{-m-\mu-1}{\lfloor\frac{m}{2}\rfloor-\mu} = (-1)^{\lfloor\frac{m}{2}\rfloor}\frac{(m)_\mu}{\mu!}\frac{(1+m+\mu)_{\lfloor\frac{m}{2}\rfloor-\mu}}{(\lfloor\frac{m}{2}\rfloor-\mu)!}$$
$$= (-1)^{\lfloor\frac{m}{2}\rfloor}\binom{\lfloor\frac{m}{2}\rfloor}{\mu}\binom{m+\lfloor\frac{m}{2}\rfloor}{m}\frac{m}{m+\mu},$$

we find finally the following quadruplicate sum expression:

$$\operatorname*{Res}_{z=z_k} T_{m,n}(z) = \binom{m+\lfloor\frac{m}{2}\rfloor}{m}\sum_{\substack{1\le j\le m\\ j\equiv_2 m}}(-1)^{j-1}\binom{n+j-2}{j-1}\sum_{\lambda=0}^{m}\sum_{\mu=0}^{\lfloor\frac{m}{2}\rfloor}\sum_{\nu=0}^{\mu}\frac{(-1)^{\mu+\nu}}{z_k^{n+j-1}}$$
$$\times\frac{m}{m+\mu}\binom{m}{\lambda}\binom{\lfloor\frac{m}{2}\rfloor}{\mu}\binom{\mu}{\nu}\frac{(m-2\lambda+\mu-2\nu)^{m+\mu-j}}{2^{m+\mu}(m+\mu-j)!}.$$

6. Conclusive theorem

To evaluate the sum of residues in Lemma 2.1, it is enough to do that for

$$\sum_{k=1}^{\infty}\frac{1}{z_k^{n+j-1}} = \left(\frac{2}{\pi i}\right)^{n+j-1}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^{n+j-1}} = \frac{2^{n+j-1}-1}{(\pi i)^{n+j-1}}\zeta(n+j-1).$$

Consequently, we have proved the following general integral identity.

Theorem 6.1. *Let m and n be the two natural numbers of the same parity with $m\ge n\ge 2$, the following integral formula holds:*

$$J(m,n) = \int_0^\infty \frac{\tanh^m(z)}{z^n}dz$$
$$= \binom{m+\lfloor\frac{m}{2}\rfloor}{m}\sum_{\substack{1\le j\le m\\ j\equiv_2 m}}(-1)^j\binom{n+j-2}{j-1}\frac{1-2^{n+j-1}}{(\pi i)^{n+j-2}}\zeta(n+j-1)$$
$$\times\sum_{\lambda=0}^{m}\sum_{\mu=0}^{\lfloor\frac{m}{2}\rfloor}\sum_{\nu=0}^{\mu}(-1)^{\mu+\nu}\frac{m}{m+\mu}\binom{m}{\lambda}\binom{\lfloor\frac{m}{2}\rfloor}{\mu}\binom{\mu}{\nu}\frac{(m-2\lambda+\mu-2\nu)^{m+\mu-j}}{2^{m+\mu}(m+\mu-j)!}.$$

From Theorem 6.1, we assert that for each pair of $m,n\in\mathbb{N}$ subject to $m\equiv_2 n$ and $m\ge n\ge 2$, the integral value $J(m,n)$ results always in a finitely linear combination of $\frac{\zeta(m+n-2k+1)}{\pi^{m+n-2k}}$ with $1\le k\le\lceil\frac{m}{2}\rceil$. Four initial formulae have been anticipated in the introduction. Further integral identities are recorded below as consequences:

$$J(5,3) = -\frac{7\zeta(3)}{\pi^2}+\frac{310\zeta(5)}{\pi^4}-\frac{1905\zeta(7)}{\pi^6},$$
$$J(5,5) = \frac{31\zeta(5)}{\pi^4}-\frac{3175\zeta(7)}{\pi^6}+\frac{35770\zeta(9)}{\pi^8},$$
$$J(6,2) = \frac{322\zeta(3)}{15\pi^2}-\frac{248\zeta(5)}{\pi^4}+\frac{762\zeta(7)}{\pi^6},$$
$$J(6,4) = -\frac{2852\zeta(5)}{15\pi^4}+\frac{5080\zeta(7)}{\pi^6}-\frac{28616\zeta(9)}{\pi^8},$$
$$J(6,6) = \frac{5842\zeta(7)}{5\pi^6}-\frac{57232\zeta(9)}{\pi^8}+\frac{515844\zeta(11)}{\pi^{10}},$$
$$J(7,7) = -\frac{127\zeta(7)}{\pi^6}+\frac{1402184\zeta(9)}{45\pi^8}-\frac{1003030\zeta(11)}{\pi^{10}}+\frac{7568484\zeta(13)}{\pi^{12}}.$$

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