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Research Article

Trails in arc-colored digraphs avoiding forbidden transitions

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Abstract

Let H be a digraph possibly with loops. Let D be a digraph without loops. An H-coloring of D is a function $c : A(D) \to V(H)$. We say that D is an H-colored digraph whenever we are taking a fixed H-coloring of D. A trail $W = (v_0, e_0, v_1, e_1, v_2, \dots, v_{n-1}, e_{n-1}, v_n)$ in D is an H-trail if and only if $(c(e_i), c(e_{i+1}))$ is an arc in H for every $i \in \{0, \dots, n-2\}$. Whenever the vertices of an H-trail are pairwise different, we say that it is an H-path. In this paper, we study the problem of finding $s-t$ H-trail in H-colored digraphs. First, we prove that finding an H-trail starting with the arc e and ending at arc f can be done in polynomial time. As a consequence, we give a polynomial time algorithm to find the shortest H-trail from a vertex s to a vertex t (if it exists). Moreover, we obtain a Menger-type theorem for H-trails. As a consequence, we show that the problem of maximizing the number of arc disjoint $s-t$ H-trails in D can be solved in polynomial time. Although finding an H-path between two given vertices is an NP-problem, it becomes a polynomial time problem in the case when H is a reflexive and transitive digraph.

Keywords: arc-colored digraph; arc-colored trails; polynomial algorithms.

2020 Mathematics Subject Classification: 05C15, 68Q25.

1. Introduction

For basic concepts, terminology, and notation not defined here, we refer the reader to [4]. Throughout this work, we will consider finite digraphs. Let D be a digraph. The sets of vertices and arcs of D are denoted by $V(D)$ and $A(D)$, respectively.

A *transition* in a digraph D is a pair of adjacent arcs of D such that the head of one is the tail of the other one. A *transition system* of a digraph D is a set of transitions in D. Let T be a transition system. We say that a transition is *permitted* if it is in T and it is *forbidden* otherwise. We say that a walk is *T-compatible* whenever all its transitions are permitted. For every vertex $v \in V(D)$, the set of allowed transitions defines a digraph $T(v)$, called *transition digraph of* v, whose vertex set is the set of arcs incident with v, and there is an arc from e to f in $T(v)$ if and only if $\{e, f\} \in T$, where v is the head of e and the tail of f.

In [24], Szeider studied the computational complexity of finding T-compatible paths between two given vertices of an undirected graph, with a transition system T, proved that the problem is in NP, and found a class of transition systems that can be solved in linear time; for example, if $T(v)$ is a complete graph for every $v \in V(G)$. Several authors have studied the existence of T-compatible trails, paths, and cycles in graphs with a given transition system, from an algorithmic point of view (see [5,6]).

A particular class of T-compatibles walks that has been extensively studied, is the class of *properly colored walks* (which are walks with no consecutive edges with the same color) in edge-colored graphs. Several authors have worked with this concept in directed and undirected graphs; for example, see [3,13,19,20]. Properly colored walks are of interest for theoretical reasons; for example, they can be considered as a generalization of walks in undirected and directed graphs [4]. Such walks may also be useful in graph theory applications; for example, in genetic and molecular biology [11,12,23], social science [8], and channel assignment in wireless networks [2,22].

In view of the relevance in applications of finding properly colored walks, the problem of finding properly colored trails and paths, between two given vertices, has been studied from an algorithmic perspective. For example, in [1] Abouelaoualim et al. proved that finding the shortest properly colored trail between two vertices can be done in polynomial time. In [24], Szeider proved that given a c-edge-colored graph, $c \geq 2$, a properly colored path between two vertices can be found in linear time on the size of the graph.

Theorem 1.1 (Abouelaoualim et al. [1])**.** *Let G be a c-edge-colored graph with $c \geq 2$. The problem of finding the shortest properly colored trail in G (if any) can be solved in polynomial time.*

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Theorem 1.2 (Szeider [24]). *Let s and t be two vertices in a c-edge-colored graph G with $c \geq 2$. Then, either we can find a properly colored path between s and t or else decide that such a path does not exist in G in linear time on the size of the graph.*

Gourvès et al. [18] proved that deciding whether a planar c-arc-colored digraph with no properly colored cycle contains a properly colored $s-t$ path is NP-complete. However, they also proved that the problem of finding a directed properly colored trail from a vertex s to a vertex t in a c-arc-colored digraph can be done in polynomial time. Moreover, they proved that the problem of maximizing the number of arc disjoint properly colored trails between two vertices can be solved in polynomial time.

Theorem 1.3 (Gourvès et al. [18]). *Deciding whether a planar c-arc-colored digraph with no properly colored cycle contains a properly colored $s-t$ path is NP-complete.*

Theorem 1.4 (Gourvès et al. [18]). *Let D be a c-arc-colored digraph with $c \geq 2$. The problem of finding a directed properly colored trail in D (if any) can be solved in polynomial time.*

Theorem 1.5 (Gourvès et al. [18]). *The problem of maximizing the number of arc disjoint properly colored trails from s to t in D can be solved in polynomial time.*

Different kinds of edge-coloring in directed and undirected graphs have been studied by several researchers; for example, in [21] the arcs of a tournament were colored with the vertices of a poset. We will consider the following arc-coloring. Let H be a digraph possibly with loops and D a digraph without loops. An H-*coloring* of D is a function $c : A(D) \to V(H)$. We will say that D is an H-*colored digraph*, whenever we are taking a fixed H-coloring of D. A directed walk $W = (v_0, e_0, v_1, \ldots, v_{k-1}, e_{k-1}, v_k)$ in D, where $e_i = (v_i, v_{i+1})$ for every i in $\{0, \ldots, k-1\}$, is an H-*walk* if and only if $(c(e_i), c(e_{i+1})) \in A(H)$ for every $i \in \{0, \ldots, k-2\}$. Let W be an H-walk. If W is a trail (path) then W will be called H-*trail* (H-*path*, respectively).

The concepts of H-coloring and H-walks were introduced, for the first time, by Linek and Sands in [21]. Such concepts have also been studied in the context of kernel theory and related topics, see [9, 10, 16].

Galeana-Sánchez et al. [15] studied the existence of H-cycles in H-colored multigraphs, and reported a polynomial time algorithm to decide whether an H-colored multigraph contains an H-cycle. In [14], they studied the existence of closed Euler H-trails in H-colored graphs. In both cases, an auxiliary graph has been of significant importance and here we include the digraph version of this auxiliary graph.

Definition 1.1. *Let D be an H-colored digraph and u a vertex of D. We define the digraph D_u as follows*

1. $V(D_u) = \{e \in A(D) \mid e$ *is incident with* $u\}$*;*
2. $(a, b) \in A(D_u)$*, with* $a \neq b$*, if and only if* u *is the head of* a *and the tail of* b*, and* $(c(a), c(b)) \in A(H)$*.*

In [17], Galeana-Sánchez and Vilchis-Alfaro defined the auxiliary digraph $L_n^H(D)$, defined below, and proved that there is a bijection between the set of closed H-trails in D and the set of directed cycles in $L_2^H(D)$.

Definition 1.2. *Let D be an H-colored digraph with $|A(D)| = q$. For $n \geq 2$, $L_n^H(D)$ is the digraph with nq vertices, obtained as follows: for each arc $e = (u, v)$ of D, we take two vertices $f(e, u)$ and $f(e, v)$ in $L_n^H(D)$, and add a directed path from $f(e, u)$ to $f(e, v)$ with $n-2$ new intermediate vertices. The rest of the arcs of $L_n^H(D)$ are defined as follows: $(f(e, u), f(g, u)) \in A(L_n^H(D))$ if and only if $e = (x, u)$ and $g = (u, y)$ for some x and y in $V(D)$ and $(c(e), c(g)) \in A(H)$.*

Notice that the digraph $L_n^H(D)$ can be constructed as follows: take the disjoint union of D_x and change the label of the vertices of D_x from e to $f(e, x)$ for every $x \in V(D)$, and for every $e = (x, y) \in A(D)$, we add a directed path from $f(e, u)$ to $f(e, v)$ with $n-2$ new intermediate vertices.

Theorem 1.6 (Galeana-Sánchez and Vilchis-Alfaro [17]). *Let D be an H-colored digraph. Then there is a bijection between the set of closed H-trails in D and the set of directed cycles in $L_2^H(D)$.*

In Section 2, we study the problem of finding $s-t$ H-trail in H-colored digraphs. We prove that determining (if there exists) an H-trail starting with the arc e and ending at the arc f can be done in polynomial time. As a consequence, we give a polynomial time algorithm to find (if there exists) the shortest H-trail from the vertex s to the vertex t. Moreover, we show that the problem of maximizing the number of arc disjoint $s-t$ H-trails in D can be solved in polynomial time. We also study the computational complexity of finding H-path between two given vertices of an H-colored digraph in terms of the digraph H.

2. Main results

In this section, we are interested in the complexity of finding H-trails in H-colored digraphs. To achieve this, we use the auxiliary graph $L_2^H(D)$. The statements given in the following observation are direct consequences of the definition.

Observation 2.1. *Let D be an H-colored digraph.*

(a). *For every $e = (x, y)$ in $A(D)$, $d^+(f(e, x)) = d^-(f(e, y)) = 1$ in $L_2^H(D)$. Moreover, $N^+(f(e, x)) = \{f(e, y)\}$ and*

$$N^-(f(e, y)) = \{f(e, x)\}.$$

(b). *For every $u \in V(D)$, D_u is a bipartite digraph with partition $\{A_u^+, A_u^-\}$, where A_u^+ (respectively, A_u^-) is the set of arcs in D with head u (respectively, with tail u). Moreover, every arc in D_u has tail in A_u^- and head in A_u^+.*

(c). *$L_2^H(D)$ is a bipartite digraph.*

Algorithm 1 is a linear time algorithm that, starting with a path in $L_2^H(D)$, obtains an H-trail in D.

Algorithm 1 Path to H-trail

Require: A path $T = (f(e_1, x_1), f(e_2, x_2), \ldots, f(e_k, x_k))$ in $L_2^H(D)$ such that $x_1 \neq x_2$ and $x_{k-1} \neq x_k$.
Ensure: An H-trail in D.

```
i ← 1
P
while i ≠ k/2 + 1 do
    if i = 1 then
        P ← P = (x_1, e_1, x_2)
    else
        P ← P = P ∪ (x_{2i−1}, e_{2i−1}, x_{2i}).
    end if
end while
```

Lemma 2.1. *Let D be an H-colored digraph. Given a path $T = (f(e_1, x_1), f(e_2, x_2), \ldots, f(e_k, x_k))$ in $L_2^H(D)$ such that $x_1 \neq x_2$ and $x_{k-1} \neq x_k$, Algorithm 1 returns an $x_1 - x_k$ H-trail in G.*

Proof. Let $T = (f(e_1, x_1), f(e_2, x_2), \ldots, f(e_k, x_k))$ be a path in $L_2^H(D)$ such that $x_1 \neq x_2$ and $x_{k-1} \neq x_k$. By the definition of $L_2^H(D)$ and the fact that $x_1 \neq x_2$, we have that $(f(e_1, x_1), f(e_2, x_2)) \notin A(G_{x_1})$. So, by Observation 2.1(a), it follows that $e_1 = e_2$ and $e_1 = (x_1, x_2)$. Hence, $P = (x_1, e_1, x_2)$ is an H-path in D. By the definition of $L_2^H(D)$ and Observation 2.1(a), we have that $x_2 = x_3$ and $(f(e_2, x_2), f(e_3, x_3)) \in A(G_{x_2})$, that is, $(c(e_1), c(e_3)) \in A(H)$. By Observation 2.1(a), it follows that $x_3 \neq x_4$. Hence, $e_3 = e_4$ and $e_3 = (x_3, x_4)$. Consequently, $P = (x_1, e_1, x_2 = x_3, e_3, x_4)$ is an H-trail (notice that $e_1 \neq e_3$, otherwise $f(e_1, x_1) = f(e_3, x_3)$, which is impossible since $x_1 \neq x_2$ and $x_2 = x_3$).

By the definition of $L_2^H(D)$ and Observation 2.1(a), we have that $x_4 = x_5$ and $(f(e_4, x_4), f(e_5, x_5)) \in A(G_{x_4})$, that is, $(c(e_3), c(e_5)) \in A(H)$. By Observation 2.1(a), it follows that $x_5 \neq x_6$, hence $e_5 = e_6$ and $e_5 = (x_5, x_6)$. Hence, $P = (x_1, e_1, x_2 = x_3, e_3, x_4 = x_5, e_5, x_6)$ is an H-trail (notice that $e_5 \notin \{e_1, e_3\}$, otherwise T is not a path).

Following this reasoning, we have that $x_{2i} = x_{2i+1}$, $x_{2i+1} \neq x_{2i+2}$, $(c(e_{2i-1}), c(e_{2i+1})) \in A(H)$ and $e_{2i+1} = (x_{2i+1}, x_{2i+2})$, for every $i \in \{1, \ldots, k/2 - 1\}$. (Notice that k is even since $x_{k-1} \neq x_k$).

Hence, $P = (x_1, e_1, x_2 = x_3, e_3, x_4 = x_5, \ldots, x_{k-1}, e_{k-1}, x_k)$ is an H-walk in D. Moreover, since T is a path, it follows that P is an H-trail.

Therefore, Algorithm 1 returns an H-trail in D in linear time. □

Lemma 2.2. *Let D be an H-colored digraph. Let $e = (x_1, y_1)$ and $g = (x_2, y_2)$ be different arcs of D. If there is no $f(e, y_1) - f(g, x_2)$ path in $L_2^H(D)$, then there is no H-trail in D starting with e and ending with g.*

Proof. Proceeding by contradiction, assume that there exists an H-trail in D starting with e and ending with g; namely, $P = (x_1, e, y_1, e_1, y_2, \ldots, y_k, e_k, x_2, g, y_2)$. Hence, $T = (f(e, x_1), f(e, y_1), f(e_1, y_2), \ldots, f(e_k, y_k), f(e_k, x_2), f(g, x_2), f(g, y_2))$ is a path in $L_2^H(D)$. Therefore, there exist an $f(e, y_1) - f(g, x_2)$ path in $L_2^H(D)$, which is a contradiction. □

Theorem 2.1. *Given an arbitrary H-colored digraph D, finding an H-trail starting with the arc e and ending with the arc g (if any) can be done in polynomial time.*

Proof. Let D be an H-colored digraph. Let $e = (x_1, y_1)$ and $g = (x_q, y_q)$ be two arcs in D. Construct the auxiliary digraph $L_2^H(D)$. Consider the vertices $f(e, y_1)$ and $f(g, x_q)$ in $L_2^H(D)$. Find a path from $f(e, y_1)$ to $f(g, x_q)$, namely P. (If there is no $f(e, y_1) - f(g, x_q)$ path in $L_2^H(D)$, then there is no H-trail starting with e and ending with g in D). Then, it follows directly from Observation 2.1(a) that $P' = (f(e, x_1), f(e, y_1), P, f(g, x_q), f(g, y_q))$ is also a path in $L_2^H(D)$. Therefore, by Algorithm 1, there is an H-trail in D starting with the arc e and finishing with the arc g. Notice that each step can be done in polynomial time. □

Corollary 2.1. *Given an arbitrary H-colored digraph D, finding a closed H-trail containing the arc e (if any) can be done in polynomial time.*

Proof. Let D be an H-colored digraph and $e = (x, y)$ an arc in $A(D)$. Construct the auxiliary digraph $L_2^H(D)$. Consider the vertices $f(e, y)$ and $f(e, x)$ in $L_2^H(D)$. Find a path from $f(e, y)$ to $f(e, x)$, namely C. (If there is no $f(e, y) - f(e, x)$ path in $L_2^H(D)$, then there is no closed H-trail containing the arc e in D). Then, it follows directly from Observation 2.1(a) that $C' = (f(e, y), C, f(e, x), f(e, y))$ is a cycle in $L_2^H(D)$. Therefore, by Theorem 1.6, there is a closed H-trail in D that contains the arc e. □

Theorem 2.2. *Let D be an H-colored digraph. The shortest H-trail (if it exists) between any pair of arcs can be found in polynomial time.*

Proof. Let D be an H-colored digraph. Let $e = (x_1, y_1)$ and $g = (x_q, y_q)$ be two arcs in D. Construct the auxiliary digraph $L_2^H(D)$. Consider the vertices $f(e, y_1)$ and $f(g, x_q)$ in $L_2^H(D)$. Find the shortest path from $f(e, y_1)$ to $f(g, x_q)$, say P. (If there is no $f(e, y_1) - f(g, x_q)$ path in $L_2^H(D)$, then there is no H-trail starting with e and ending with g). Then, $P' = (f(e, x_1), f(e, y_1), P, f(g, x_q), f(g, y_q))$ is also a path in $L_2^H(D)$ because of Observation 2.1(a). Therefore, by Algorithm 1, there is an H-trail in D starting with the arc e and finishing with the arc g, namely T. Notice that each step can be done in polynomial time.

Notice also that if there is a shorter H-trail from e to g in D than T, namely

$$Q = (v_1 = x_1, g_1 = e, v_2 = y_1, g_2, v_3, \ldots, v_{j-1} = x_q, g_{j-1} = g, v_j = y_q),$$

then $W = (f(g_1, v_2), f(g_2, v_2), f(g_2, v_3), \ldots, f(g_{j-1}, v_{j-1}))$ is a shorter $f(e, y_1) - f(g, x_q)$ path than P in $L_2^H(D)$, which is a contradiction. □

It follows from Theorem 2.1 that finding an $s - t$ H-trail in D (if any) can be done in polynomial time. This can be done using Theorem 2.1 with all the possible pairs of arcs, one with tail s and the other one with head t. This method can be improved using the following variation of the auxiliary digraph $L_2^H(D)$.

Definition 2.1. *Let D be an H-colored digraph. Let s, t, be a pair of distinct vertices in $V(D)$. The digraph $L_2^H(s, t)$ is the digraph with the vertex set $V(L_2^H(s, t)) = V(L_2^H(D) \cup \{x_s, x_t\}$ and the arc set*

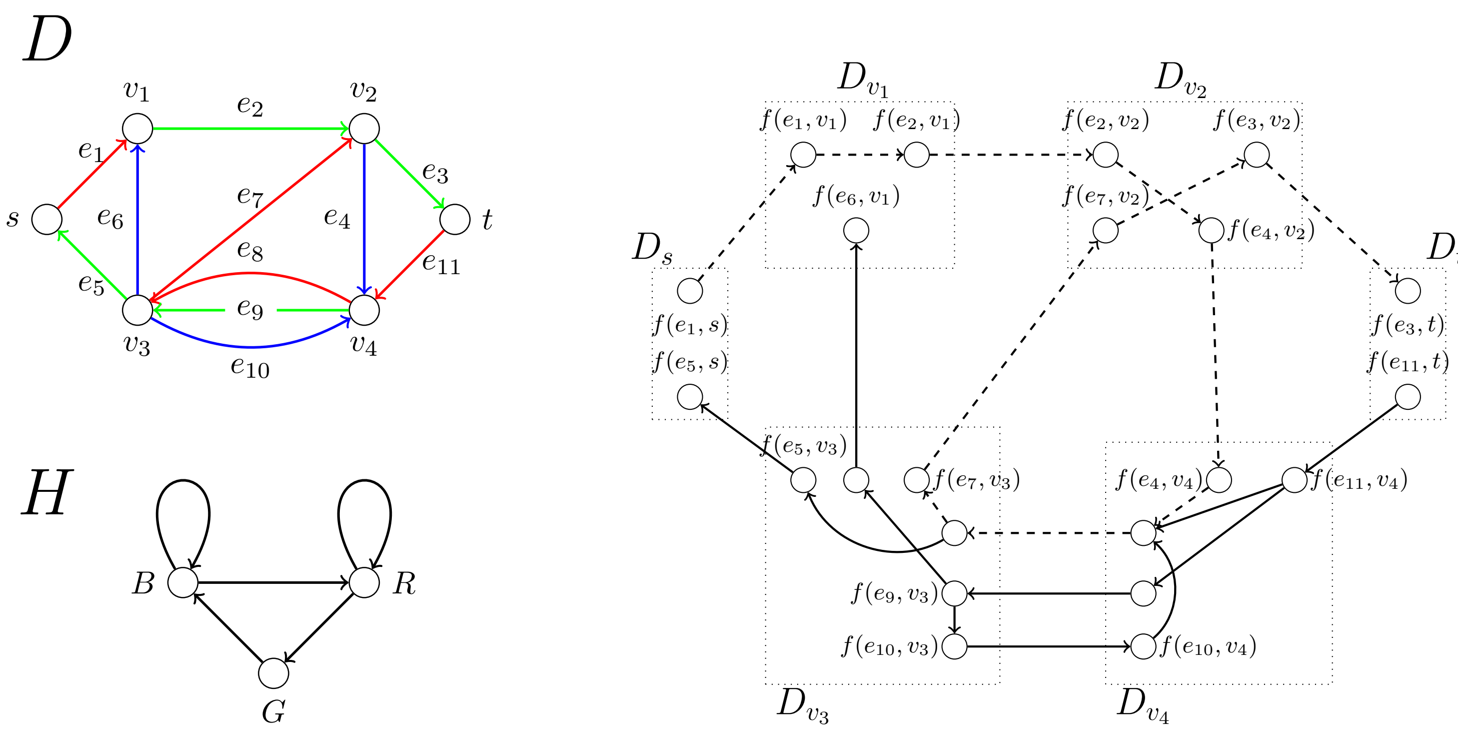
$$A(L_2^H(s, t)) = A(L_2^H(D)) \cup \{(x_s, f(e, s)) : e = (u, s) \in A(D)\} \cup \{(f(e, t), x_t) : e = (u, t) \in A(D)\}.$$

Theorem 2.3. *Given an arbitrary H-colored digraph D, finding an $s - t$ H-trail in D (if any) can be done in polynomial time.*

Proof. Let D be an H-colored digraph. Let s and t be two vertices in D. Construct the auxiliary digraph $L_2^H(s, t)$. Find a path from x_s to x_t in the digraph $L_2^H(s, t)$ (if any), namely P. (If there is no $x_s - x_t$ path in $L_2^H(s, t)$, then there is no $s - t$ H-trail in D). Hence, $P' = P \setminus \{x_s, x_t\}$ is a path in $L_2^H(D)$, and by Algorithm 1, there is an $s - t$ H-trail in D. □

Corollary 2.2. *Let D be an H-colored digraph. The shortest H-trail (if it exists) between any pair of distinct vertices can be found in polynomial time.*

Let D be a digraph. Let s, t, be a pair of different vertices in $V(D)$. An (s, t)-*separator* is a subset $X \subseteq V(D) \setminus \{s, t\}$ with the property that $D - X$ has no $s - t$ paths. If D is an H-colored digraph, an (s, t)-*H-trails-separator by arcs* is a subset $X \subseteq A(D)$ with the property that $D - X$ has no $s - t$ H-trails.



(a) D is an H-colored digraph (b) $L_2^H(D)$

Figure 2.1: $P = (s, e_1, v_1, e_2, v_2, e_4, v_4, e_8, v_3, e_7, v_2, e_3, t)$ is an $s-t$ H-trail in D. But, there is no H-path from s to t in G. Moreover, P is obtained by applying Algorithm 1 to the dashed path of $L_2^H(D)$

Theorem 2.4. *Let D be an H-colored digraph. Let s, t, be a pair of different vertices in $V(D)$. Then, the following assertions are equivalent.*

(a). *The maximum number of arc-disjoint $s-t$ H-trails in D is equal to k.*

(b). *The maximum number of internally disjoint $x_s - x_t$-paths in $L_2^H(s,t)$ is equal to k.*

(c). *The minimum number of vertices in an (x_s, x_t)-separator in $L_2^H(s,t)$ is equal to k.*

(d). *The minimum number of arcs in an (s,t)-H-trails-separator by arcs in D is equal to k.*

Proof. Let k_1 be the maximum number of arc-disjoint $s-t$ H-trail in D, k_2 the maximum number of internally disjoint $x_s - x_t$-paths in $L_2^H(s,t)$, k_3 the minimum number of vertices in an (x_s, x_t)-separator in $L_2^H(s,t)$, and k_4 the minimum number of arcs in an (s,t)-H-trails-separator by arcs in D. We will prove that $k_1 = k_2 = k_3 = k_4$.

Claim 1. $k_1 = k_2$.
It follows by Algorithm 1 that $k_2 \leq k_1$. Let $\{P_1, \ldots, P_{k_1}\}$ be a set of k_1 arc-disjoint $s-t$ H-trails in D. For each i in $\{1, \ldots, k_1\}$, we can construct an $x_s - x_t$-path in $L_2^H(s,t)$ from P_i as follows: Let $P_i = (s, e_0^i, x_1^i, e_1^i, x_2^i, \ldots, x_{j_i}^i, e_{j_i}^i, t)$. Then, $T_1 = (x_s, f(e_0^i, s), f(e_0^i, x_1^i), f(e_1^i, x_1^i), f(e_1^i, x_2^i), \ldots, f(e_{j_i}^i, x_{j_i}^i), f(e_{j_i}^i, t), x_t)$ is a path in $L_2^H(s,t)$. It follows from the construction of each T_i and Observation 2.1(a) that T_i and T_j are internally disjoint $x_s - x_t$-paths in $L_2^H(s,t)$, for every $\{i,j\} \subseteq \{1, \ldots, k_1\}$. Thus, $k_1 \leq k_2$ and hence the claim holds.

Claim 2. $k_3 \leq k_4$.
Let $A = \{e_i = (x_i, y_i) \in A(D) : i \in \{1, \ldots, k_4\}\}$ be an (s,t)-H-trails-separator by arcs in D with k_4 arcs. Consider $B = \{f(e_i, y_i) \in L_2^H(s,t)\}$. Suppose that there exists an $x_s - x_t$-path in $L_2^H(s,t) \setminus B$, namely P. Hence, by applying Algorithm 1 to the path P, we conclude that there is T, an H-trail from s to t in D, such that $e_i \notin A(T)$, which is a contradiction. Therefore, B is an (x_s, x_t)-separator in $L_2^H(s,t)$ with k_4 vertices, and hence the claim holds.

Claim 3. $k_4 \leq k_3$.
Let $A = \{f(e_i, x_i) : i \in \{1, \ldots, k_3\}\}$ be an (x_s, x_t)-separator in $L_2^H(s,t)$. It follows from Observation 2.1(a) that for every $e = (x,y) \in A(D)$, at most one of the vertices $f(e,x)$ and $f(e,y)$ is in a minimum (x_s, x_t)-separator in $L_2^H(s,t)$. Hence, $B = \{e_i \in A(D) : f(e_i, x_i) \in A\}$ has k_3 arcs. Notice that B is an (s,t)-H-trails-separator by arcs in D. Otherwise, there is an (s,t) H-trail in D, say P, and we can find T, an $x_s - x_t$-path in $L_2^H(s,t)$ from P (as in Claim 1), such that $V(T) \cap A = \emptyset$, which is a contradiction. Therefore, $k_4 \leq k_3$ and hence the claim holds.

Notice that $k_2 = k_3$ follows from Menger's Theorem. Therefore, $k_1 = k_2 = k_3 = k_4$. □

Corollary 2.3. *The problem of maximizing the number of arc disjoint $s - t$ H-trails in D can be solved in polynomial time.*

Recall that if H is a complete digraph without loops, then every H-trail is a properly colored trail.

Corollary 2.4. *Given an arbitrary c-arc-colored digraph D, finding a properly colored trail starting with the arc e and ending with the arc f (if any) can be done in polynomial time.*

Corollary 2.5. *Given an arbitrary c-arc-colored digraph D and two vertices s, t, of D, finding a properly colored $s - t$ trail in D (if any) can be done in polynomial time.*

Corollary 2.6. *The problem of maximizing the number of arc disjoint properly colored $s - t$ trails in D can be solved in polynomial time.*

Recall that if H is a complete digraph without loops, then every H-path is a properly colored path. Hence, the next result follows immediately from Theorem 1.3.

Corollary 2.7. *Deciding whether an H-colored digraph contains an $s - t$ H-path is NP-complete.*

Benítez-Bobadilla et al. [7] gave a characterization of digraphs H such that for every digraph D and every H-coloring of D, every H-walk between two vertices in D contains an H-path with the same endpoints.

Theorem 2.5 (Benítez-Bobadilla et al. [7])**.** *Let H be a reflexive digraph. Then, H is transitive if and only if for every H-colored digraph D, and every pair of different vertices s and t of D, every $s - t$ H-walk in D contains an $s - t$ H-path in D.*

The following result is a direct consequence of Corollary 2.3 and Theorem 2.5.

Corollary 2.8. *Let H be a transitive and reflexive digraph. Given an arbitrary H-colored digraph D, finding an $s - t$ H-path in D (if any) can be done in polynomial time.*

Corollary 2.9. *Given an arc-colored digraph D, finding a monochromatic $s-t$ path in D (if any) can be done in polynomial time.*

In what follows, we note that H-walks in H-colored digraphs and T-compatibles walks in a digraph with a transition system T are equivalent.

Let D be an H-colored digraph. We define the transition system of $v \in V(D)$ as the set

$$T(v) = \{(e, f) : e = (x, v), f = (v, y), \text{ for some } x, y \in V(D) \text{ and } (c(e), c(f)) \in A(H)\}.$$

Hence, if $T = \{T(v) : v \in V(D)\}$, then every T-compatible walk is an H-walk.

Let D be a digraph with arc set $A(D) = \{f_1, \ldots, f_m\}$. Let $T = \{T(v) : v \in V(D)\}$ be a transition system of D. Consider the digraph H with the vertex set $V(H) = \{c_1, \ldots, c_m\}$ such that $(c_i, c_j) \in A(H)$ if and only if $\{f_i, f_j\} \in T$. If we color the arc f_i with the vertex c_i, then every H-walk is a T-compatible walk.

The following problem arises in a natural way. Given a digraph D with transition system T, find a digraph H with the minimum number of vertices such that there exists an H-coloring of D, where a walk is an H-walk if and only if it is a T-compatible walk. Notice that such a digraph H exists and $|V(H)| \leq |A(D)|$.

From the above discussion, we can conclude that all the results presented in this section can be stated in terms of T-compatible walks. It is important to note that transition systems provide local information about allowed transitions at every vertex; on the other hand, H-coloring provides global information about allowed transitions.

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