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Research Article

The total number of descents and levels in (cyclic) tensor words

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Abstract

We obtain an explicit formula for the total number of descents and levels in (cyclic) tensor words of arbitrary dimension. We also determine the maximal number of cyclic descents in cyclic tensor words. Furthermore, we establish a lower bound and an upper bound on the maximal number of descents in tensor words.

Keywords: word; generating function.

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1. Introduction

Let k and n be two natural numbers and let $[k] = \{1, 2, \ldots, k\}$. Recall that if $w = w_1 \cdots w_n \in [k]^n$ is a word over the alphabet $[k]$, then $1 \le i \le n-1$ is a descent (respectively, level) of w if $w_i > w_{i+1}$ (respectively, $w_i = w_{i+1}$). If the word is cyclic, then also $i = n$ is a descent (respectively, level) of w if $w_n > w_1$ (respectively, $w_n = w_1$). One possibility for a generalization, that we pursue in this work, is to consider words in higher dimensions, thus regarding (standard) words as one-dimensional words. For example, if m is an additional natural number, then two-dimensional words over $[k]$ are matrices $w = (w_{i,j})_{1\le i\le m, 1\le j\le n} \in [k]^{mn}$. Here, a descent (respectively, level) of w is a pair of double indices $((i_1, j_1), (i_2, j_2))$ such that $w_{i_1,j_1} > w_{i_2,j_2}$ (respectively, $w_{i_1,j_1} = w_{i_2,j_2}$) and either $(i_2, j_2) = (i_1 + 1, j_1)$ or $(i_2, j_2) = (i_1, j_1 + 1)$.

The primary purpose of this work is to establish a formula for the total number of descents (respectively, levels) in multidimensional words of an arbitrary (but fixed) dimension. We refer to such words as tensor words, as formalized in Definition 1.1.

Descents and levels in words were studied, for example, in [1,2]. Especially related to our work is the work of Mansour and Shattuck [4], which studied common occurrences of patterns in matrix (i.e., two-dimensional) words. Nevertheless, they have not considered vertical descents (respectively, levels), nor have they considered cyclic words. In contrast, Knopfmacher et al. [3] considered cyclic words, but in a different context, namely that of staircase words.

Before we begin, let us introduce some notation. Vectors are written in bold font. We denote by $\mathbb{N}$ (respectively, $\mathbb{R}$) the set of natural (respectively, real) numbers. For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$ and, for $d \in \mathbb{N}$ and $j \in [d]$, we denote by $\boldsymbol{e_j}$ the jth vector in the standard basis of $\mathbb{R}^d$. If p is a condition, then 1_p equals 1 if p holds and 0 otherwise. Fix $k, d \in \mathbb{N}$ and $\boldsymbol{m^{(d)}} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, to be used throughout this work. We write $p_d = \prod_{j\in[d]} m_j$ and, for $i \in [d]$, we write

$$p_{d,i} = \prod_{j\in[d], j\neq i} m_j.$$

Definition 1.1. *An $\boldsymbol{m^{(d)}}$-tensor word over $[k]$ is a function $w\colon [m_1] \times \cdots \times [m_d] \to [k]$. The set of all $\boldsymbol{m^{(d)}}$-tensor words over $[k]$ is denoted by $T(\boldsymbol{m^{(d)}}, k)$. A* descent *(respectively,* level*) of an $\boldsymbol{m^{(d)}}$-tensor word w is a pair $(\boldsymbol{i}, \boldsymbol{i'}) \in ([m_1] \times \cdots \times [m_d])^2$, such that $w(\boldsymbol{i}) > w(\boldsymbol{i'})$ (respectively, $w(\boldsymbol{i}) = w(\boldsymbol{i'})$) and $\boldsymbol{i'} = \boldsymbol{i} + \boldsymbol{e_j}$, for some $j \in [d]$. We denote by $\operatorname{des}(w)$ (respectively, $\operatorname{lev}(w)$) the number of descents (respectively, levels) of w. If the word w is regarded as cyclic, then additional* cyclic descents *(respectively,* cyclic levels*) are allowed, namely all $(\boldsymbol{i}, \boldsymbol{i'}) \in ([m_1] \times \cdots \times [m_d])^2$, where $\boldsymbol{i} = (i_1, \ldots, i_d)$ and $\boldsymbol{i'} = (i'_1, \ldots, i'_d)$, such that there exists a unique $j \in [d]$ with $i_j = n, i'_j = 1$, and $w(\boldsymbol{i}) > w(\boldsymbol{i'})$ (respectively, $w(\boldsymbol{i}) = w(\boldsymbol{i'})$). Finally, we denote by $\operatorname{cycdes}(w)$ (respectively, $\operatorname{cyclev}(w)$) the number of descents and cyclic descents (respectively, levels and cyclic levels) of w.*

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Example 1.1. *Let*

$$w = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \end{pmatrix}.$$

Then w is a $(3,4)$-tensor word over $[k]$, for every $k \geq 3$. The set of descents of w is given by

$$\{((1,1),(1,2)),\ ((1,3),(1,4)),\ ((1,1),(2,1)),\ ((2,1),(3,1))\}$$

and the set of levels of w is given by

$$\{((2,1),(2,2)),\ ((2,2),(2,3)),\ ((2,3),(2,4)),\ ((3,3),(3,4)),\ ((2,2),(3,2)),\ ((1,3),(2,3))\}.$$

Thus, $\mathrm{des}(w) = 4$ *and* $\mathrm{lev}(w) = 6$. *The set of cyclic descents of w is given by*

$$\{((3,4),(3,1)),\ ((3,2),(1,2)),\ ((3,3),(1,3)),\ ((3,4),(1,4))\}$$

and the set of cyclic levels of w is given by $\{((2,4),(2,1))\}$. Thus, $\mathrm{cycdes}(w) = 8$ *and* $\mathrm{cyclev}(w) = 7$.

2. Main results

Our main results are as follows: Let a_d (respectively, b_d) denote the total number of descents (respectively, levels) of all $\boldsymbol{m^{(d)}}$-tensor words over $[k]$. Then

$$a_d = \frac{1}{2}\left(dp_d - \sum_{i\in[d]} p_{d,i}\right)(k-1)k^{p_d-1}$$

and $b_d = 2a_d/(k-1)$. Similarly, let f_d (respectively, g_d) denote the total number of descents and cyclic descents (respectively, levels and cyclic levels) of all $\boldsymbol{m^{(d)}}$-tensor words over $[k]$. Then

$$f_d = \frac{1}{2}dp_d(k-1)k^{p_d-1}$$

and $g_d = 2f_d/(k-1)$.

In the last subsection of this section, we establish the maximal number of cyclic descents in cyclic tensor words and obtain a lower and an upper bound on the maximal number of descents in tensor words. Namely, we prove that, for every $w \in T(\boldsymbol{m^{(d)}}, k)$, we have

$$\mathrm{cycdes}(w) = \sum_{i\in[d]} p_{d,i}\left\lfloor \frac{m_i(k-1)}{k}\right\rfloor \text{ and}$$

$$\sum_{i\in[d]} p_{d,i}\left(\left\lfloor \frac{m_i(k-1)}{k}\right\rfloor - 1\right) \leq \mathrm{des}(w) \leq \sum_{i\in[d]} p_{d,i}\left\lfloor \frac{m_i(k-1)}{k}\right\rfloor.$$

2.1. Noncyclic words

Let $m_{d+1} \in \mathbb{N}$ and let ℓ be a nonnegative integer. Set $\boldsymbol{m^{(d+1)}} = (m_1, \ldots, m_{d+1})$. We denote by $D_{\boldsymbol{m^{(d)}}, m_{d+1}, \ell}$ the number of $\boldsymbol{m^{(d+1)}}$-tensor words w such that $\mathrm{des}(w) = \ell$. For $w \in T(\boldsymbol{m^{(d)}}, k)$, we denote by $D_{\boldsymbol{m^{(d)}}, m_{d+1}, \ell}(w)$ the number of $\boldsymbol{m^{(d+1)}}$-tensor words w', such that $\mathrm{des}(w') = \ell$ and such that $w'(\boldsymbol{i}, 1) = w(\boldsymbol{i})$, for every $\boldsymbol{i} \in [m_1] \times \cdots \times [m_d]$. Let

$$F_{d+1}(t,x) = \sum_{m_{d+1}\geq 1}\sum_{\ell\geq 0} D_{\boldsymbol{m^{(d)}}, m_{d+1}, \ell}\, t^{\ell} x^{m_{d+1}}$$

be the generating function for the numbers $D_{\boldsymbol{m^{(d)}}, m_{d+1}, \ell}$, and, for $w \in T(\boldsymbol{m^{(d)}}, k)$, we denote by

$$F_{d+1}(t,x,w) = \sum_{m_{d+1}\geq 1}\sum_{\ell\geq 0} D_{\boldsymbol{m^{(d)}}, m_{d+1}, \ell}(w)\, t^{\ell} x^{m_{d+1}}$$

the generating function for the numbers $D_{\boldsymbol{m^{(d)}}, m_{d+1}, \ell}(w)$. Notice that

$$F_{d+1}(1,x,w) = \frac{x}{1-k^{p_d}x}.$$

Lemma 2.1.1. *For $w \in T(\boldsymbol{m^{(d)}}, k)$, we have*

$$F_{d+1}(t,x,w) = \frac{xt^{\mathrm{des}(w)}}{1-xt^{\mathrm{des}(w)}}\left(1 + \sum_{w'\in T(\boldsymbol{m^{(d)}},k), w'\neq w} t^{\sum_{\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]} 1_{w(\boldsymbol{i})>w'(\boldsymbol{i})}} F_{d+1}(t,x,w')\right).$$

Proof. It suffices to show that the generating function $F_{d+1}(t,x,w)$ satisfies the equation

$$F_{d+1}(t,x,w) = t^{\mathrm{des}(w)}x + x\sum_{w'\in T(\boldsymbol{m}^{(d)},k)} t^{\mathrm{des}(w)+\sum_{\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]}1_{w(\boldsymbol{i})>w'(\boldsymbol{i})}}F_{d+1}(t,x,w'). \tag{1}$$

Indeed, let $v\in T(\boldsymbol{m}^{(d+1)},k)$ such that $v(\boldsymbol{i},1)=w(\boldsymbol{i})$, for every $\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]$. If $m_{d+1}=1$ then there are no descents in the direction of $\boldsymbol{e}_{d+1}$. This case corresponds to the first term on the right-hand side of (1). Assume that $m_{d+1}>1$ and let $w'\in T(\boldsymbol{m}^{(d)},k)$ such that $v(\boldsymbol{i},2)=w'(\boldsymbol{i})$, for every $\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]$. Then u, defined by $u(\boldsymbol{i},j)=v(\boldsymbol{i},j+1)$ for every $\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]$ and $j\in[m_{d+1}-1]$, is an $\boldsymbol{m}^{(d+1)}$-tensor word, such that $u(\boldsymbol{i},1)=w'(\boldsymbol{i})$, for every $\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]$. Now, for $\ell\geq 0$, we have $\mathrm{des}(v)=\ell$ if and only if

$$\ell = \mathrm{des}(w) + \sum_{\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]} 1_{w(\boldsymbol{i})>w'(\boldsymbol{i})} + \mathrm{des}(u).$$

This corresponds to the second term on the right-hand side of (1) and the proof is complete. □

Theorem 2.1.1. *Define* $a_d=\sum_{w\in T(\boldsymbol{m}^{(d)},k)}\mathrm{des}(w)$ *and let* $A_d(x)$ *be the corresponding generating function. Then*

$$a_d = \frac{1}{2}\left(dp_d - \sum_{i\in[d]} p_{d,i}\right)(k-1)k^{p_d-1} \tag{2}$$

and

$$A_{d+1}(x) = \frac{a_d x + \frac{1}{2}(k-1)k^{2p_d-1}p_d x^2}{(1-k^{p_d}x)^2}. \tag{3}$$

Proof. We proceed by induction on d. The case $d=1$ is similar to the general case and the details are omitted. Assume that (2) holds for d. In order to prove that it holds for $d+1$, we first prove (3). To this end, let $w\in T(\boldsymbol{m}^{(d)},k)$. Differentiating (1) with respect to t and substituting $t=1$, we obtain

$$\begin{aligned} A_{d+1}(x,w) &= \mathrm{des}(w)x + x\sum_{w'\in T(\boldsymbol{m}^{(d)},k)}\left(A_{d+1}(x,w') + \left(\mathrm{des}(w) + \sum_{\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]} 1_{w(\boldsymbol{i})>w'(\boldsymbol{i})}\right)F(1,x,w')\right)\\ &= \mathrm{des}(w)x + xA_{d+1}(x) + \left(k^{p_d}\mathrm{des}(w) + \sum_{w'\in T(\boldsymbol{m}^{(d)},k)}\sum_{\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]} 1_{w(\boldsymbol{i})>w'(\boldsymbol{i})}\right)\frac{x^2}{1-k^{p_d}x}. \end{aligned} \tag{4}$$

Summing (4) over $w\in T(\boldsymbol{m}^{(d)},k)$ and solving for $A_{d+1}(x)$, we obtain

$$A_{d+1}(x) = \frac{a_d x}{1-k^{p_d}x} + \left(a_d k^{p_d} + \sum_{w,w'\in T(\boldsymbol{m}^{(d)},k)}\sum_{\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]} 1_{w(\boldsymbol{i})>w'(\boldsymbol{i})}\right)\frac{x^2}{(1-k^{p_d}x)^2}.$$

Now, due to symmetry,

$$\begin{aligned} \sum_{w,w'\in T(\boldsymbol{m}^{(d)},k)}\sum_{\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]} 1_{w(\boldsymbol{i})>w'(\boldsymbol{i})} &= p_d\sum_{w'\in T(\boldsymbol{m}^{(d)},k)}\sum_{w\in T(\boldsymbol{m}^{(d)},k)} 1_{w(1,\dots,1)>w'(1,\dots,1)}\\ &= p_d k^{p_d-1}\sum_{w'\in T(\boldsymbol{m}^{(d)},k)}(k-w'(1,\dots,1))\\ &= p_d k^{2p_d-2}\sum_{w'(1,\dots,1)=1}^{k}(k-w'(1,\dots,1))\\ &= \frac{1}{2}(k-1)k^{2p_d-1}p_d. \end{aligned}$$

It follows that

$$A_{d+1}(x) = \frac{a_d x}{1-k^{p_d}x} + \left(a_d k^{p_d} + \frac{1}{2}(k-1)k^{2p_d-1}p_d\right)\frac{x^2}{(1-k^{p_d}x)^2},$$

from which (3) immediately follows.

Now, we can prove that (2) holds for $d+1$. Indeed, by (3), we have

$$\begin{aligned}A_{d+1}(x) &= \frac{a_d x + \frac{1}{2}(k-1)k^{2p_d-1}p_d x^2}{(1-k^{p_d}x)^2}\\ &= \left(a_d x + \frac{1}{2}(k-1)k^{2p_d-1}p_d x^2\right)\sum_{m_{d+1}\ge 1} m_{d+1}k^{(m_{d+1}-1)p_d}x^{m_{d+1}-1}\\ &= \sum_{m_{d+1}\ge 1}\frac{1}{2}\left(\left(dp_d - \sum_{i\in[d]} p_{d,i}\right)m_{d+1} + p_d(m_{d+1}-1)\right)(k-1)k^{p_{d+1}-1}x^{m_{d+1}}\\ &= \sum_{m_{d+1}\ge 1}\frac{1}{2}\left((d+1)p_{d+1} - \sum_{i\in[d+1]} p_{d+1,i}\right)(k-1)k^{p_{d+1}-1}x^{m_{d+1}}.\end{aligned}$$

Thus,

$$a_{d+1} = \frac{1}{2}\left((d+1)p_{d+1} - \sum_{i\in[d+1]} p_{d+1,i}\right)(k-1)k^{p_{d+1}-1}$$

and the proof is complete. □

Remark 2.1.1. *For the proof of the result regarding the total number of levels, only a few minor modifications are necessary. First, Equation (1) needs to be replaced with*

$$F_{d+1}(t,x,w) = t^{\mathrm{lev}(w)}x + x\sum_{w'\in T(\boldsymbol{m^{(d)}},k)} t^{\mathrm{lev}(w)+\sum_{\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]}1_{w(\boldsymbol{i})=w'(\boldsymbol{i})}}F_{d+1}(t,x,w').$$

Second, it is not hard to see that

$$\sum_{w,w'\in T(\boldsymbol{m^{(d)}},k)}\ \sum_{\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]} 1_{w(\boldsymbol{i})=w'(\boldsymbol{i})} = k^{2p_d-1}p_d.$$

2.2. Cyclic tensor words

Let $m_{d+1}\in\mathbb{N}$ and let ℓ be a nonnegative integer. Set $\boldsymbol{m^{(d+1)}} = (m_1,\ldots,m_{d+1})$. We denote by $D_{\boldsymbol{m^{(d)}},m_{d+1},\ell}$ the number of $\boldsymbol{m^{(d+1)}}$-tensor words w such that $\mathrm{cycdes}(w)=\ell$. For $w_1,w_2\in T(\boldsymbol{m^{(d)}},k)$, we denote by $D_{\boldsymbol{m^{(d)}},m_{d+1},\ell}(w_1,w_2)$ the number of $\boldsymbol{m^{(d+1)}}$-tensor words w', such that $\mathrm{cycdes}(w')=\ell$ and such that $w'(\boldsymbol{i},j)=w_j(\boldsymbol{i})$, for every $\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]$ and $j=1,2$. Let

$$F_{d+1}(t,x) = \sum_{m_{d+1}\ge 1}\sum_{\ell\ge 0} D_{\boldsymbol{m^{(d)}},m_{d+1},\ell}t^{\ell}x^{m_{d+1}}$$

be the generating function for the numbers $D_{\boldsymbol{m^{(d)}},m_{d+1},\ell}$, and, for $w_1,w_2\in T(\boldsymbol{m^{(d)}},k)$, we denote by

$$F_{d+1}(t,x,w_1,w_2) = \sum_{m_{d+1}\ge 1}\sum_{\ell\ge 0} D_{\boldsymbol{m^{(d)}},m_{d+1},\ell}(w)t^{\ell}x^{m_{d+1}}$$

the generating function for the numbers $D_{\boldsymbol{m^{(d)}},m_{d+1},\ell}(w_1,w_2)$. Notice that

$$F_{d+1}(1,x,w_1,w_2) = \frac{x^2}{1-k^{p_d}x}.$$

The following lemma and theorem are the analogues of Lemma 2.1.1 and Theorem 2.1.1, respectively. We state them without proof.

Lemma 2.2.1. *For $w_1,w_2\in T(\boldsymbol{m^{(d)}},k)$, the generating function $F(t,x,w_1,w_2)$ satisfies the equation*

$$F_{d+1}(t,x,w_1,w_2) = t^{\mathrm{cycdes}(w_1w_2)}x^2 + x\sum_{w'\in T(\boldsymbol{m^{(d)}},k)} t^{\sum_{\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]}\left(1_{w_1(\boldsymbol{i})>w_2(\boldsymbol{i})}+1_{w_2(\boldsymbol{i})>w'(\boldsymbol{i})}-1_{w_1(\boldsymbol{i})>w'(\boldsymbol{i})}\right)+\mathrm{cycdes}(w')}F_{d+1}(t,x,w_1,w'),$$

where w_1w_2 is the tensor word $w'\in T(\boldsymbol{m^{(d+1)}},k)$, with $m_{d+1}=2$ and $w'(\boldsymbol{i},j)=w_j$, for every $\boldsymbol{i}\in[m_1]\times\cdots\times[m_d]$ and $j=1,2$.

Theorem 2.2.1. *Define $g_d=\sum_{w\in T(\boldsymbol{m^{(d)}},k)}\mathrm{cycdes}(w)$ and let $G_d(x)$ be the corresponding generating function. Then*

$$g_d = \frac{1}{2}dp_d(k-1)k^{p_d-1}$$

and

$$G_{d+1}(x) = \frac{2\left(\frac{1}{2}(k-1)k^{2p_d-1}p_d + k^{p_d}g_d\right)}{(1-k^{p_d}x)^2}x^2 - \frac{\left(\frac{1}{2}(k-1)k^{3p_d-1}p_d + k^{2p_d}g_d\right)}{(1-k^{p_d}x)^2}x^3.$$

2.3. The maximal number of descents

In this subsection, we establish the maximal number of cyclic descents in cyclic tensor words. This yields an upper bound on the maximal number of descents in tensor words, which we prove to be tight. First, we state the following results, concerning descents and cyclic descents in (standard) words.

Lemma 2.3.1. *We have*

$$\max\{\mathrm{des}(x)\ :\ x\in[k]^n\}=\max\{\mathrm{cycdes}(x)\ :\ x\in[k]^n\}=\left\lfloor\frac{n(k-1)}{k}\right\rfloor.$$

Proof. Let $C(n,k)$ denote the set of all binary sequences of length n having at most k consecutive 1s. For $y\in C(n,k)$ let us denote by $|y|$ the number of 1s in y. We first show that

$$\max\{|y|\ :\ y\in C(n,k)\}=\left\lfloor\frac{(n+1)k}{k+1}\right\rfloor \tag{5}$$

(see A182210 in [5]). Write $n=\sigma(k+1)+\rho$, where $0\le\rho<k+1$. We then have the sequence

$$y=\overbrace{\overbrace{1\cdots1}^{k\text{ times}}0\cdots\overbrace{1\cdots1}^{k\text{ times}}0}^{\sigma\text{ times}}\overbrace{1\cdots1}^{\rho\text{ times}}\in C(n,k),$$

satisfying

$$|y|=n-\sigma=n-\left\lfloor\frac{n}{k+1}\right\rfloor=\left\lfloor\frac{(n+1)k}{k+1}\right\rfloor.$$

If $\sigma=0$ then $n\le k$ and y consists solely of n 1s, which is obviously maximal. Assume that $\sigma\ge1$. Then, treating the 0s as separators between pigeonholes and the 1s as pigeons, for every $t\in[\sigma]$, we have

$$n-(\sigma-t)=(\sigma-t+1)k+\overbrace{(t-1)k+\rho+t}^{\ge1}.$$

By the pigeonhole principle, in any binary sequence consisting of $\sigma-t$ 0s and $n-(\sigma-t)$ 1s, there exists at least one subsequence of $k+1$ consecutive 1s.

Having proved (5), we now show the connection to descents in words. To this end, we construct maps

$$\varphi\colon[k]^n\to C(n-1,k-1)\quad\text{and}\quad\theta\colon C(n-1,k-1)\to[k]^n$$

such that $|\varphi(x)|=\mathrm{des}(x)$ and $\mathrm{des}(\theta(y))=|y|$. Let $x=x_1\cdots x_n\in[k]^n$ and define a binary sequence $\varphi(x)=y=y_1\cdots y_{n-1}$ of length $n-1$ as follows: For $i\in[n-1]$ we set $y_i=1_{x_i>x_{i+1}}$. Clearly, $y\in C(n-1,k-1)$ and $\mathrm{des}(x)=|y|$. Conversely, let $y=y_1\cdots y_{n-1}\in C(n-1,k-1)$ and let z_i (respectively, o_i) be the length of the ith sequence of consecutive 0s (respectively, 1s) in y, where $i\in[r]$ for some $r\in\mathbb{N}$. We define $\theta(y)=x=v_1\cdots v_r$, where

$$v_i=\begin{cases}\overbrace{o_1+1,\ldots,o_1+1}^{z_1+1\text{ times}},o_1,o_1-1,\ldots,1 & \text{if } i=1;\\ \overbrace{o_i+1,\ldots,o_i+1}^{z_i\text{ times}},o_i,o_i-1,\ldots,1 & \text{otherwise.}\end{cases}$$

First, we notice that v_1 is of length z_1+o_1+1 and v_i is of length z_i+o_i, for $2\le i\le r$. Thus, x is of length n, since $\sum_{i\in[r]}(z_i+o_i)=n-1$. Second. $0\le o_i\le k-1$, for every $i\in[r]$. Thus, $x\in[k]^n$. Finally, since $z_i>0$ for every $2\le i\le r$, the number of descents occurring in v_i is o_i. Notice that every v_i ends with 1, so no descents occur in the transition between v_i and v_{i+1}. It follows that $\mathrm{des}(x)=\sum_{i\in[r]}o_i=|y|$. We have thus proved that $\max\{\mathrm{des}(x)\ :\ x\in[k]^n\}=\lfloor n(k-1)/k\rfloor$. Trivially, $\max\{\mathrm{des}(x)\ :\ x\in[k]^n\}\le\max\{\mathrm{cycdes}(x)\ :\ x\in[k]^n\}$. On the other hand, let $x\in[k]^n$. Necessarily, there exists $i\in[n]$ such that $x_i\le x_j$, where $j=i+1$ if $i<n$ and $j=1$, otherwise. Let x' be the rotation of x such that the first letter of x' is x_j. Then $\mathrm{cycdes}(x)=\mathrm{cycdes}(x')=\mathrm{des}(x')$. We conclude that $\max\{\mathrm{cycdes}(x)\ :\ x\in[k]^n\}\le\max\{\mathrm{des}(x)\ :\ x\in[k]^n\}$ and hence the proof is complete. □

Corollary 2.3.1. *We have*

$$\max\left\{\mathrm{cycdes}(w)\ :\ w\in T(\boldsymbol{m}^{(\boldsymbol{d})},k)\right\}=\sum_{i\in[d]}p_{d,i}\left\lfloor\frac{m_i(k-1)}{k}\right\rfloor. \tag{6}$$

Proof. First, we show that the left-hand side of (6) is less than or equal to the right-hand side. Proceeding by induction on d, the case $d = 1$ follows from Lemma 2.3.1. Assume that the inequality holds for d and let $w \in T(\boldsymbol{m}^{(\boldsymbol{d+1})}, k)$. Then

$$\begin{aligned}\text{cycdes}(w) &\leq m_{d+1} \max\left\{\text{cycdes}(w') \ : \ w' \in T(\boldsymbol{m}^{(\boldsymbol{d})}, k)\right\} + p_d \left\lfloor \frac{m_{d+1}(k-1)}{k} \right\rfloor \\ &\leq m_{d+1} \sum_{i\in[d]} p_{d,i} \left\lfloor \frac{m_i(k-1)}{k} \right\rfloor + p_d \left\lfloor \frac{m_{d+1}(k-1)}{k} \right\rfloor \\ &= \sum_{i\in[d+1]} p_{d+1,i} \left\lfloor \frac{m_i(k-1)}{k} \right\rfloor,\end{aligned}$$

as required. To prove the reversed inequality, consider the word $v = k, k-1, k-2, \ldots, 1, k, k-1, \ldots$, of length m_1, that obtains the maximal number of cyclic descents. Due to cyclicity, we may rotate v without losing cyclic descents. Writing v and its $m_2 - 1$ rotations, we obtain the matrix word of size $m_1 \times m_2$:

$$u = \begin{pmatrix} k & k-1 & k-2 & \cdots \\ k-1 & k-2 & k-3 & \cdots \\ \vdots & \vdots & \ddots & \end{pmatrix}.$$

Rotating u, i.e., decreasing by 1 each of its entries, and replacing 0s with ks, we may write u and its $m_3 - 1$ rotations along the third axis, obtaining an (m_1, m_2, m_3)-tensor word. In this manner we successively construct an $\boldsymbol{m}^{(\boldsymbol{d})}$-tensor word over $[k]$ that, by induction, may be shown to have the necessary number of cyclic descents. □

Remark 2.3.1. *Since the number of descents is never larger than the number of cyclic descents, the right-hand side of* (6) *gives an upper bound on the maximal number of descents in tensor words. Here, the situation is more complex and the naive approach of Corollary 2.3.1 does not work. Indeed, we have*

$$\text{des}\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = 13, \ \text{while des}\begin{pmatrix} 4 & 3 & 4 & 3 \\ 3 & 2 & 3 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix} = 14,$$

which is maximal for $(3, 4)$*-tensor words over* $[4]$*. Nevertheless, by rotating the (standard) word* $v = k, k-1, k-2, \ldots, 1, k, k-1, \ldots$*, we lose, at most, one descent. It follows that the tensor word, constructed in the proof of Corollary 2.3.1, has at least*

$$\sum_{i\in[d]} p_{d,i} \left(\left\lfloor \frac{m_i(k-1)}{k} \right\rfloor - 1 \right)$$

descents.

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