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Research Article

Vertex-pancyclism in edge-colored complete graphs with restrictions in color transitions

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Abstract

Let H be a graph possibly with loops. Let G be a simple graph. We say that G is an H-colored graph whenever every edge of G has assigned a vertex of H as a color. A cycle C in an H-colored graph G is an H-cycle if and only if the colors of consecutive edges in C are adjacent vertices in H, including the last and first edges of C. An H-colored graph G is said to be vertex H-pancyclic if every vertex of G is contained in an H-cycle of length l for every l in $\{3, \dots, |V(G)|\}$. A properly colored cycle in an edge-colored graph is a particular case of H-cycles in H-colored graph, namely when H is a complete graph with no loops. In this paper, we show sufficient conditions on an H-colored complete graph G to be vertex H-pancyclic. As a consequence, we obtain a well-known result about properly vertex pancyclicism in edge-colored complete graphs.

Keywords: edge-colored graph; vertex H-panciclicity; H-cycle; properly colored cycle.

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1. Introduction

For basic concepts, terminology, and notation not defined here, we refer the reader to [7]. Throughout this work, we consider finite simple graphs, unless otherwise specified. Let G be a graph. The sets of vertices and edges of G are denoted by $V(G)$ and $E(G)$, respectively.

We say that a graph G is *pancyclic* whenever G contains a cycle of length l for every l in $\{3, \dots, |V(G)|\}$. Moreover, G is said to be *vertex pancyclic* if and only if each vertex of G is contained in a cycle of length l for every l in $\{3, \dots, |V(G)|\}$. In 1971, Bondy [4] conjectured that almost every condition implying that a graph G is Hamiltonian, also implies that the graph G is pancyclic. Since then, several authors have extensively studied whether a given graph is pancyclic or vertex-pancyclic, see for example [5,6,8]. In particular, conditions on the minimum degree of the graph have proven to be efficient in knowing if a graph is vertex pancyclic, see Theorem 1.1. In [18], Ming-Chi Li et al. proved that deciding whether a graph is (vertex) pancyclic is NP-complete, even for 3-connected cubic planar graphs.

Theorem 1.1 (see [17]). *If G is a graph of order n, with $n \geq 3$, such that $\delta(G) \geq \frac{n+1}{2}$, then G is vertex pancyclic.*

Let $I_k = \{1, \dots, k\}$ be a given set of colors, with $k \geq 2$. A graph G is a *k-edge-colored graph* whenever every edge has a color in I_k. A walk W in G is *properly colored* if and only if no consecutive edges have the same color; in particular, when W is a cycle, we say that W is a *properly colored cycle*. Properly colored walks are of interest for theoretical reasons (for example, in undirected and directed graphs [2]) as well as for graph theory application (for example, in genetic and molecular biology [11, 12, 20, 22], engineering and computer science [1, 21, 23], and management science [24, 25]). In particular, Dorninger in [11] studied a model of cell division where a properly colored cycle that contains all the vertices of a 2-edge-colored graph is needed.

Let G be a k-edge-colored graph with n vertices. We say that G is *properly vertex pancyclic* if and only if each vertex of G is contained in a properly colored cycle of length l for every l in $\{3, \dots, n\}$. This concept has been studied by considering the following definitions: for every vertex v in $V(G)$, the color degree of v, denoted by $\delta^c(v)$, is the number of different colors on the edges incident with the vertex v in G, and $\delta^c(G)$ is the minimum value of $\delta^c(v)$ over all vertices v in G.

Fujita and Magnant [13] conjectured that every edge-colored complete graph with n vertices, $n \geq 3$, such that $\delta^c(G) \geq \frac{n+1}{2}$ is properly vertex pancyclic. Chen et al. in [9] partially solve this conjecture by adding the condition that the graph has no monochromatic cycles of length three (a monochromatic cycle is a cycle where all of its edges are color alike).

Theorem 1.2 (see [9]). *Let G be an edge-colored complete graph on n vertices, $n \geq 3$, such that $\delta^c(G) \geq \frac{n+1}{2}$. If G contains no monochromatic cycles of length 3, then G is properly vertex pancyclic.*

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Let H be a graph possibly with loops and G a graph. We say that G is an *H-colored graph* if there exists a function $c : E(G) \longrightarrow V(H)$. These types of colorings were introduced by Linek and Sands in the context of kernels of a digraph to codify allowed color transitions in the walks of G (see [19]). A walk $W = (x_0, \ldots, x_k)$ in G is an *H-walk* if and only if $(c(x_0x_1), c(x_1x_2), \ldots, c(x_{k-1}c_k))$ is a walk in H, and in particular, a cycle $C = (x_0, \ldots, x_k = x_0)$ in G is an *H-cycle* if and only if $(c(x_0x_1), c(x_1x_2), \ldots, c(x_{k-1}x_k), c(x_0x_1))$ is a walk in H. Notice that when H is a complete graph without loops, C is an H-cycle if and only if C is a properly colored cycle, moreover, H decides what color transitions are allowed in a cycle to be an H-cycle. We say that G is *vertex H-pancyclic* if and only if each vertex of G is contained in a H-cycle of length l for every l in $\{3, \ldots, n\}$.

The study of H-colorings in graphs began in [15] when Galeana-Sánchez, Rojas-Monroy, Sánchez-López and Villareal-Valdés characterized the H-colored multigraphs containing Euler H-trails. Later in [16], they gave an algorithm to determine whether an H-colored multigraph has an H-cycle. They work with an auxiliary graph, introduced in [3] by Benkour et al., as follows: Let G be an H-colored graph. For each non-isolated vertex v in $V(G)$, G_v is the graph with vertex set $V(G_v) = \{vx : vx \in E(G)\}$, and $ab \in E(G_v)$, with $a \neq b$, if and only if $c(a)c(b) \in E(H)$. Note that G_v is a simple graph for every non-isolated vertex v of G.

Let $W = (x_0, x_1, \ldots, x_k)$ a walk in an H-colored graph G, and i in $\{1, \ldots, k-1\}$. We say that x_i is an *obstruction* of the walk W if and only if $c(x_{i-1}x_i)c(x_ix_{i+1}) \notin E(H)$. When $x_0 = x_k$ we say that x_0 is an *obstruction* of the closed walk W if and only if $c(x_{k-1}x_k)c(x_0x_1) \notin E(H)$. We denote by $O_H(W)$ to the set of obstructions of the walk W. Notice that W is an H-walk if and only if $O_H(W) = \emptyset$.

In [14], Galeana-Sánchez, Hernández-Lorenzana and Sánchez-López proved the following result as a first approach to the study of vertex H-pancyclism.

Theorem 1.3 (see [14]). *Let H be a graph possibly with loops and G be an H-colored complete graph of order n, such that: for every x in $V(G)$, G_x is a complete k_x-partite graph for some k_x in $\mathbb{N}$; for any cycle C of length 4 in G, $|O_H(C)| \neq 3$; and for every x in $V(G)$, $k_x \geq \frac{n+1}{2}$. Then:*

1. *Each vertex of G is contained in an H-cycle of length 3 ($n \geq 3$).*

2. *Each vertex of G is contained in an H-cycle of length 4, whenever for every cycle C of length 3 in G, $|O_H(C)| \neq 2$ and $n \geq 4$.*

This work is organized as follows: Section 2 gives the basic concepts, terminology, and some results that we will use in the rest of the paper. In Section 3, we give sufficient conditions, similar to those stated in Theorem 1.3, for an H-colored complete graph to be vertex H-pancyclic. It is worth mentioning that the conditions in the main theorem can be verified in polynomial time. Finally, Theorem 1.2 is proven as a direct consequence of the main result.

2. Preliminaries

For a better understanding for the reader, we start with some notation, and some observations introduced in [14].

Let G be a graph. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X. A walk will be denoted by the sequence of its vertices $W = (v_0, v_1, \ldots, v_k)$. If $V = (u_0, \ldots, u_n)$ and $W = (u_n, v_1, v_2, \ldots, v_k)$ are two walks, and $\{i, j\}$ is a subset of $\{0, \ldots, n\}$, with $i < j$, the concatenation $(u_0, \ldots, u_n, v_1, v_2 \ldots, v_k)$ of the walks V and W is denoted by $V \cup W$, the subwalk $(u_i, u_{i+1}, \ldots, u_j)$ is denoted by (u_i, V, u_j), and the walk $(u_n, \ldots, u_0)$ is denoted by V^{-1}. A subset I of $V(G)$ is independent if and only if the subgraph $G[I]$ has no edges.

Observation 2.1. *Let G be an H-colored graph, such that for every x in $V(G)$, G_x is a complete k_x-partite graph for some k_x in $\mathbb{N}$. Suppose that $\{ux, vx\}$ is a subset of $E(G)$. Then, $x \notin O_H((u, x, v))$ if and only if ux and vx are in different partite sets of the k_x-partition of $V(G_x)$.*

As a direct consequence of Observation 2.1 and the definition of H-cycle, we have the following observations.

Observation 2.2. *Let G be an H-colored graph, such that for every x in $V(G)$, G_x is a complete k_x-partite graph for some k_x in $\mathbb{N}$. Suppose that $C = (u_1, \ldots, u_{n-1}, u_n, u_1)$ is a cycle in G. Then, C is an H-cycle in G if and only if $u_{i-1}u_i$ and $u_{i+1}u_i$ are in different partite sets of the k_{u_i}-partition of $V(G_{u_i})$ for every i in $\{1, \ldots, n\}$ (the subindices are taken modulo n).*

Observation 2.3. *Let G be an H-colored graph, such that for every x in $V(G)$, G_x is a complete k_x-partite graph for some k_x in $\mathbb{N}$. If (u, v, w) is an H-path, then for every x adjacent with v, $x \notin \{u, w\}$, we have that (x, v, u) or (x, v, w) is an H-path.*

Observations 2.1, 2.2, and 2.3 will be frequently used along with the proof of the main result.

Notice that if G is an H-colored graph, D is an induced subgraph of G and x is a vertex of D such that G_x is a complete k_x-partite graph for some k_x in $\mathbb{N}$, then D_x is a complete k_x^D-partite graph for some k_x^D in $\mathbb{N}$. Moreover, if $\{P_1^x, P_2^x, \ldots, P_{k_x}^x\}$ is the k_x-partition of $V(G_x)$ into independent sets, then $\{P_i^x \cap V(D_x) : P_i^x \cap V(D_x) \neq \emptyset, i \in \{1, 2, \ldots, k_x\}\}$ is the k_x^D-partition of $V(D_x)$ into independent sets.

Let G be an H-colored complete graph, A a subset of $V(G)$ and v in $V(G) \setminus A$. We say that A has the *H-dependency property* with respect to the vertex v if and only if for every subset $\{a, a'\}$ of A, (v, a, a') or (v, a', a) is not an H-path in G.

Proposition 2.1 (see [14]). *Suppose that for every x in $V(G)$, G_x is a complete k_x-partite graph for some k_x in $\mathbb{N}$. Let A be a subset of $V(G)$ and v be a vertex in $V(G) \setminus A$. If A has the H-dependence property with respect to the vertex v, then there exists some vertex a in A such that:*

1. *$k_a^D \leq \frac{|A|+1}{2}$, where $D = G[A]$.*
2. *If $|A| \geq 2$, then $a \in O_H((v, a, a'))$ for some a' in $N_D(a)$.*

Let H be a graph possibly with loops, G an H-colored complete graph, $C = (x_1, \ldots, x_l, x_1)$ a cycle in G and $v \in V(G) \setminus V(C)$. We say that C has *increasing* (*decreasing*) *obstruction with respect to* v if and only if for every $i \in \{1, \ldots, l\}$, x_i is an obstruction of the path (v, x_i, x_{i+1}) ((v, x_i, x_{i-1}), respectively), where the indices are taken modulo l. In any case, we say that C has *obstruction with respect to* v.

3. Main theorem

Theorem 3.1. *Let G be an H-colored complete graph of order n, with $n \geq 3$, such that for every x in $V(G)$, G_x is a complete k_x-partite graph for some $k_x \geq 2$. Suppose that:*

1. *For any cycle C of length 3 in G, $|O_H(C)| \leq 1$.*
2. *For any cycle C of length 4 in G, $|O_H(C)| \neq 3$.*
3. *For every x in $V(G)$, $k_x \geq \dfrac{n+1}{2}$.*

Then, G is vertex H-pancyclic.

Proof. To prove that G is vertex H-pancyclic, it is sufficient to show that if a vertex is contained in an H-cycle of length l in G, for some $3 \leq l \leq n-1$, then it is also contained in an H-cycle of length $l+1$ in G. It follows from Theorem 1.3 that the assertion is true for $l = 3$. So, in what follows we can assume that $l \geq 4$ and $n \geq 5$.

Proceeding by contradiction, suppose that there is a vertex v which is contained in an H-cycle of length l, for some $4 \leq l \leq n-1$, but is not contained in any H-cycle of length $l+1$. Let $C = (v = x_1, x_2, \ldots, x_l, x_1)$ be an H-cycle of length l containing v in G.

Let $W_1 = \{w \in V(G) \setminus V(C) : \{wx_i : 1 \leq i \leq l\}$ is an independent set in $G_w\}$, meaning that for each vertex $w \in W_1$, all the wx_i are in the same partite set of the k_w-partition of $V(G_w)$. Let $W_2 = V(G) \setminus (V(C) \cup W_1)$, that is, W_2 is the set of vertices in $V(G) \setminus V(C)$ such that, for every $w \in W_2$, there exist x_i and x_j in $V(C)$, $i \neq j$, such that wx_i and wx_j are in different partite sets of the partition of $V(G_w)$. Notice that every vertex of G is in one and only one of the sets $V(C), W_1$ and W_2. From now on, x_i is the same vertex as x_{i+l} and x_{i-l}, for every $i \in \{1, \ldots, l\}$ (the indices are taken modulo l).

Claim 1. The cycle C has obstruction with respect to each vertex w in W_2.

Proof of Claim 1. Let w in W_2. It follows from the definition of W_2 that $w \notin V(C)$ and there exists $i \in \{1, \ldots, l\}$ such that wx_i and wx_{i+1} are in different partite sets of the partition of $V(G_w)$. If $c(x_{i-1}x_i)c(x_iw)$ and $c(wx_{i+1})c(x_{i+1}x_{i+2})$ are edges in H, then $(w, x_{i+1}) \cup (x_{i+1}, C, x_i) \cup (x_i, w)$ is an H-cycle of length $l+1$ containing v, a contradiction. Hence, $c(x_{i-1}x_i)c(x_iw)$ is not in $E(H)$ or $c(wx_{i+1})c(x_{i+1}x_{i+2})$ is not in $E(H)$. By symmetry, suppose without loss of generality that $c(wx_{i+1})c(x_{i+1}x_{i+2})$ is not in $E(H)$. Then, $x_{i+1} \in O_H(C')$, where $C' = (x_{i+1}, x_{i+2}, w, x_{i+1})$, and since $|O_H(C')| \leq 1$ (by hypothesis), we conclude that $x_{i+2} \notin O_H(C')$ and $w \notin O_H(C')$, that is, $c(x_{i+1}w)c(wx_{i+2})$ and $c(wx_{i+2})c(x_{i+2}x_{i+1})$ are edges in H. Now, as $c(wx_{i+1})c(x_{i+1}x_{i+2})$ is not in $E(H)$, we have that (w, x_{i+1}, x_{i+2}) is not an H-path, and by Observation 2.3, (x_i, x_{i+1}, w) is an H-path (since (x_i, x_{i+1}, x_{i+2}) is an H-path). Notice that if $c(wx_{i+2})c(x_{i+2}x_{i+3})$ is an edge in H, then $(w, x_{i+2}) \cup (x_{i+2}, C, x_{i+1}) \cup (x_{i+1}, w)$ is an H-cycle of length $l+1$ containing v, a contradiction. Therefore, $c(wx_{i+2})c(x_{i+2}x_{i+3})$ is not in $E(H)$. Applying the same reasoning, it follows that for every $j \in \{1, \ldots, l\} \setminus \{i\}$, $c(wx_j)c(x_jx_{j+1})$ is not in $E(H)$, that is, x_j is an obstruction of the path (w, x_j, x_{j+1}). It remains to prove that $x_i \in O_H((w, x_i, x_{i+1}))$.

Recall that for every $j \in \{1, \ldots, l\} \setminus \{i\}$, G_{x_j} is a complete k_j-partite graph, so $c(wx_j)c(x_jx_{j-1})$ is an edge in H. In particular, when $j = i-1$ we have that $x_{i-1} \in O_H(C'')$, where $C'' = (w, x_{i-1}, x_i, w)$. Since $|O_H(C'')| \leq 1$ (by hypothesis), we have that $x_i \notin O_H(C'')$ and $w \notin O_H(C'')$, thus $c(x_{i-1}w)c(wx_i)$ is an edge in H. If $c(wx_i)c(x_ix_{i+1})$ is an edge in H, then $(w, x_i) \cup (x_i, C, x_{i-1}) \cup (x_{i-1}, w)$ is an H-cycle of length $l+1$ containing v, a contradiction. Hence, $c(wx_i)c(x_ix_{i+1})$ is not an edge in H, that is, x_i is an obstruction of the path (w, x_i, x_{i+1}). Thus, C has increasing obstruction with respect to w. □

Claim 2. C has increasing obstruction with respect to each vertex in W_2 or C has decreasing obstruction with respect to each vertex in W_2.

Proof of Claim 2. It follows from Claim 1 that C has obstruction with respect to each vertex in W_2. Proceeding by contradiction, suppose that there exist w_1 and w_2 in W_2 such that C has increasing obstruction with respect to w_1 and C has decreasing obstruction with respect to w_2. Hence, for every $i \in \{1, \ldots, l\}$, $c(w_1x_i)c(x_ix_{i+1}) \notin E(H)$ and $c(w_2x_i)c(x_ix_{i-1}) \notin E(H)$. Since G_{x_i} is a complete k_{x_i}-partite graph and (x_{i-1}, x_i, x_{i+1}) is an H-path, we have that $c(w_1x_i)c(x_iw_2)$ is an edge in H, that is, $c(w_1x_i)$ and $c(x_iw_2)$ are in different partite sets of the partition of $V(G_{x_i})$.

Case 1. There exists $x_i \in V(C) \setminus \{x_l\}$ such that (x_i, w_1, x_{i+2}) is an H-path.
Consider the cycle $T = (x_{i+2}, w_2, x_{i+3}, x_{i+2})$, since C has decreasing obstruction with respect to w_2, we have that $x_{i+3} \in O_H(T)$. Hence, $x_{i+2} \notin O_H(T)$ and $w_2 \notin O_H(T)$, otherwise $|O_H(T)| \geq 2$, a contradiction. Therefore, $(w_1, x_{i+2}, w_2, x_{i+3}) \cup (x_{i+3}, C, x_i) \cup (x_i, w_1)$ is an H-cycle of length $l+1$ containing v, which is impossible.

Case 2. For every $x_i \in V(C) \setminus \{x_l\}$, (x_i, w_1, x_{i+2}) is not an H-path, that is, $w_1 \in O_H((x_i, w_1, x_{i+2}))$.
Consider the cycle $C' = (x_i, w_1, x_{i+2}, x_i)$, for some $x_i \in V(C) \setminus \{x_l\}$, by the assumption of this case, we have that $w_1 \in O_H(C')$. Hence, $x_i \notin O_H(C')$ and $x_{i+2} \notin O_H(C')$, otherwise $|O_H(C')| \geq 2$, which is not possible. In particular, (w_1, x_2, x_4) is an H-path in G.

Consider the cycle $C'' = (x_j, w_1, x_{j+1}, x_j)$, where $x_j \in V(C)$. Since C has increasing obstruction with respect to w_1, we have that $x_j \in O_H(C'')$. So, $w_1 \notin O_H(C'')$ and $x_{j+1} \notin O_H(C'')$, otherwise $|O_H(C'')| \geq 2$, which is impossible. Similarly, we have that $w_2 \notin O_H((x_j, w_2, x_{j+1}, x_j))$, for every $x_j \in V(C)$. In particular, (x_1, w_1, x_2) and (x_4, w_2, x_5) are H-paths in G.

Subcase 2.1 (x_2, x_4, w_2) is an H-path.
Note that $(w_1, x_2, x_4, w_2, x_5) \cup (x_5, C, x_1) \cup (x_1, w_1)$ is an H-cycle of length $l+1$ containing v, a contradiction.

Subcase 2.2 (x_2, x_4, w_2) is not an H-path.
Consider the cycle $T = (w_2, x_2, x_4, w_2)$, by the assumption of this subcase, we have that $x_4 \in O_H(T)$. So, $w_2 \notin O_H(T)$, otherwise $|O_H(T)| \geq 2$, which is impossible.

We have that $\{w_1x_2, x_2x_3\}$ and $\{w_2x_2, x_2x_1\}$ are independent sets in G_{x_2}, since C has increasing obstruction (respectively decreasing) with respect to w_1 (w_2). Moreover, since G_{x_2} is a complete k_{x_2}-partite graph and the vertices x_1x_2 and x_2x_3 are adjacent in G_{x_2}, we have that w_2x_2 and x_2w_1 are adjacent in G_{x_2}. Therefore, (w_2, x_2, w_1) is an H-path in G.

Now, we can conclude that $(w_1, x_2, w_2, x_4) \cup (x_4, C, x_1) \cup (x_1, w_1)$ is an H-cycle of length $l+1$ containing v, a contradiction.

Therefore, Claim 2 holds. □

Suppose, without loss of generality, that C has increasing obstruction with respect to each vertex in W_2.

Claim 3. Let $w \in W_2$ and x_a and x_b be two distinct vertices of C, with $a < b$, such that wx_a and wx_b are not adjacent in G_w. Then, wx_{a-i} and wx_{b-i} are not adjacent in G_w, for every $i \in \{0, \ldots, l-1\}$. Moreover, wx_y and $wx_{y+k(b-a)}$ are not adjacent in G_w for every $x_y \in V(C)$ and for every nonnegative integer k.

Proof of Claim 3. Since the result follows for $i = 0$, proceeding by the strong inductive method, it is sufficient to show that if k is an index with $0 \leq k \leq l-2$, and $i \in \{0, \ldots, k\}$ are such that wx_{a-i} and wx_{b-i} are not adjacent in G_w, then $wx_{a-(k+1)}$ and $wx_{b-(k+1)}$ are not adjacent in G_w.

Since wx_{a-k} and wx_{b-k} are not adjacent in G_w and $|O_H(C')| \leq 1$ for every cycle C' of length 3 in G, we have that $w \notin O_H((x_{a-k}, w, x_{b-k}, x_{a-k}))$. Hence, $x_{a-k}x_{b-k}$ and $x_{a-k}w$ are adjacent in $G_{x_{a-k}}$ and $x_{a-k}x_{b-k}$ and $x_{b-k}w$ are adjacent in $G_{x_{b-k}}$. Moreover, since $G_{x_{a-k}}$ is a complete $k_{x_{a-k}}$-partite graph and C has increasing obstruction with respect to w, we have that $x_{a-k}x_{b-k}$ and $x_{a-k}x_{a-k+1}$ are adjacent in $G_{x_{a-k}}$. Similarly, $x_{a-k}x_{b-k}$ and $x_{b-k}x_{b-k+1}$ are adjacent in $G_{x_{b-k}}$.

If $x_{a-k-1}w$ and wx_{b-k-1} are adjacent in G_w, then $(w, x_{b-k-1}) \cup (x_{b-k-1}, C^{-1}, x_{a-k}) \cup (x_{a-k}, x_{b-k}) \cup (x_{b-k}, C, x_{a-k-1}) \cup (x_{a-k-1}, w)$ is an H-cycle of length $l+1$ containing v, a contradiction. Hence, $x_{a-(k+1)}w$ and $wx_{b-(k+1)}$ are not adjacent in G_w.

Therefore, for every $i \in \{0, \ldots, l-1\}$, wx_{a-i} and wx_{b-i} are not adjacent in G_w.

Notice that for any two vertices x_p and x_q in $V(C)$ such that $q-p=b-a$, we have that $\{wx_q, wx_p\}$ is an independent set in G_w. Consequently, since G_w is a complete k_w-partite graph, it follows that for every vertex $x_q \in V(C)$ and for every positive integer k, we have that $\{wx_q, wx_{q+(b-a)}, \ldots, wx_{q+k(b-a)}\}$ is an independent set in G_w. □

Claim 4. If $|W_2| \geq 2$, then:

1. For every vertex $w \in W_2$ and for every pair of different vertices x_i and x_j in $V(C)$, we have that wx_i and wx_j are adjacent in G_w.
2. For each $w \in W_2$, $V(C)$ has the H-dependence property with respect to w.

Proof of Claim 4. 1. Proceeding by contradiction, suppose that there exist different vertices x_p and x_q in $V(C)$, with $1 \leq p < q \leq l$, such that wx_p and wx_q are not adjacent in G_w and x_p and x_q are chosen in a way $q-p$ is as small as possible.

Let $l = k(q-p)+r$, where $0 \leq r < q-p < l$. Recall that x_p and x_{p+l} are the same vertex in C. Hence, by Claim 3, we have that wx_{p+r} and $wx_{p+k(q-p)+r}$ are not adjacent in G_w, where $x_{p+k(q-p)+r} = x_{p+l} = x_p$, that is, wx_{p+r} and wx_p are not adjacent in G_w. Notice that $r=0$, otherwise wx_p and wx_{p+r} are not adjacent in G_w, where $(p+r)-p = r < q$, contradicting the choice of p and q.

Again, by Claim 3 we have that wx_{p+2} and $wx_{p+(k-1)(q-p)+2}$ are not adjacent in G_w, where

$$x_{p+(k-1)(q-p)+2} = x_{p+l-(q-p)+2} = x_{p-(q-p)+2}.$$

Thus, wx_{p+2} and $wx_{p-(q-p)+2}$ are not adjacent in G_w, that is, $w \in O_H(C')$, where $C' = (w, x_{p+2}, x_{p-(q-p)+2}, w)$. By hypothesis, $|O_H(C')| \leq 1$, hence $x_{p+2} \notin O_H(C')$, implying that wx_{p+2} and $x_{p+2}x_{p-(q-p)+2}$ are in different partite sets of the partition of $V(G_{x_{p+2}})$; also since $x_{p-(q-p)+2} \notin O_H(C')$, we get $wx_{p+(q-p)+2}$ and $x_{p-(q-p)+2}x_{p+2}$ are in different partite sets of the partition of $V(G_{x_{p-(q-p)+2}})$. Recall that C has increasing obstruction with respect to w, thus we have that wx_{p+2} and $x_{p+2}x_{p+3}$ are in the same partite set of $V(G_{x_{p+2}})$, and $wx_{p-(q-p)+2}$ and $x_{p-(q-p)+2}x_{p-(q-p)+3}$ are in the same partite set of $V(G_{x_{p-(q-p)+2}})$. Therefore, $x_{p+2}x_{p+3}$ and $x_{p+2}x_{p-(q-p)+2}$ are in different partite sets of the partition of $V(G_{x_{p+2}})$, and $x_{p+2}x_{p-(q-p)+2}$ and $x_{p-(q-p)+2}x_{p-(q-p)+3}$ are in different partite sets of the partition of $V(G_{x_{p-(q-p)+2}})$, that is, $x_{p+2}x_{p-(q-p)+2}$ and $x_{p+2}x_{p+3}$ are adjacent in $G_{x_{p+2}}$, and $x_{p+2}x_{p-(q-p)+2}$ and $x_{p-(q-p)+2}x_{p-(q-p)+3}$ are adjacent in $G_{x_{p-(q-p)+2}}$. Since wx_p and wx_q are not adjacent in G_w, by Claim 3 we have that $w_{x_{p+1}}$ and $w_{x_{q+1}}$ are not adjacent in G_w, and following the same reasoning we obtain that $x_{p+3}x_{p+1-(q-p)+2}$ and $x_{p+3}x_{p+4}$ are not adjacent in $G_{x_{p+3}}$, and $x_{p+3}x_{p+1-(q-p)+2}$ and $x_{p+1-(q-p)+2}x_{p+1-(q-p)+3}$ are not adjacent in $G_{x_{p+1-(q-p)+2}}$.

Let w' be a vertex in $W_2 \setminus \{w\}$ (w' exists because $|W_2| \geq 2$). By Claim 2, we have that for every $x_i \in V(C)$, wx_i and x_ix_{i+1} (respectively $w'x_i$ and x_ix_{i+1}) are not adjacent in G_{x_i}. Since G_{x_i} is a complete k_{x_i}-partite graph, we have that wx_i,x_ix_{i+1} and $w'x_i$ are in the same partite set of the partition of $V(G_{x_i})$, in particular, wx_i and x_iw' are not adjacent in G_{x_i}. Hence, $x_i \in O_H(C'')$, where $C'' = (w, x_i, w', w)$, and by hypothesis, $w \notin O_H(C'')$ and $w' \notin O_H(C'')$. Therefore, for every $x_i \in V(C)$, ww' and $w'x_i$ are not adjacent in $G_{w'}$, and $w'w$ and wx_i are not adjacent in G_w. Hence, $(w, x_p) \cup (x_p, C^{-1}, x_{p-(q-p)+2}) \cup (x_{p-(q-p)+2}, x_{p+2}) \cup (x_{p+2}, C, x_{p-(q-p)+1}) \cup (x_{p-(q-p)+1}, w', w)$ and $(w, x_{p+1}) \cup (x_{p+1}, C^{-1}, x_{p+1-(q-p)+2}) \cup (x_{p+1-(q-p)+2}, x_{p+3}) \cup (x_{p+3}, C, x_{p+1-(q-p)+1}) \cup (x_{p+1-(q-p)+1}, w', w)$ are H-cycles of length $l+1$, where at least one of them contains v, a contradiction.

Therefore, for every vertex $w \in W_2$ and every pair of distinct vertices x_i and x_j in $V(C)$, we have wx_i and wx_j are adjacent in G_w.

2. Proceeding by contradiction, suppose that there exists w in W_2 such that $V(C)$ does not have H-dependence property with respect to the vertex w, that is, there exist two vertices x_i and x_j in $V(C)$ such that wx_i and x_ix_j are adjacent in G_{x_i}, and wx_j and x_ix_j are adjacent in G_{x_j}. Since C has increasing obstruction with respect to w, we have that wx_i and x_ix_{i+1} are not adjacent in G_{x_i}. Given that G_{x_i} is a complete k_{x_i}-partite graph, we have that x_ix_{i+1} and x_ix_j are adjacent in G_{x_i}. Similarly, x_jx_{j+1} and x_ix_j are adjacent in G_{x_j}. By item 1, we have that wx_{i-1} and wx_{j-1} are adjacent in G_w. Hence, $(w, x_{j-1}) \cup (x_{j-1}, C^{-1}, x_i) \cup (x_i, x_j) \cup (x_j, C, x_{i-1}) \cup (x_{i-1}, w)$ is an H-cycle of length $l+1$ containing v, a contradiction.

Therefore, Claim 4 holds. □

Recall that W_1 is the set of all w in $V(G) \setminus V(C)$ such that all the wx_i are in the same partite set of the k_w-partition of $V(G_w)$.

Claim 5. If $|W_1| \geq 2$, then W_1 has the H-dependence property with respect to the vertex v.

Proof of Claim 5. Suppose by contradiction that W_1 has no H-dependence property with respect to the vertex $v = x_1$, that is, there exists w_1 and w_2 in W_1 such that vw_1 and w_1w_2 are adjacent in G_{w_1} (vw_2 and w_1w_2 are adjacent in G_{w_2}, respectively). Given that G_{w_1} is a complete k_{w_1}-partite graph and x_1w_1 and w_1x_2 are in the same partite set of the partition of $V(G_{w_1})$, we have that w_1w_2 and w_1x_2 are adjacent in G_{w_1}, that is, (w_2, w_1, x_2) is an H-path in G. Now, consider the cycle $C' = (w_1, x_3, x_4, w_1)$, since w_1 is in W_1, it follows that $w_1 \in O_H(C')$. Thus, by hypothesis $x_3 \notin O_H(C')$ and $x_4 \notin O_H(C')$, in particular, (w_1, x_3, x_4) is an H-path in G. Similarly, (w_2, x_1, x_l) is an H-path in G. Therefore, $(x_1 = v, w_2, w_1, x_3) \cup (x_3, C, x_1)$ is an H-cycle of length $l+1$ containing v, a contradiction. So Claim 5 follows. $\square$

Recall that $l \leq n-1$, so $W_1 \neq \emptyset$ or $W_2 \neq \emptyset$. We consider the following three cases.

Case 1. $W_1 \neq \emptyset$ and $W_2 = \emptyset$.
If $|W_1| = 1$, then $k_x = 1 \leq \frac{n+1}{2}$, where x is the only vertex in W_1, which is impossible. Hence, $|W_1| \geq 2$. It follows from Claim 5 and Proposition 2.1 that there exists a vertex w in W_1 such that

$$k_w^{G[W_1]} \leq \frac{|W_1|+1}{2}.$$

In this case, $V(G) = V(C) \cup W_1$, so $|W_1| \leq n-4$. Therefore,

$$k_w \leq k_w^{G[W_1]} + k_w^{G[V(C)\cup\{w\}]} = k_w^{G[W_1]} + 1 \leq \frac{n-4+1}{2} + 1 = \frac{n-1}{2} < \frac{n+1}{2},$$

which is a contradiction. Therefore, this case is not possible.

Case 2. $W_1 \neq \emptyset$ and $W_2 \neq \emptyset$.
Let w_1 in W_1 and w_2 in W_2. Given that $w_1 \in W_1$, we have that $w_1 \in O_H(C')$, where $C' = (w_1, x_i, x_{i+1}, w_1)$ and, by hypothesis, it follows that $x_i \notin O_H(C')$, that is, (w_1, x_i, x_{i+1}) is an H-path in G, for every $i \in \{1, \ldots, l\}$. Recall that C has increasing obstruction with respect to w_2, so (w_2, x_i, x_{i-1}) is an H-path in G and (w_2, x_i, x_{i+1}) is not an H-path in G, for every $i \in \{1, \ldots, l\}$. In particular, for $i = 1$ we have that $x_1 \in O_H(C'')$, where $C'' = (w_2, x_1, x_2, w_2)$ and, by hypothesis, $w_2 \notin O_H(C'')$. So, (x_1, w_2, x_2) is an H-path in G and, by Observation 2.3, (x_1, w_2, w_1) or (x_2, w_2, w_1) is an H-path in G. We claim that $\{w_1w_2\} \cup \{w_1x_i : x_i \in V(C)\}$ is an independent set in G_{w_1}. Otherwise, $(x_1, w_2, w_1, x_3) \cup (x_3, C, x_1)$ or $(x_2, w_2, w_1, x_4) \cup (x_4, C, x_2)$ is an H-cycle of length $l+1$ containing v, a contradiction. Therefore, $\{w_1w_2\} \cup \{w_1x_i : x_i \in V(C)\}$ is an independent set in G_{w_1}, for every w_2 in W_2.

If $|W_1| = 1$, then $k_x = 1 \leq \frac{n+1}{2}$, where x is the only vertex in W_1, which is impossible. Hence, $|W_1| \geq 2$. It follows from Claim 5 and Proposition 2.1 that there exists a vertex w_1 in W_1 such that

$$k_{w_1}^{G[W_1]} \leq \frac{|W_1|+1}{2}.$$

In this case $V(G) = V(C) \cup W_1 \cup W_2$, so $|W_1| \leq n-4$. Therefore,

$$k_{w_1} \leq k_{w_1}^{G[W_1]} + k_{w_1}^{G[V(C)\cup W_2\cup\{w\}]} = k_{w_1}^{G[W_1]} + 1 \leq \frac{n-4+1}{2} + 1 = \frac{n-1}{2} < \frac{n+1}{2},$$

which is a contradiction.

Case 3. $W_1 = \emptyset$ and $W_2 \neq \emptyset$.
If $|W_2| \geq 2$, then by Claim 4 we have that C has H-dependence property with respect to the vertex w, for every $w \in W_2$. It follows from Proposition 2.1 that there exists a vertex $x_i \in V(C)$ such that

$$k_{x_i}^{G[V(C)]} \leq \frac{|V(C)|+1}{2} < \frac{n+1}{2}.$$

Recall that for every vertex $w \in W_2$, C has increasing obstruction with respect to w, that is, wx_i and x_ix_{i+1} are in the same partite set of the partition of $V(G_{x_i})$. Therefore,

$$k_{x_i} = k_{x_i}^{G[V(C)]} < \frac{n+1}{2},$$

which is a contradiction. Hence, $|W_2| = 1$ and $|V(C)| = n-1$.

Notice that if there exists a pair of different vertices x_a and x_b, with $a < b$, in $V(C)$ such that wx_a and wx_b are not adjacent in G_w, then by Claim 3 we have that wx_y and $wx_{y+k(b-a)}$ are in the same partite set of the partition of $V(G_w)$, for every positive integer k. Thus, $k_w \leq \frac{n-1}{2} < \frac{n+1}{2}$, a contradiction (observe that when $b - a = \frac{n-1}{2}$, we have that $k_w = \frac{n-1}{2}$). Therefore, for every pair of different vertices x_a and x_b in $V(C)$, we conclude that wx_a and wx_b are not adjacent in G_w.

If $V(C)$ has no the H-dependence property with respect to the vertex w, then there exist x_i and x_j, with $i < j$, in $V(C)$ such that $x_i \notin O_H((w, x_i, x_j, w))$ and $x_j \notin O_H((w, x_i, x_j, w))$, in particular, wx_i and x_ix_j are adjacent in G_{x_i}. Since C has increasing obstruction with respect to w, we have that (w, x_i, x_{i+1}) is not an H-path, that is, wx_i and x_ix_{i+1} are not adjacent in G_{x_i}. Moreover, since G_{x_i} is a complete k_{x_i}-partite graph, it follows that $x_{i+1}x_i$ and x_ix_j are adjacent in G_{x_i}, that is, (x_{i+1}, x_i, x_j) is an H-path in G. Applying the same reasoning, we have that (x_i, x_j, x_{j+1}) is an H-path in G. So, $(w, x_{j-1}) \cup (x_{j-1}, C^{-1}, x_i) \cup (x_i, x_j) \cup (x_j, C, x_{i-1}) \cup (x_{i-1}, w)$ is an H-cycle of length $l + 1$ containing v, a contradiction. Therefore, $V(C)$ has the H-dependence property with respect to the vertex w and, by Proposition 2.1, there exists a vertex x_i in $V(C)$ such that

$$k_{x_i} = k_{x_i}^{G[V(C)]} \leq \frac{|V(C)| + 1}{2} = \frac{(n-1) + 1}{2} < \frac{n+1}{2},$$

which is a contradiction. Hence, we conclude that the case under consideration is not possible.

Since Cases 1, 2, and 3 are impossible, we conclude that there exists a cycle of length $l + 1$ containing v. Therefore, G is vertex H-pancyclic. This completes the proof of Theorem 3.1. □

Notice that, given a graph H possibly with loops and a complete H-colored graph G, we can check in polynomial time whether G_x is a complete multipartite graph for each x in $V(G)$ (see [10]); whether a vertex is an obstruction of a given cycle (by Observations 2.1 and 2.2); and whether $|O_H(C)| \leq 1$ for every cycle C of length 3 in G and $|O_H(C')| \neq 3$ for every cycle C' of length 4 in G (as there exists $\binom{n}{3}$ cycles of length three and $3\binom{n}{4}$ cycles of length four in G). Therefore, it is possible to verify in polynomial time whether the hypotheses of Theorem 3.1 are satisfied.

In the particular case, when H is a complete graph without loops, we obtain as a direct consequence of Theorem 3.1 the following result.

Corollary 3.1 (see [9]). *Let G be an k-edge-colored complete graph on n vertices, $n \geq 3$, such that $\delta^c(G) \geq \frac{n+1}{2}$. If G contains no monochromatic cycles of length 3, then G is properly vertex panclycic.*

The following construction given by Fujita and Magnant in [13] can be used to show that the degree condition on Corollary 3.1 cannot be improved or dropped, and therefore, the condition on k_x of Theorem 3.1 cannot be improved or dropped either.

Construction 3.1 (see [13]). *Consider a complete graph $G = K_{2m}$ with set of vertices $\{x, v_1, \ldots, v_{2m-1}\}$. Color the edge xv_i with color i, for all $i \in \{1, \ldots, 2m-1\}$. Let $H = G - \{x\}$ and arbitrarily partition $E(H)$ into $m - 1$ Hamiltonian cycles. Also, we arbitrarily orient these Hamiltonian cycles in such a way every Hamiltonian cycle is a directed cycle. Color the edge v_iv_j of G with color j if (v_i, v_j) is an arc of one of the oriented Hamiltonian cycles. This provides an edge-coloring of G such that $\delta^c(G) = \frac{n}{2}$ and x is not contained in an properly colored cycle in G.*

Let n be a positive integer, $n \geq 3$, and $k = \binom{n-1}{2} + 1$. Consider G a complete graph of order n with $V(G) = \{v_1, \ldots, v_{n-1}, x\}$. Color the edges of the graph $G - x$ in such a way every two different edges have different color; and color every edge incident with x with color k. Hence, G is a k-edge-colored complete graph. Notice that G is not a properly vertex-pancyclic graph, since x is contained in no properly colored cycle. Nevertheless, if we see G as an H-colored graph, where H is the complete graph with vertex set $I_k = \{1, \ldots, k\}$, with a loop in the vertex k, then G fulfills the hypotheses of Theorem 3.1, and G is a vertex H-pancyclic graph.

The above discussion shows that Theorem 3.1 is more general than Theorem 1.2.

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