Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **13** (2024) 58–65
DOI: 10.47443/dml.2024.014

Research Article

All-path convexity: two characterizations, general position number, and one algorithm

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(Received: 17 January 2024. Received in revised form: 29 March 2024. Accepted: 2 April 2024. Published online: 5 April 2024.)

Abstract

We present two characterizations for the all-path convex sets in graphs. Using the first criterion, we obtain a new characterization of connected block graphs and compute the general position number in a graph with respect to the all-path convexity. The second criterion allows us to provide a new algorithm for testing a set on all-path convexity.

Keywords: all-path convexity; graph convexity; interval space; block graph; gated set; general position number.

2020 Mathematics Subject Classification: 05C69, 52A01, 05C75, 05C85, 52A37.

1. Introduction

Abstract convexity theory is a well-established (though not a mainstream) branch of mathematics, with applications in various settings including both "continuous" and discrete structures [14]. This versatility arises partly because the definition of a convexity on a set resembles that of a topology. Specifically, a convexity on a set X is any collection $\mathcal{C}$ of its subsets satisfying three simple axioms: $\emptyset, X \in \mathcal{C}$; $\mathcal{C}$ is closed under arbitrary intersections; $\mathcal{C}$ is closed under nested unions. The elements of $\mathcal{C}$ are called convex sets.

One way to establish a convexity on a set X is to start with an interval operator, which is a map I from $X \times X$ to the powerset of X (such maps are also called binary hyperoperations) satisfying conditions: $x, y \in I(x, y)$ and $I(x, y) = I(y, x)$ for all $x, y \in X$. We interpret $I(x, y)$ as the set of all elements that lie "between" two given $x, y \in X$. Subsequently, I naturally induces a convexity on X by declaring a set $A \subset X$ convex provided $I(x, y) \subset A$ for all $x, y \in A$. The most well-known examples of convexities arising this way are convexities induced by metric intervals $[x, y]_d = \{z \in X : d(x, z) + d(z, y) = d(x, y)\}$ in metric spaces and linear intervals $[x, y]_l = \{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}$ in normed spaces. In fact, there is a Galois connection between all convexities on a fixed set X and all the interval operators on X (see Proposition 2.2.1).

Graph theory, due to the numerous classes of paths between pairs of vertices, naturally defines several interval operators (which induce the corresponding convexities). Shortest paths, induced paths, locally shortest paths, chordless paths, and other families of paths produce interval operators as follows. If $\mathcal{P}$ is a collection of paths in a graph G with the property that every pair of vertices in G is joined by at least one element from $\mathcal{P}$, then put $I_{\mathcal{P}}(x, y) = \{z \in V(G) : z \text{ lies on some path from } \mathcal{P} \text{ joining } x, y\}$.

In this paper, we focus on the all-path convexity induced by the interval operator $I_{\mathcal{P}}$, where $\mathcal{P}$ is the collection of all (simple) paths in a given graph. Initially, this particular convexity was considered in [9], and an algorithmic approach for the classical problems related to this convexity was established in [8]. We also refer to the work [3] where the corresponding interval operator was characterized in an abstract manner.

This paper is structured as follows. In Section 2, we outline all the basic definitions and preliminary results that will be used later in the work. In particular, Section 2.1 covers the basics of graph theory and Section 2.2 presents all the necessary background on convex spaces, interval operators, and all-path convexity in graphs. In Section 3 we present our main results. At first, we give a new characterization of all-path convex sets in Section 3.1. Namely, Theorem 3.1.1 provides more theoretical criterion than that in [8], which can be easily used to obtain all the known important properties of all-path convex sets. Moreover, Theorem 3.1.1 allows us to obtain a new characterization of block graphs (Theorem 3.1.2) and to compute the general position number for the all-path convexity (Theorem 3.2.1) in Section 3.2. The criterion for all-path

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convex sets proved in Theorem 3.3.1 will be used in developing a new algorithm for testing a set for all-path convexity in Section 3.3.

2. Definitions and preliminary results

2.1. Basics of graph theory

In this paper, all graphs are assumed to be finite, simple, and undirected. For convenience, we simply write uv for an edge $\{u, v\}$. Two vertices $u, v \in V(G)$ in a graph $G = (V(G), E(G))$ are *adjacent* provided $uv \in E(G)$. The *neighborhood* of a vertex $u \in V(G)$ is the set $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. The *closed neighborhood* is the set $N_G[u] = N_G(u) \cup \{u\}$. The *degree* $d_G(u)$ of a vertex u is the cardinality of its neighborhood $N_G(u)$.

A *walk* in a graph is a sequence of vertices $u_1, \ldots, u_m$ such that the vertices u_i and u_{i+1} are adjacent. A *path* is a walk with pairwise distinct vertices (note that some authors prefer to use the words *simple path*). A *cycle* in a graph is a walk $u_1, \ldots, u_m$ with $m \geq 3$ having pairwise distinct vertices except for $u_1 = u_m$. A graph is *connected* provided every pair of its vertices is joined by a path. Otherwise, the graph is *disconnected*. A graph H is a subgraph of a graph G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. A *connected component* in a graph is a maximal connected subgraph.

For a set of vertices $A \subset V(G)$, by $E_G(A)$ we denote the set of edges with both endpoints from A. A subgraph H of G is called *induced* provided $E(H) = E_G(V(H))$. A set of vertices $A \subset V(G)$ is *connected* if the corresponding induced subgraph $G[A]$ is connected. For $A \subset V(G)$ we put $G - A = G[V(G) \backslash A]$ for the graph obtained from G by deleting the vertices of A. We also write $G - u$ instead of $G - \{u\}$ for a vertex $u \in V(G)$.

A vertex of degree one is called a *leaf*. A vertex whose deletion increases the number of connected components in a graph is called a *cut vertex*. Hence, for a connected graph G, a vertex $u \in V(G)$ is a cut vertex if and only if $G - u$ is disconnected. It is clear that a leaf is a non-cut vertex. A graph is 2*-connected* if it has no cut vertices. A *block* in a graph is a maximal 2-connected subgraph. The next basic result is convenient when working with vertices in blocks.

Lemma 2.1.1. [5, Theorem 3.3(2)] *In a graph, two vertices lie in the same block with at least three vertices if and only if they lie on a common (simple) cycle.*

The vertex set of a connected graph G is endowed with a natural "shortest path" metric d_G (where $d_G(u, v)$ equals the length of a shortest $u - v$ path in G). For a vertex $x \in V(G)$ and a set $A \subset V(G)$ we put $d_G(x, A) = \min\{d_G(x, a) : a \in A\}$ and $\mathrm{pr}_A(x) = \{a \in A : d_G(x, a) = d_G(x, A)\}$ for the *distance from* x *to* A and the corresponding *projection of* x *on* A, respectively. A set of vertices $A \subset V(G)$ is called *Chebyshev* if for every $x \in V(G)$ we have $|\mathrm{pr}_A(x)| = 1$ (in other words, if any vertex x has a unique projection on A). A subgraph $H \subset G$ of a connected graph G is called *isometric* if H is connected with $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$ (note that, the inequality $d_H(u, v) \geq d_G(u, v)$ always holds for all $u, v \in V(H)$).

Lemma 2.1.2. [1] *Let G be a non-complete 2-connected graph. Then G contains a cycle of length at least four, and for each cycle C of minimal length $m \geq 4$ either $m = 4$ and $G[V(C)] \simeq K_4 - e$, or else C is an isometric subgraph in G.*

Let G be a connected graph. The *metric interval* between a pair of vertices $u, v \in V(G)$ in a connected graph G is the set $[u, v]_G = \{w \in V(G) : d_G(u, w) + d_G(w, v) = d_G(u, v)\}$.

Let $A \subset V(G)$ and $x \in V(G)$. A vertex $a \in A$ is called an *x-gate in A* if for all $b \in A$ it holds $a \in [x, b]_G$. It can be easily proved that for any fixed x, there can be at most one x-gate in A. A set $A \subset V(G)$ is called *gated* provided for any vertex $x \in V(G)$ there exists a (unique) x-gate in A. Note that each gated set is Chebyshev.

Let $\mathcal{F}$ be a collection of sets. The corresponding *intersection graph* has the vertex set $\mathcal{F}$ with two vertices $A, B \in \mathcal{F}$ being adjacent provided $A \cap B \neq \emptyset$. The *block graph* of a graph G is the intersection graph on the collection of vertex sets of all blocks in G. The next classical characterization of block graphs is frequently used as their definition.

Theorem 2.1.1. [4] *A graph is a block graph if and only if each of its blocks is complete.*

A connected graph without cycles is called a *tree*. Since in a tree blocks are its edges (see Lemma 2.1.1), Theorem 2.1.1 immediately implies that every tree is a block graph.

2.2. Convex structures

A *convexity* on a set X is a collection of its subsets $\mathcal{C}$ which satisfies the next three simple conditions (see [14, p. 3]):

1. $\emptyset, X \in \mathcal{C}$;
2. $\mathcal{C}$ is stable under intersections: for all subcollections $\mathcal{C}' \subset \mathcal{C}$, it holds $\bigcap_{A \in \mathcal{C}'} A \in \mathcal{C}$;
3. $\mathcal{C}$ is stable under nested unions: for any subcollection $\mathcal{C}' \subset \mathcal{C}$ which is totally ordered by inclusion, we have $\bigcup_{A \in \mathcal{C}'} A \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is called a *convex structure*, and the elements of $\mathcal{C}$ are called *convex sets*. The *convex hull* of a set $A \subset X$ is the smallest convex set containing it, i.e., $\mathrm{co}_{\mathcal{C}}(A) := \bigcap_{B \in \mathcal{C}, A \subset B} B$. Note that condition 2 from the definition of a convexity ensures that $\mathrm{co}_{\mathcal{C}}(A)$ is indeed a convex set.

An *interval operator* on X is a map of the form $I : X \times X \to 2^X$ which satisfies two conditions (see [14, p. 71]):

1. $a, b \in I(a, b)$ for all $a, b \in X$;
2. $I(a, b) = I(b, a)$ for all $a, b \in X$.

The pair (X, I) is called an *interval space*. Convexities and interval operators are connected by the following constructions. Having a convexity $\mathcal{C}$ on a set X, put $I_{\mathcal{C}}(a, b) := \mathrm{co}_{\mathcal{C}}(\{a, b\})$ for all pairs $a, b \in X$. It is easy to see that $I_{\mathcal{C}}$ is an interval operator on X. Conversely, each interval operator I on X induces a convexity by declaring a set $A \subset X$ to be convex provided $I(a, b) \subset A$ for all $a, b \in A$ (the fact that the collection of these convex sets indeed form a convexity on X can be easily verified). Denote the obtained convexity as $\mathcal{C}_I$.

Recall that given two partially ordered sets $(X, \leq_1)$ and $(Y, \leq_2)$, an *antitone Galois connection* between them is a pair of antitone maps $p : X \to Y$ and $q : Y \to X$ such that for all $x \in X$, $y \in Y$ we have $y \leq_2 p(x)$ if and only if $x \leq_1 q(y)$.

Now, the collection of all convexities on a set X is naturally partially ordered by inclusion. The collection of all interval functions on X also possesses a partial ordering: we declare $I_1 \leq I_2$ if $I_1(a, b) \subset I_2(a, b)$ for all $a, b \in X$. It turns out that the above-mentioned "connection" between these two collections is in fact an antitone Galois connection.

Proposition 2.2.1. *Let X be a set. Then the maps $\mathcal{C} \mapsto I_{\mathcal{C}}$ and $I \mapsto \mathcal{C}_I$ establish an antitone Galois connection between the posets of all convexities and all interval operators on X.*

Proof. First, let us show that these maps are antitone. Indeed, if $\mathcal{C} \subset \mathcal{C}'$ for two convexities $\mathcal{C}$ and $\mathcal{C}'$ on X, then for all $a, b \in X$ it holds $I_{\mathcal{C}'}(a, b) = \mathrm{co}_{\mathcal{C}'}(\{a, b\}) \subset \mathrm{co}_{\mathcal{C}}(\{a, b\}) = I_{\mathcal{C}}(a, b)$. Hence, the map $\mathcal{C} \mapsto I_{\mathcal{C}}$ is antitone. Similarly, let $I \leq I'$ for two interval operators on X. Fix a set $A \in \mathcal{C}_{I'}$. Then for all $a, b \in A$ we have $I(a, b) \subset I'(a, b) \subset A$. Thus, $A \in \mathcal{C}_I$. Hence, $\mathcal{C}' \subset \mathcal{C}$ implying that the map $I \mapsto \mathcal{C}_I$ is also antitone.

Now assume that $I' \leq I_{\mathcal{C}}$ for a convexity $\mathcal{C}$ and an interval operator I' on X. Fix a set $A \in \mathcal{C}$. Then for all $a, b \in A$ we have $I'(a, b) \subset I_{\mathcal{C}}(a, b) = \mathrm{co}_{\mathcal{C}}(\{a, b\}) \subset A$. This means that $\mathcal{C} \subset \mathcal{C}_{I'}$.

Conversely, suppose $\mathcal{C} \subset \mathcal{C}_{I'}$ for a convexity $\mathcal{C}$ and an interval operator I' on X. Fix a pair $a, b \in X$ and a convex set $A \in \mathcal{C}$ with $a, b \in A$. Then $A \in \mathcal{C}_{I'}$ meaning that $I'(a, b) \subset A$. Recalling the definition of a convex hull, we can conclude that $I'(a, b) \subset \mathrm{co}_{\mathcal{C}}(\{a, b\})$. Hence, $I' \leq I_{\mathcal{C}}$. Therefore, $I' \leq I_{\mathcal{C}}$ if and only if $\mathcal{C} \subset \mathcal{C}_{I'}$. □

Let $p : X \to Y$ and $q : Y \to X$ be two antitone maps between posets $(X, \leq_1)$ and $(Y, \leq_2)$. The fact that the pair p, q establish an antitone Galois connection is equivalent to the following condition: for all $x \in X$ and $y \in Y$ it must hold

$$x \leq_1 q(p(x)) \text{ and } y \leq_2 p(q(y)).$$

Hence, Proposition 2.2.1 immediately asserts that for any convexity $\mathcal{C}'$ on a set X we have $\mathcal{C}' \subset \mathcal{C}_{I_{\mathcal{C}'}}$, and for any interval operator I' on X it holds $I' \leq I_{\mathcal{C}_{I'}}$. Finally, we note that for an antitone Galois connection p, q, the further iterations of p and q are trivial, i.e. $p(q(p(x))) = p(x)$ and $q(p(q(y))) = q(y)$ for all $x \in X$, $y \in Y$.

Two prominent examples of interval operators I that define the corresponding convexities $\mathcal{C}_I$ come from metric spaces and real vector spaces. Namely, for a metric space (X, d), the metric interval $[a, b]_d = \{x \in X : d(a, x) + d(x, b) = d(a, b)\}$ defines a standard *geodesic convexity*. For a real vector space X, we consider the *linear interval* $[a, b]_l = \{ta + (1 - t)b : t \in [0, 1]\}$ between the vectors $a, b \in X$. It is easily seen that both $[\cdot, \cdot]_d$ and $[\cdot, \cdot]_l$ are indeed interval operators.

A lot of known graph convexities come from interval operators, which in turn, are constructed using collections of paths in a graph. Namely, let $\mathcal{P}$ be some collection of paths in a connected graph G such that for every pair of vertices in G there exists an element of $\mathcal{P}$ that joins them. For all $a, b \in V(G)$ put

$$I_{\mathcal{P}}(a, b) = \{x \in V(G) : x \text{ lies on some path } P \in \mathcal{P} \text{ between } a, b\}.$$

Among the well-studied families of paths $\mathcal{P}$ are shortest paths (*geodesic convexity*), induced paths (*monophonic convexity*) and simple paths (*all-path convexity*). Note that if $\mathcal{P}$ is the collection of all shortest paths in a connected graph G, then $I_{\mathcal{P}}(\cdot, \cdot) = [\cdot, \cdot]_G = [\cdot, \cdot]_{d_G}$. Also, one can prove that gated sets are geodesically convex. Further, as each shortest path is necessarily induced, every monophonically convex set is geodesically convex as well.

In this paper, we are studying the all-path convexity in connected graphs (see paper [3] for an axiomatic characterization of the all-path interval operator on graphs). Thus, a set $A \subset V(G)$ is called *all-path convex* (shortly, *AP-convex*) [9] provided for all $a, b \in A$ the vertex sets of all simple paths between a, b lie in A. One can prove that AP-convex sets are gated (and thus, geodesically convex).

Using AP-convex sets, in [9, Theorem 5], the characterization of trees was established. Also, it was shown [9, Theorem 10] that the family of AP-convex sets satisfies the Helly property (each subcollection with pairwise intersecting members has a non-empty intersection). The article [8] presents a more algorithmic view of the problems related to AP-convexity. At first, the authors prove a criterion for AP-convex sets. To state it, we need one more definition. For $X, Y \subset V(G)$, put

$$N_G(X, Y) = \{v \in Y : v \text{ is a neighbor of some } x \in X\}.$$

Theorem 2.2.1. [8] *Let $A \subset V(G)$ be a set of vertices in a connected graph G. Then A is AP-convex if and only if either $A = V(G)$ or for every connected component G_i of $G - A$ it holds that $|N_G(V(G_i), A)| = 1$.*

This characterization allows one to tackle all of the important algorithmic problems about AP-convexity including problems of determining whether a set is AP-convex, finding the convex hull of a given set, and problems concerning the calculation of various invariants from abstract convex theory (such as convexity number, hull number, interval number, and geodetic iteration number).

3. Main results

3.1. New characterization of AP-convex sets and its application to block graphs

Here we present our first characterization of AP-convex sets.

Theorem 3.1.1. *Let G be a connected graph, $A \subset V(G)$ and $|A| \geq 2$. The set A is AP-convex if and only if the induced subgraph $G[A]$ is a connected union of blocks of G.*

Proof. Sufficiency. We prove it by contradiction. Thus, let $G[A] = \bigcup_{k=1}^{m} B_k$ be a connected subgraph, where $B_1, \ldots, B_m$ are some blocks in G. Assume that there exist two vertices $u, v \in A$ and a simple path $P_1 = P(u, v)$ joining them, such that $V(P_1) \not\subset A$. Without loss of generality, we can also assume that $V(P_1) \cap A = \{u, v\}$. Since $G[A]$ is connected, there exists a simple path $P_2 = P(u, v)$ such that $V(P_2) \subset A$. Denote by $\mathcal{B} = \{B_k : V(P_2) \cap V(B_k) \neq \emptyset\}$, a non-empty class of blocks in G. Then $(\cup_{B \in \mathcal{B}} B) \cup P_1$ is also a 2-connected subgraph in G, which is a contradiction.

Necessity. Let A be an AP-convex set. Since G is connected, it is easy to see that $G[A]$ is also connected.

Now consider some vertex $u \in A$. Since $|A| \geq 2$, there exists a vertex $v \in A \backslash \{u\}$ with the edge $uv \in E(G)$. But every edge from a graph lies in some block, so there exists a block B of G such that $u, v \in V(B)$. If $|B| = 2$, then clearly, $V(B) \subset A$. Further, assume $|B| \geq 3$ and fix a vertex $w \in V(B) \backslash \{u, v\}$. Then the edge uv and the vertex w both lie on a common cycle in B (see [5, Theorem 3.3(3)]). Therefore, there exists a simple path $P = P(u, v) \subset B$ such that $w \in V(P)$. But A is an AP-convex set. This implies $V(P) \subset A$, which asserts $w \in A$. However, since w was chosen arbitrarily from $V(B) \backslash \{u, v\}$ and $u, v \in B$, we have $V(B) \subset A$. Hence, $G[A]$ is the union of blocks which contain edges of $E_G(A)$. □

Corollary 3.1.1. *The vertex set of any connected union of blocks in a connected graph is gated.*

Using Theorem 3.1.1, we can also give simpler proofs to all the results on AP-convex sets obtained in [9]. For example, we can establish the following characterization of trees in terms of AP-convex sets rather easily.

Proposition 3.1.1. [9] *A connected graph is a tree if and only if each of its connected sets of vertices is AP-convex.*

Proof. Necessity. If T is a tree, then its blocks are the edges. Hence, any connected set A of vertices in T is the union of edges from $E_T(A)$, implying that by Theorem 3.1.1, it is an AP-convex set.

Sufficiency. Assume that in a connected graph G all its connected sets of vertices are AP-convex. Since the edges in G are connected sets, they are AP-convex. Theorem 3.1.1 now asserts that all the edges in G are blocks. Hence, G does not have cycles (see Lemma 2.1.1) meaning that G is a tree. □

In the next theorem, we use AP-convex sets to characterize a generalization of trees, namely, connected block graphs.

Theorem 3.1.2. *For a graph G, the following statements are equivalent:*

1. *G is a connected block graph;*
2. *$N_G[u]$ is AP-convex for all $u \in V(G)$;*
3. *$N_G[u]$ is gated for all $u \in V(G)$.*

Proof. Let G be a block graph. Since each block in G is complete (see Theorem 2.1.1), for any vertex $u \in V(G)$ the closed neighborhood $N_G[u]$ induces a connected union of blocks in G. Therefore, by Theorem 3.1.1, $N_G[u]$ is AP-convex. Thus, the first statement implies the second. Further, since each AP-convex set is necessarily gated, the second statement trivially implies the third.

Finally, we show that the third statement implies the first. Assume that $N_G[u]$ is gated for all $u \in V(G)$. Consider an arbitrary block B in G. If B is a non-complete graph, then Lemma 2.1.2 asserts that B contains a cycle of length at least four, and for each cycle C of minimal length $m \geq 4$ we have $m = 4$ and $G[V(C)] \simeq K_4 - e$, or C is an isometric subgraph in G. Let $V(C) = \{x_1, \ldots, x_m\}$ and $E(C) = \{x_i x_{i+1} : 1 \leq i \leq m-1\} \cup \{x_1 x_m\}$ for $m \geq 4$. If $m = 4$ and $G[V(C)] \simeq K_4 - e$, then, without loss of generality, we can assume that $E(G[V(C)]) = \{x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_2x_4\}$. In this case, $x_2, x_4 \in \mathrm{pr}_{N_G[x_1]}(x_3)$ implying $N_G[x_1]$ is not Chebyshev, and therefore, not a gated set.

Now suppose C is an isometric subgraph in B. If m is an even number, then $x_2, x_m \in \mathrm{pr}_{N_G[x_1]}(x_{\frac{m}{2}+1})$. Hence, in this case, $N_G[x_1]$ is also not gated. If m is odd, then $d_G(x_{\frac{m+1}{2}}, x_1) = d_C(x_{\frac{m+1}{2}}, x_1) = \frac{m-1}{2}$ implies that $d_G(x_{\frac{m+1}{2}}, N_G[x_1]) = \frac{m-3}{2}$. We have $x_2, x_m \in N_G[x_1]$ and $d_G(x_{\frac{m+1}{2}}, x_2) = d_C(x_{\frac{m+1}{2}}, x_2) = \frac{m-3}{2}$, $d_G(x_{\frac{m+1}{2}}, x_m) = d_C(x_{\frac{m+1}{2}}, x_m) = \frac{m-1}{2}$. Therefore, x_2 must be the $x_{\frac{m+1}{2}}$-gate in $N_G[x_1]$. However, since $x_2x_m \notin E(G)$, we obtain $x_2 \notin [x_{\frac{m+1}{2}}, x_m]_G$. Hence, $N_G[x_1]$ is not gated in this case as well. The obtained contradiction asserts that each block in G is complete, implying that G is a block graph by Theorem 2.1.1. □

3.2. General position sets for the AP-convexity

Let (X, I) be a finite interval space. We say that a set $A \subset X$ is in *general position* (or, it is a *gp-set*) provided for all $a, b \in A$ it holds that $I(a, b) \cap A = \{a, b\}$. In other words, A is a gp-set if and only if none of its elements lie on an interval between two other elements. The *gp-number* of a finite interval space is the cardinality of the largest gp-set in it. We also note that the notions of a gp-set and gp-number for a finite convexity space $(X, \mathcal{C})$ are defined in terms of $I_{\mathcal{C}}$.

A study on gp-sets and gp-number for the shortest path interval functions in graphs can be found in many various recent articles (which indicates that this is an active field of research), see for example [2, 6, 7, 12, 15] and references therein. For the corresponding study on gp-sets for monophonic interval functions on graphs see [13], where, in particular, it was shown that in this setting, the gp-number of triangle-free graphs is bounded above by the independence number. The exact formulas for monophonic gp-number for complements of bipartite graphs and split graphs, several realization results, and a discussion on the computation complexity of the monophonic position problem can also be found in [13].

In this section, we will calculate the gp-number for the AP-convexity on finite graphs. To do this, we need two preliminary results. The first one is a generalization of the fact that each subtree in a tree T has at most the same number of leaves as T.

Lemma 3.2.1. *Any induced connected subgraph in a graph G has at most the same number of non-cut vertices as G.*

Proof. Suppose $A \subset V(G)$ is a connected set of vertices and put $H = G[A]$. If every non-cut vertex in H is a non-cut vertex in G, then we are done. Hence, let $u \in A$ be a non-cut vertex in H which is a cut vertex in G. Then there exists a block B in G with $V(B) \cap A = \{u\}$. Let G' be a connected component in a graph $G - A$ which contains the subgraph $B - u$. Fix a non-cut vertex x_u in G'. Then x_u is a non-cut vertex in G as well. Moreover, $\mathrm{pr}_A(x_u) = \{u\}$. By construction, the vertices x_u are distinct for distinct u. Hence, H has at most the same number of non-cut vertices as G. □

The second preliminary result establishes a basic fact about gp-sets for the AP-convexity in graphs. In what follows, the words "gp-set" and "gp-number" will refer only to this particular convexity.

Lemma 3.2.2. *Let $A \subset V(G)$ be a gp-set in G with $|A| \geq 3$. Then A has at most one common vertex with any block in G.*

Proof. To the contrary, suppose there exists a block B in G with $|A \cap V(B)| \geq 2$. Choose two different vertices $a, b \in A \cap V(B)$. Since $|A| \geq 3$, there is a third vertex $c \in A \backslash \{a, b\}$. If $c \in V(B)$, then B has at least 3 vertices implying that a, b lie on a common simple cycle C in B (see Lemma 2.1.1). Hence, taking the shortest path from c to b and concatenating it with one of the two paths from b to a on C, we can assure that a lies on a simple path between c and b. This is a contradiction. Similarly, if c does not belong to B, then we can take the shortest path from c to B and concatenate it in a similar way inside B to ensure that a lies on a simple path between c and b, or b lies on a simple path between c and a. In all cases, we arrive at a contradiction with the fact that A is a gp-set. □

It is clear that the gp-number of a graph G equals 1 if and only if $G \simeq K_1$. Further, for any 2-connected graph G with at least 2 vertices, its gp-number equals 2. The next theorem explains other (non-trivial) cases. A block in a graph is called a *leaf block* provided it contains a unique cut vertex.

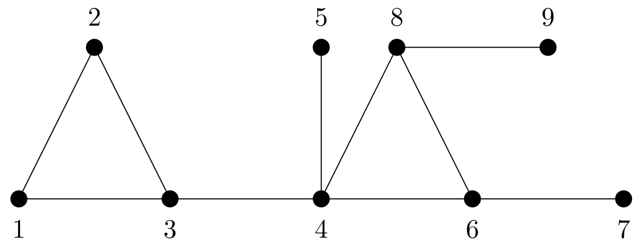

Figure 3.1: The graph G whose gp-number is 4.

Theorem 3.2.1. *Let G be a connected, but not a 2-connected graph with $|V(G)| \geq 2$. Then the gp-number of G equals the number of leaf blocks in G.*

Proof. First, we note that the gp-number of G is at least the number of its leaf blocks. Indeed, we can just pick a non-cut vertex from any such block to form a gp-set in G. Therefore, the main difficulty is to establish the other inequality. For this, let $A \subset V(G)$ be a gp-set in G. Since $|V(G)| \geq 2$, we have $|A| \geq 2$ as well. If $|A| = 2$, then as G is not 2-connected, we can conclude that G has at least two blocks. In this case, the statement of the theorem is clear. Hence, in what follows we assume that $|A| \geq 3$. Denote by $\mathcal{B}_A = \{B : B \text{ is a block in } G \text{ and } V(B) \cap A \neq \emptyset\}$ the collection of blocks from G that intersect A. Now consider the set $\mathcal{B}_A$ as the subset of vertices in the block graph $B(G)$. Put $X = \text{co}(\mathcal{B}_A)$ for the usual convex hull (for geodesic convexity) of the set $\mathcal{B}_A$ in the block graph $B(G)$. Then X is a connected set in $B(G)$, hence by Lemma 3.2.1, the induced subgraph $B(G)[X]$ has at most the same number of non-cut vertices as $B(G)$. However, the non-cut vertices in $B(G)$ correspond to leaf blocks in G. And non-cut vertices in $B(G)[X]$, by Lemma 3.2.2, correspond to pairwise distinct blocks which contain the vertices of A. Therefore, $|A|$ is at most the number of leaf blocks in G. □

Example 3.2.1. *Consider the graph G with $V(G) = \{1, \ldots, 9\}$ and $E(G) = \{12, 13, 23, 34, 45, 46, 48, 67, 89\}$ depicted in Figure 3.1. Then the set $A = \{1, 2, 8\}$ is not a gp-set, the set $B = \{3, 5, 7\}$ is a gp-set, and the set $C = \{1, 5, 7, 9\}$ is a maximum gp-set (i.e., the gp-number of G equals $|C| = 4$, which is the number of leaf blocks in G).*

For convexities on finite sets, several other classical number invariants are known. The *convexity number* is the size of a maximum proper convex set, the *interval number* is the size of the smallest interval set (a set A such that $X = \{x \in X : x \in I(a, b) \text{ for some } a, b \in A\}$), and the *hull number* is the size of the smallest hull set (a set A with $\text{co}(A) = X$). For the AP-convexity on finite graphs these numbers were explicitly calculated in [8] (see Theorems 12 and 14 therein). We also note that the recent paper [10] conducts a study on Radon and tolerant Radon partitions for AP-convexity.

3.3. Another characterization of AP-convex sets and the new algorithm

The criterion in Theorem 2.2.1 gives a simple tool for checking whether a set is AP-convex, realized in a two-step algorithm (proposed in [8]). The first step of the algorithm is finding all connected components of a graph, and the second is applying the mentioned criteria to them. Considering well-known graph algorithms from [11], this two-step algorithm requires two DFS traversals through the graph. The most basic DFS algorithm consists of the following steps:

- Fix any vertex of a graph.
- If possible, move to the neighboring unchecked vertex and check it.
- If not possible, move to the previously checked vertex.

It could be represented by the following pseudocode:

Algorithm 1 DFS

```
function DFS(v)
    visited[v] ← True                                ▷ Happens |V| times
    for u ∈ N(v) do
        if u is not visited then                     ▷ Happens 2|E| times
            DFS(u)
        end if
    end for
end function
```

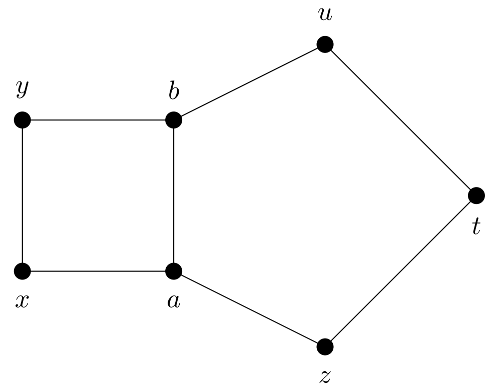

Figure 3.2: The vertex a is an x-gate, but not a strong x-gate in $A = \{a, b\}$.

To elaborate more, the algorithm checks a vertex on being visited once for each incident edge; hence, the number of such visits equals $|E|$ if E is a set of undirected edges and $2|E|$ otherwise. It can be easily seen that its time complexity is $T(G) = |V| + 2|E|$ as long as the condition statement inside the *if* statement has constant time complexity. Therefore, it is linear with respect to $|V| + |E|$. Thus, the time complexity of the algorithm from [8] for checking a set for AP-convexity is at least twice as slow, or $T(G) \geq 2|V| + 4|E|$.

The main objective of this section is to propose a new algorithm for testing a set for AP-convexity. This algorithm is based on yet another characterization of AP-convex sets, which was briefly mentioned in [3] as Fact 5 without a proof. To present this result, we need the following new definition. Let G be a connected graph, $A \subset V(G)$ and $x \in V(G)$. A vertex $a \in A$ is called a *strong x-gate in A* provided a lies on *all* shortest paths between x and the vertices in A. It is clear that a strong x-gate is an x-gate (as the definition of an x-gate requires that for every $b \in A$, the vertex a lies on *some* shortest $x - b$ path). For example, consider the graph depicted in Figure 3.2. Then for the set $A = \{a, b\}$, the vertex a is an x-gate, but not a strong x-gate; a is a strong z-gate; there is no (strong) t-gate in A.

Now we are ready to state the mentioned result from [3] (we give the detailed proof for the sake of completeness).

Theorem 3.3.1. [3, Fact 5] *A non-empty set $A \subset V(G)$ is AP-convex if and only if for any $x \in V(G)$ there exists a strong x-gate in A.*

Proof. Necessity. To the contrary, suppose $A \subset V(G)$ is AP-convex, but there is a vertex $x \in V(G)$ with no strong x-gate in A. It is clear that $x \notin A$. Without loss of generality, we can assume that x is such a vertex with minimum distance $d_G(x, A)$. Fix a vertex $a \in \mathrm{pr}_A(x)$. Since a is not a strong x-gate in A, there exists $b \in A$ and a shortest $x - b$ path P which does not contain a. We can also assume that $V(P) \cap A = \{b\}$. Now, fix a shortest $x - a$ path Q. It is clear that b does not lie on Q as $d_G(x, a) = d_G(x, A)$. Since $d_G(x, A)$ is minimal among all such vertices x, we have $V(P) \cap V(Q) = \{x\}$. Hence, the concatenation of Q and P provides a path between a and b. As A is AP-convex, this implies $x \in A$, a contradiction.

Sufficiency. The proof in this direction relies on the following claim.
Claim: for every $u \notin A$ and each of its neighbors $v \in N_G(u)$, the strong u-gate and the strong v-gate in A coincide.
Let $a \in A$ be the strong u-gate in A and $b \in A$ be the strong v-gate in A. As u and v are adjacent, we have $d_G(u, A) - 1 \leq d_G(v, A) \leq d_G(u, A) + 1$. If $d_G(v, A) = d_G(u, A) + 1$, then $d_G(v, a) = 1 + d_G(u, a) = 1 + d_G(u, A) = d_G(v, A)$ which implies that a is the only candidate for a strong v-gate in A. Similarly, one can consider the case $d_G(v, A) = d_G(u, A) - 1$. Thus, assume $d_G(v, A) = d_G(u, A)$. If $a \neq b$, then $d_G(u, b) = 1 + d_G(v, b)$. However, then the concatenation of an edge uv with any shortest $v - b$ path provides a shortest $u - b$ path not containing a. This is a contradiction.

Now we will use the above Claim as follows. If A is not AP-convex, then there are two vertices $a, b \in A$ and a path between them $P = \{a, x_1, \ldots, x_m, b\}$ that does not lie completely in A. We can also assume that $V(P) \cap A = \{a, b\}$. Then it is clear that a is a strong x_1-gate in A and b is a strong x_m-gate in A. However, using the Claim, we can prove by induction that all the vertices of P have a common strong gate in A. The obtained contradiction proves the theorem. □

Theorem 3.3.1 allows us to introduce another algorithm for testing a set for AP-convexity. Namely, we present a linear time algorithm that requires only one DFS traversal. The algorithm consists of the next two procedures for every $v \in A$:

- Start a DFS traversal through vertices from $V \setminus A$.
- Stop the algorithm if we meet the vertex $u \in V : u \neq v$.

In case the algorithm goes through all vertices in A and the second procedure is never triggered, A is considered to be AP-convex. We also give a pseudocode of the algorithm in Algorithm 2. There, E_A means the set of directed edges that start from a vertex of A. Therefore, if $ab \in E$ for $a, b \in A$, then $(a, b), (b, a) \in E_A$.

Algorithm 2 Algorithm for checking a set A on being AP-convex

```
function triggerDFS(A)
    visited ← A
    ap-convex ← True
    for v ∈ A do
        strongGate ← v                                   ▷ Happens |A| times
        for u ∈ N(v) do                                  ▷ Happens |E_A| times
            if u is not visited then
                DFS(u)
            end if
        end for
    end for
return ap-convex
end function
function DFS(v)
    visited[v] ← True                                    ▷ Happens |V\A| times
    for u ∈ N(v) do
        if u is not visited then                         ▷ Happens |E_{V\A}| times
            DFS(u)
        else
            if u is in A and u ≠ strongGate then         ▷ Happens |E_{V\A}|/2 times
                ap-convex ← False
            end if
        end if
    end for
end function
```

With two constant time assignments at the start of $triggerDFS$ function and with constant time complexity conditional statements inside each *if* statement, the time complexity of the algorithm is:

$$T(G) = |V \setminus A| + |A| + |E_A| + |E_{V\setminus A}| + 2\frac{|E_{V\setminus A}|}{2} + 2 = |V| + |E_V| + \frac{|E_{V\setminus A}|}{2} \leq |V| + 2|E| + 2|E| + 2 = |V| + 4|E| + 2.$$

The algorithm requires memory for an array of edges, an array for visited vertices, and an array for vertices of set A. Therefore, the memory complexity of the algorithm is linear relatively to $|E| + |V|$ or $M(G) = |E| + |V| + |A| \leq |E| + 2|V|$.

4. Acknowledgments

We are deeply thankful to the Armed Forces of Ukraine for keeping the cities of Kyiv and Mena safe during the work on this paper. We also thank K. Antoshyna and A. Serdiuk for reading earlier versions of the paper and providing insightful comments. Our sincere gratitude goes to the two anonymous referees for their invaluable feedback and suggestions, which helped to improve the clarity of the exposition. Finally, we acknowledge the partial support from the Fund (endowment) of projects of the MBFVKMA for the development of science, innovations, and scientific education at NaUKMA.

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