Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **13** (2024) 36–43
DOI: 10.47443/dml.2023.215

Research Article

Independent domination polynomial for the cozero divisor graph of the ring of integers modulo n

Bilal Ahmad Rather*

Department of Mathematical Sciences, College of Science, United Arab Emirates University, Al Ain 15551, Abu Dhabi, UAE

(Received: 14 November 2023. Received in revised form: 16 February 2024. Accepted: 27 February 2024. Published online: 4 April 2024.)

© 2024 the author. This is an open-access article under the CC BY (International 4.0) license (www.creativecommons.org/licenses/by/4.0/).

Abstract

The cozero divisor graph $\Gamma'(R)$ of a commutative ring R is a simple graph whose vertex set is the set of non-zero non-unit elements of R such that two distinct vertices x and y of $\Gamma'(R)$ are adjacent if and only if $x \notin Ry$ and $y \notin Rx$, where Rx is the ideal generated by x. In this article, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is found for $n \in \{p_1p_2, p_1p_2p_3, p_1^{n_1}p_2\}$, where p_i's are primes, n_1 is an integer greater than 1, and $\mathbb{Z}_n$ is the integer modulo ring. It is shown that the independent domination polynomial of $\Gamma'(\mathbb{Z}_{p_1p_2})$ has only one real root. It is also proved that these polynomials are not unimodal but are log-concave under certain conditions.

Keywords: cozero divisor graphs; commutative ring; independent domination polynomial; unimodal; log-concave.

2020 Mathematics Subject Classification: 05C25, 05C31, 05C69, 05C90.

1. Introduction

Only finite, simple, and undirected graphs are considered in this paper. A graph is denoted by $G = G(V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The numbers $n = |V(G)|$ and $m = |E(G)|$ are order and size of G, respectively. An edge between two vertices u and v is represented by $u \sim v$. A vertex of degree 0 is an isolated vertex and a vertex of degree one is a pendent vertex. The *degree* $d_{v_i}(G)$ (or simply d_i, if G is clear) of a vertex v_i is the number of vertices incident with it. The *union* of two graphs $G_1 = G_1(V_1(G_1), E_1(G_1))$ and $G_2 = G_2(V_2(G_2), E_2(G_2))$, denoted by $G_1 \cup G_2$, is defined as a graph with vertex set $V_1(G_1) \cup V_2(G_2)$ and edge set $E_1(G_1) \cup E_2(G_2)$. The *join* of G_1 and G_2 is denoted by $G_1 \vee G_2$ and is defined as a graph with vertex set $V_1(G_1) \cup V_2(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{u \sim v \mid u \in V(G_1), v \in V(G_2)\}$.

A non-empty set $S \subseteq V(G)$ is said to be a *dominating* set if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The minimum cardinality among all dominating sets of G is known as the *domination number* of G, denoted by $\gamma(G)$. The domination theory of graphs is very well developed, see [16]. An *independent* set in a graph G is a set of pairwise non-adjacent vertices. The cardinality of the largest independent set is known as the *independence number* of G, denoted by $\alpha(G)$. An independent dominating set of G is a vertex subset that is both dominating and independent in G. The independent domination number, denoted by $\gamma_i(G)$, is the minimum size of all independent dominating sets of G. The relation between γ, α and γ_i of G is $\gamma(G) \leq \gamma_i(G) \leq \alpha(G)$ (see, [16]). The independent set problem is a strongly NP-hard problem while the dominating set problem is an NP-complete problem, which are well well-studied both in mathematics and theoretical computer science. A star graph of order n is denoted by $K_{1,n-1}$ and a complete bipartite graph by $K_{a,b}$, with $n = a + b$. A graph G of order n is said to be totally disconnected if G is isomorphic to the complement of a complete graph.

Let $d_k(G, k)$ denote the number of independent dominating sets of carnality k in G. The independent domination polynomial of G is defined as

$$D_i(G, x) = \sum_{k=\gamma_i(G)}^{\alpha(G)} d_i(G, k)x^k.$$

A root of the equation $D_i(G, x) = 0$ is known as the independent domination root of G. The independent domination polynomial $D_i(G, x)$ is a generating function of the number of the independent dominating sets of certain cardinalities of G. The independent domination polynomials and their zeros have attracted many researchers, see [8,13,14,18]. Jahari and Alikhani [17] gave the independent domination polynomials of generalized compound graphs and constructed graphs whose independent domination polynomials have real zeros. Recently, the authors in [15,22] presented the results related to the

*E-mail address: bilalahmadrr@gmail.com

independent domination polynomial of zero-divisor graphs of commutative rings. More about independent domination polynomials can be found in [5–7]. Although it is hard to find the independent domination polynomial of a general graph, we can often find a closed expression for such polynomials for certain classes of graphs. Motivated by the above-mentioned work, especially the one related to zero-divisor graphs of commutative rings in [15, 22], we consider the independent domination polynomial theory for cozero divisor graphs of the commutative ring of integer modulo n.

In Section 2, we present the closed expression for the independent domination polynomial of cozero divisor graphs of commutative rings and Section 3 is concerned with their unimodal and log-concave properties. We end this article with the conclusion section

2. Independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$

We first discuss the structure of cozero divisor graphs. The cozero divisor graphs are motivated by zero divisor graphs, which is defined as a graph $\Gamma(R)$ associated to a ring R, with vertex set as non-zero zero divisors of R such that two distinct vertices are adjacent if and only if their product is zero. The cozero divisor graph of a commutative ring R (with unity $1 \neq 0$) is a simple graph with vertex set as non-zero non-unit elements of R such that two vertices x and y $(x \neq y)$ are adjacent if and only if $x \notin Ry$ and $y \notin Rx$, where Rx is the ideal generated by x. The cozero divisor graph of R is denoted by $\Gamma'(R)$. The basic properties of cozero divisor graphs including their complement graphs, planarity, characterization of commutative rings with forest, star, or unicyclic cozero divisor graphs, their relations with comaximal graphs of rings, and zero divisor graph were investigated by Afkhami and Khashyarmanesh [1–4]. Cozero divisor graphs of polynomial rings were discussed in [9], and spectral analysis of cozero divisor graphs was carried out in [19]. For some other progress of cozero divisor, see [10, 11, 21] and the references cited therein.

In general, it is not easy to find the structure of $\Gamma'(R)$ completely, though for some special cases we can have some information about the structure of $\Gamma'(R)$ (especially for $\Gamma'(\mathbb{Z}_n)$), where $\mathbb{Z}_n$ is the integral modulo ring. Depending on the proper divisors $d_i, i \notin \{1, n\}$ of n, we divide $V(\Gamma'(\mathbb{Z}_n))$ into mutually disjoint vertex cells as (a similar concept is used in [12, 20, 23–25] for studying other algebraic graphs):

$$A_{d_i} = \{a \in \mathbb{Z}_n \ : \ (a, n) = d_i\},$$

where (a, n) is the greatest common divisor of a and n. Clearly A_{d_i} are mutually pairwise disjoint and

$$V(\Gamma'(\mathbb{Z}_n)) = \bigcup_{i=1}^{t} A_{d_i},$$

where t is the number of proper divisor of n. Furthermore, for $a, b \in A_{d_i}$, we have $\langle a\rangle = \langle b\rangle$. The cardinality of A_{d_i} is $\phi\left(\frac{n}{d_i}\right)$ (see [25]), for $i = 1, 2, \dots, t$, where $\phi(\cdot)$ is an Euler function. Also, if $a \in A_{d_i}$ and $b \in A_{d_j}$ then a and b are adjacent in $\Gamma'(\mathbb{Z}_n)$ if and only $d_i \nmid d_j$ and $d_j \nmid d_i$, for $i, j \in \{1, 2, \dots, \tau(n) - 2\}$, where $\tau(\cdot)$ is divisor function. For $i \in \{1, 2, \dots, \tau(n) - 2\}$, the induced subgraph of A_{d_i} is $\overline{K}_{\phi\left(\frac{n}{d_i}\right)}$. For more above the structural properties of $\Gamma'(\mathbb{Z}_n)$, we refer to [19].

Our first result gives the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ when n is the product of two distinct primes.

Proposition 2.1. *For $n = p_1p_2$, with $p_1 < p_2$, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is*

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^{p_1-1} + z^{p_2-1}.$$

Proof. We partition the vertex set of $\Gamma'(\mathbb{Z}_n)$ into the following subsets

$$A_1 = \{kp_1 \mid k = 1, 2, \dots, p_2 - 1\},$$
$$A_2 = \{kp_2 \mid k = 1, 2, \dots, p_1 - 1\}.$$

Clearly $A_1 \cap A_2 = \emptyset$ and each $x \in A_1$ does not divide each $y \in A_2$. So it follows that $\Gamma'(\mathbb{Z}_n)$ is a complete bipartite graph and its independent domination polynomial is given by $D_i(\Gamma'(\mathbb{Z}_n), z) = z^{p_1-1} + z^{p_2-1}$. □

The next result gives the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ when n is a product of three primes.

Theorem 2.1. *The independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ for $n = p_1p_2p_3$ with $p_1 < p_2 < p_3$ is*

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^{p_2p_3-p_3} + z^{p_2p_3-p_2} + z^{p_1p_3-p_3} + z^{p_1p_3-p_1} + z^{p_1p_2-p_1} + z^{p_1p_2-p_2}.$$

Proof. Depending on the proper divisors of n, the vertex set of $\Gamma'(\mathbb{Z}_n)$ can be partition into following mutually disjoint subsets

$$A_1 = \{kp_1 \mid k = 1, 2, \ldots, p_2p_3 - 1, p_2 \nmid k, p_3 \nmid k\}, A_2 = \{kp_2 \mid k = 1, 2, \ldots, p_1p_3 - 1, p_1 \nmid k, p_3 \nmid k\},$$
$$A_3 = \{kp_3 \mid k = 1, 2, \ldots, p_1p_2 - 1, p_1 \nmid k, p_2 \nmid k\}, A_4 = \{kp_1p_2 \mid k = 1, 2, \ldots, p_3 - 1\},$$
$$A_5 = \{kp_1p_3 \mid k = 1, 2, \ldots, p_2 - 1\}, A_6 = \{kp_2p_3 \mid k = 1, 2, \ldots, p_1 - 1\}.$$

Clearly, $x \in A_i$ does not divide $y \in A_i$ ($x \neq y$), for each $i = 1, 2, \ldots, 6$. It follows that induced subgraphs of each A_i is null graph (non-empty edgeless graph). Furthermore, $|A_1| = \phi(p_2p_3) = (p_2 - 1)(p_3 - 1), |A_2| = (p_1 - 1)(p_3 - 1), |A_3| = (p_1 - 1)(p_2 - 1), |A_4| = p_3 - 1, |A_5| = p_2 - 1$ and $|A_6| = p_1 - 1$. Also, we note that $x \in A_1$ divides some $y \in A_4$ and some $l \in A_5$, so it follows that no vertex of A_1 is adjacent to any vertex of A_4 and A_5. Likewise $x \in A_2$ divides some $y \in A_4$ and some $l \in A_6$, implying that each vertex of A_2 is not adjacent to any vertex of A_4 and A_6. Similarly, each vertex of A_3 is not adjacent to any vertex of A_5 and A_6, there are edges between each vertex of A_6 and A_4, between each vertex of A_5 with A_6 and A_4. Depending on these subsets and their adjacent relations, we have the following cases:

Case (i). Suppose $D = A_1 \cup A_5$. Then the vertices of A_1 dominate the vertices of A_2, A_3 and A_6 and A_5 dominates the vertices of A_2, A_4 and A_6. Thus, all vertices in $(\Gamma'(\mathbb{Z}_n)) \setminus D$ are dominated by D and it implies that D is an independent dominating set of cardinality $(p_2 - 1)(p_3 - 1) + p_2 - 1 = p_2p_3 - p_3$. Besides, in this case $d_i(\Gamma'(\mathbb{Z}_n), p_2p_3 - p_3) = 1$.

Case (ii). Consider $D = A_1 \cup A_4$. Then as in (i), A_1 dominates the vertices of A_2, A_3 and A_6 and A_4 dominates the vertices of A_3, A_5 and A_6. Thus, the vertices in $(\Gamma'(\mathbb{Z}_n)) \setminus D$ are dominated by D and the cardinality of such an independent dominating set is $(p_2 - 1)(p_3 - 1) + p_3 - 1 = p_2p_3 - p_2$.

Case (iii). Take $D = A_2 \cup A_6$ and note that these two subsets dominates all vertices in A_1, A_3, A_4 and A_5. So, D is another independent dominating set of cardinality $(p_1 - 1)(p_3 - 1) + p_1 - 1 = p_1p_3 - p_3$.

Case (iv). Take $D = A_2 \cup A_4$ and observe that A_2 dominates A_1, A_3 and A_5 and A_4 dominates A_6 along with already dominated sets A_5 and A_3. Thus, D is another independent dominating set of cardinality $(p_1 - 1)(p_3 - 1) + p_3 - 1 = p_1p_3 - p_1$.

Case (v). Consider $A_3 \cup A_5$. Then A_3 dominates A_1, A_2 and A_4 while A_5 dominates A_2, A_4 and A_6 and the vertices in $(\Gamma'(\mathbb{Z}_n)) \setminus (A_3 \cup A_5)$ are dominated by $A_3 \cup A_5$. So, it is another independent dominating set with cardinality:

$$(p_1 - 1)(p_2 - 1) + p_2 - 1 = p_1p_2 - p_1.$$

Case (vi). Lastly, consider $D = A_3 \cup A_6$. Then A_3 dominates A_1, A_2 and A_4 while A_6 dominates A_1, A_4 and A_5 and the vertices in $(\Gamma'(\mathbb{Z}_n)) \setminus D$ are dominated by $A_3 \cup A_5$. So, it follows that D is an independent dominating set of cardinality $(p_1 - 1)(p_2 - 1) + p_1 - 1 = p_1p_2 - p_2$.

Therefore, by the above cases, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^{p_2p_3 - p_3} + z^{p_2p_3 - p_2} + z^{p_1p_3 - p_3} + z^{p_1p_3 - p_1} + z^{p_1p_2 - p_1} + z^{p_1p_2 - p_2}.$$

□

We illustrate Theorem 2.1 with the help of the following example.

Example 2.1. *For $n = 2 \cdot 3 \cdot 5 = 30$, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is*

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^3 + z^4 + z^5 + z^8 + z^{10} + z^{12}.$$

For $n = 30$, the order of $\Gamma'(\mathbb{Z}_n)$ is $n - \phi(n) - 1 = 21$. The independent vertex partitions are

$$A_1 = \{2, 4, 8, 14, 16, 22, 26, 28\}, A_2 = \{3, 9, 21, 27\}, A_3 = \{5, 25\}, A_4 = \{6, 12, 18, 24\}, A_5 = \{10, 20\}, A_6 = \{15\}.$$

The graph is shown in Figure 2.1. According to these six subsets and their independent domination combinations in Theorem 2.1, we have

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^3 + z^4 + z^5 + z^8 + z^{10} + z^{12}.$$

In the next couple of results, we find the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ for $n = p^{n_1}p_2$ with $n_1 \geq 2$.

Theorem 2.2. *For $n = p_1^2p_2$, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is*

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^{p_1p_2 - p_1} + z^{p_1p_2 - p_2} + z^{p_1^2 - 1}.$$

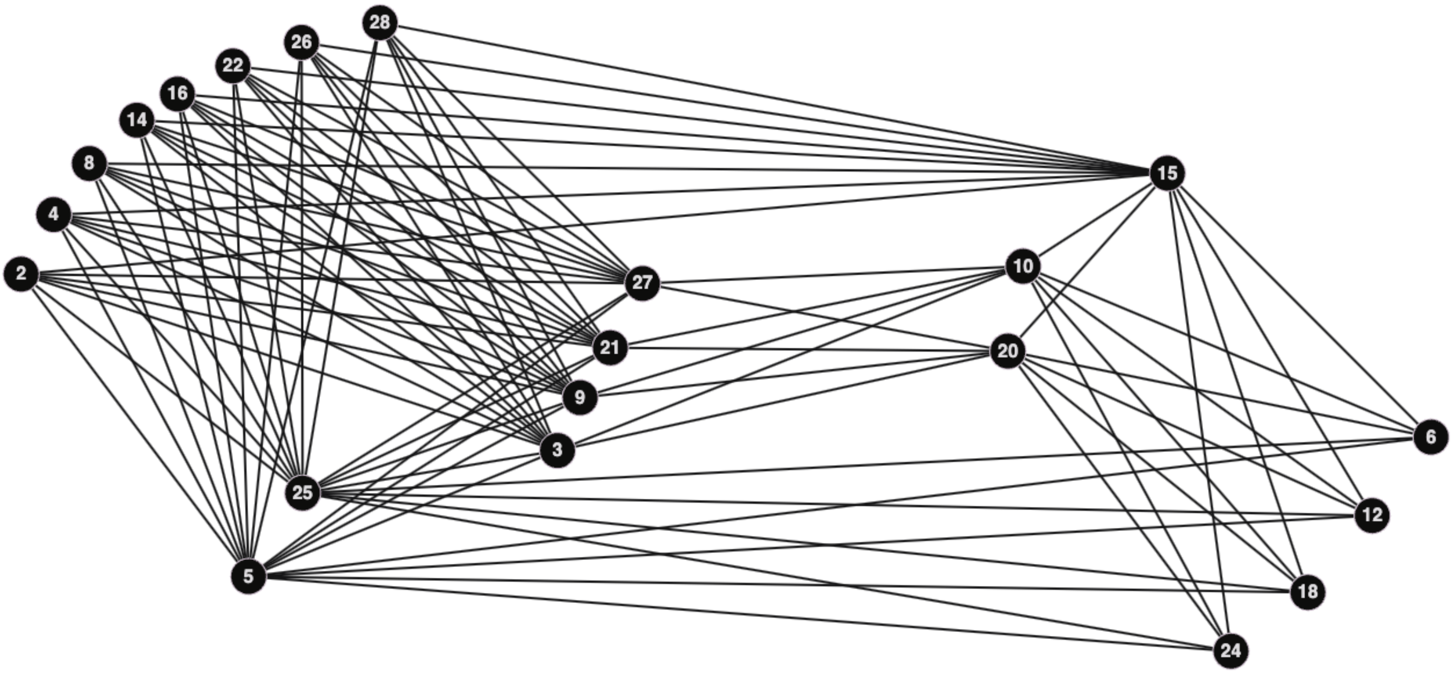

Figure 2.1: Cozero divisor graph $\Gamma'(\mathbb{Z}_{30})$.

Proof. For $n = p_1^2 p_2$, based on proper divisors p_1, p_2, p_1^2 and p_1p_2, we divide $V(\Gamma'(\mathbb{Z}_n))$ into following subsets

$$A_{p_1} = \{kp_1 \mid k = 1, 2, \ldots, p_1p_2 - 1, p_1 \nmid k, p_2 \nmid k\}, A_{p_2} = \{kp_2 \mid k = 1, 2, \ldots, p_1^2 - 1, p_1 \nmid k\},$$
$$A_{p_2^2} = \{kp_2^2 \mid k = 1, 2, \ldots, p_2 - 1\}, A_{p_1p_2} = \{kp_1p_2 \mid k = 1, 2, \ldots, p_1 - 1\}.$$

The induces subgraphs of A_i's are non-empty null graph and their cardinalities are $(p_1 - 1)(p_2 - 1), (p_1^2 - p_1), p_2 - 1$ and $p_1 - 1$, respectively. Also, each vertex of A_{p_1} is adjacent to every vertex of A_{p_2}, since A_{p_1} contains some multiplies of p_1 and A_{p_2} contains some multiples of p_2 and p_1 does not divide p_2. Likewise, each vertex of A_{p_2} is adjacent to each vertex of $A_{p_1^2}$ and each vertex of $A_{p_1^2}$ is adjacent to every vertex of $A_{p_1p_2}$. As each of A_i is an independent set, so there are total $\binom{4}{2} = 3$ independent dominating sets namely $A_{p_1} \cup A_{p_1^2}, A_{p_1} \cup A_{p_1p_2}$ and $A_{p_2} \cup A_{p_1p_2}$ each with cardinalities $p_1p_2 - p_1, p_1p_2 - p_2$ and $p_1^2 - 1$, respectively. Therefore, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is $D_i(\Gamma'(\mathbb{Z}_n), z) = z^{p_1p_2 - p_1} + z^{p_1p_2 - p_2} + z^{p_1^2 - 1}$. $\square$

Theorem 2.3. *For the cozero divisor graph $\Gamma'(\mathbb{Z}_n)$, the following hold:*

(i) *If $n = p_1^3 p_2$, then the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is*

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^{p_1^2(p_2 - 1)} + z^{(p_1 - 1)(p_1p_2 + p_2 - p_1)} + z^{(p_1 - 1)(p_1p_2 + 1)} + z^{p_1^3 - 1}.$$

(ii) *If $n = p_1^4 p_2$, then the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is*

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^{p_1^3(p_2 - 1)} + z^{(p_2 - 1)(p_1^3 - 1) + p_1 - 1} + z^{(p_2 - 1)(p_1^3 - p_1) + p_1^2 - 1} + z^{(p_2 - 1)(p_1^3 - p_1^2) + p_1^3 - 1} + z^{p_1^4 - 1}.$$

Proof. We prove only (ii), and (i) can be similarly proved. For $n = p_1^4 p_2$, the vertex set of $\Gamma'(\mathbb{Z}_n)$ can be portioned as

$$A_1 = \{kp_1 \mid k = 1, 2, \ldots, p_1^3 p_2 - 1, p_1 \nmid k, p_2 \nmid k\}, A_2 = \{kp_1^2 \mid k = 1, 2, \ldots, p_1^2 p_2 - 1, p_1 \nmid k, p_2 \nmid k\},$$

$$A_3 = \{kp_1^3 \mid k = 1, 2, \ldots, p_1p_2 - 1, p_1 \nmid k, p_2 \nmid k\}, A_4 = \{kp_1^4 \mid k = 1, 2, \ldots, p_2 - 1\},$$

$$A_5 = \{kp_2 \mid k = 1, 2, \ldots, p_1^4 - 1, p_1 \nmid k\}, A_6 = \{kp_1p_2 \mid k = 1, 2, \ldots, p_1^3 - 1, p_1 \nmid k\},$$

$$A_7 = \{kp_1^2 p_2 \mid k = 1, 2, \ldots, p_1^2 - 1, p_1 \nmid k\}, A_8 = \{kp_1^3 p_2 \mid k = 1, 2, \ldots, p_1 - 1\}.$$

By the definition of the cozero divisor graph, each vertex of A_1 is adjacent to each vertex of A_5, each vertex of A_2 is adjacent to each vertex of A_5 and A_6, each vertex of A_3 is adjacent to each vertex of A_5, A_6 and A_7, each vertex of A_4 is adjacent to each vertex of $A_i, i = 5, 6, 7$ and 8.

Case (i). Let $D = A_1 \cup A_2 \cup A_3 \cup A_4$. Then D is an independent set since each of A_i induces a complement of clique. Also, A_4 dominates each vertex of A_5, A_6, A_7 and A_8. Thus each vertex of $V(\Gamma'(\mathbb{Z}_n)) \setminus D$ is adjacent to at least one vertex of D. So D is an independent dominating set of cardinality $\phi(p_1^3p_2) + \phi(p_1^2p_2) + \phi(p_1p_2) + \phi(p_2) = p_1^3(p_2 - 1)$.

Case (ii). If $D = A_1 \cup A_2 \cup A_3 \cup A_8$, then A_1 dominates A_5, A_8 dominates A_4 and $A_2 \cup A_3$ dominates A_5, A_6 and A_7. In this way all vertices of $\Gamma'(\mathbb{Z}_n)$ are dominated by D and it follows that D is an independent dominating set of order $\phi(p_1^3p_2) + \phi(p_1^2p_2) + \phi(p_1p_2) + \phi(p_1) = (p_2 - 1)(p_1^3 - 1) + p_1 - 1$.

Case (iii). If $D = A_1 \cup A_2 \cup A_7 \cup A_8$, then A_1 dominates A_5, A_2 dominates A_5 and A_6, A_7 dominates A_3 and A_4. So each vertex of $V(\Gamma'(\mathbb{Z}_n)) \setminus D$ is adjacent to at least one vertex of D. It implies that D is an independent dominating set of order $\phi(p_1^3p_2) + \phi(p_1^2p_2) + \phi(p_1^2) + \phi(p_1) = (p_2 - 1)(p_1^3 - p_1) + p_1^2 - 1$.

Case (iv). If $D = A_1 \cup A_6 \cup A_7 \cup A_8$, then A_1 dominates A_5, and A_6 dominates A_2, A_3 and A_4. It is clear that each vertex of $V(\Gamma'(\mathbb{Z}_n)) \setminus D$ is adjacent to at least one vertex of D. So, D is an independent dominating set of order $\phi(p_1^3p_2) + \phi(p_1^3) + \phi(p_1^2) + \phi(p_1) = (p_2 - 1)(p_1^3 - p_1^2) + p_1^3 - 1$.

Case (v). Clearly each vertex of $\Gamma'(\mathbb{Z}_n)$ is dominated by $A_5 \cup A_6 \cup A_7 \cup A_8$ and it is an independent dominating set of cardinality $\phi(p_1^4) + \phi(p_1^3) + \phi(p_1^2) + \phi(p_1) = p_1^4 - 1$.

Based on the above discussion, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^{p_1^3(p_2-1)} + z^{(p_2-1)(p_1^3-1)+p_1-1} + z^{(p_2-1)(p_1^3-p_1)+p_1^2-1} + z^{(p_2-1)(p_1^3-p_1^2)+p_1^3-1} + z^{p_1^4-1}.$$

□

Next, we illustrate Theorem 2.1 by the following example:

Example 2.2. *For $n = 2^4 \cdot 3 = 48$, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is $D_i(\Gamma'(\mathbb{Z}_n), z) = z^{15}(z+4)$. For $n = 48$, the order of $\Gamma'(\mathbb{Z}_n)$ is $n - \phi(n) - 1 = 31$. The independent vertex partitions of $\Gamma'(\mathbb{Z}_n)$ are*

$$A_1 = \{2, 10, 14, 22, 26, 34, 38, 46\}, A_2 = \{4, 20, 28, 44\}, A_3 = \{8, 40\}, A_4 = \{16, 32\},$$
$$A_5 = \{3, 9, 15, 21, 27, 33, 39, 45\}, A_6 = \{6, 18, 30, 42\}, A_7 = \{12, 36\}, A_8 = \{24\}.$$

The graph is shown in Figure 2.2. With the independent dominating combination of A_i's as in Theorem 2.3, we have

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^{16} + 4z^{15}.$$

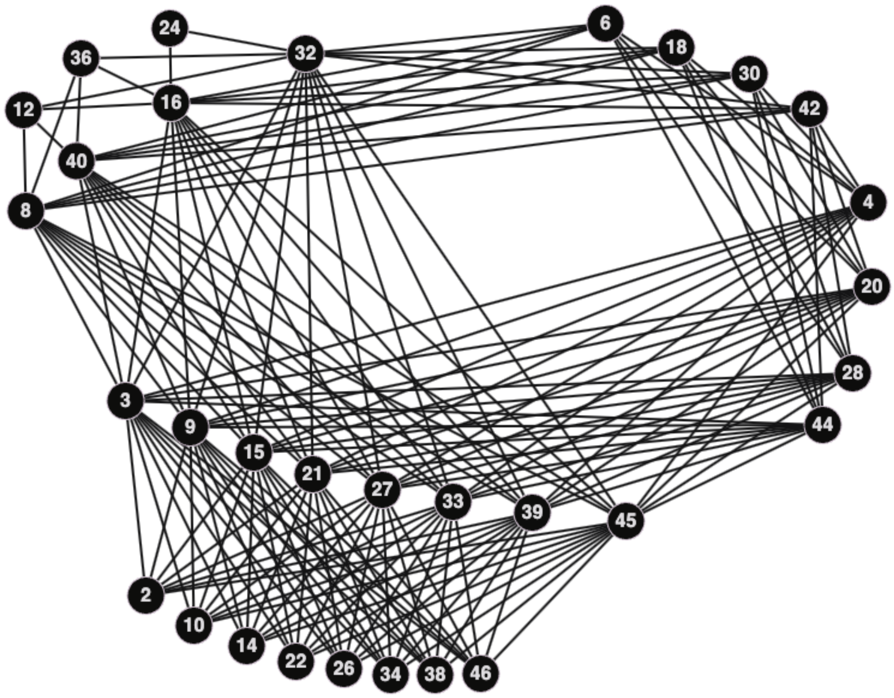

Figure 2.2: Cozero divisor graph $\Gamma'(\mathbb{Z}_{48})$.

Next, we generalize Theorem 2.1 for $n = p_1^{n_1}p_2$, where p_1, p_2 are primes and n_1 is a positive integer. A similar analysis can be carried for $n = p_1p_2^{n_2}$. In order to make calculations simple, we denote p_1 by p and p_2 by q.

Theorem 2.4. *If $n = p^{n_1}q$, then the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is*

$$\begin{aligned} D_i(\Gamma'(\mathbb{Z}_n), z) &= z^{p^{n_1-1}(q-1)} + z^{(q-1)(p^{n_1-1}-1)+p-1} + z^{(q-1)(p^{n_1-1}-p)+p^2-1} + z^{(q-1)(p^{n_1-1}-p^2)+p^3-1} \\ &\quad + \cdots + z^{(p-1)(p^{n_1-1}-p^{i-1})+p^i-1} + \cdots + z^{(q-1)(p^{n_1-1}-p^{n_1-4})+p^{n_1-3}-1} \\ &\quad + z^{(q-1)(p^{n_1-1}-p^{n_1-3})+p^{n_1-2}-1} + z^{(q-1)(p^{n_1-1}-p^{n_1-2})+p^{n_1-1}-1} + z^{p^{n_1}-1}. \end{aligned}$$

Proof. Let $n = p^{n_1}q$ with $p < q$. Then the proper divisors of n are $p^i, i = 1, 2, \ldots, n_1, q$ and p^jq, for $j = 1, 2, \ldots, n_1 - 1$. We partition the vertex set of $\Gamma'(\mathbb{Z}_n)$ as $A_{p^i} = \{a \in \mathbb{Z}_n : (a, n) = p^i\}$ and $A_{p^{i-1}q} = \{b \in \mathbb{Z}_n : (b, n) = p^{i-1}q\}$, where $i = 1, 2, \ldots, n_1$. We denote these sets by $A_i = A_{p^{n_1-i+1}}$ and $B_i = A_{p^{i-1}q}$, for $i = 1, 2, \ldots, n_1$. The cardinality of A_i is $\phi(p^{i-1}q)$ and that of B_i is $\phi(p^{n_1-i+1})$, for $i = 1, 2, \ldots, n_1$. Also A_i's induce a non-empty totally disconnected graph of order $\sum_{i=1}^{n_1} \phi(p^{n_1-i}q) = p^{n_1}(q-1)$, since $\sum_{i=1}^{\eta} \phi(p^i) = p^\eta - 1$, for prime p. Likewise B_i's induce a totally disconnected graph of order $\sum_{i=1}^{n_1} p^i = p^{n_1} - 1$. This implies that no vertex of any A_i is adjacent to any vertex of A_j, for each $i < j$, since $p^j = cp^i$, where c is some scaler. Similarly, no vertex of B_i is adjacent to any vertex of B_j, for each i and j. Thus, there are adjacency relation only between A_i's and B_j's for some i and j. The divisor p^{n_1} is not multiple of any $p^{n_1-i}q$, for $i = 1, 2, \ldots, n_1$. So, the vertices of A_1 are adjacent to all B_i, $i = 1, 2, \ldots, n_1$. For $i = 1, 2, \ldots, n_1 - 2$, the divisor p^{n_1-1} is adjacent to $p^{n_1-i}q$ except $p^{n_1-1}q$, it implies that the vertices of A_2 are adjacent to all B_i except $i = n_1$. Similarly, the set A_{n_1} containing some multiplies of p is adjacent only to set B_1, the set A_{n_1-1} is adjacent to sets B_1 and B_2 and so on. Thus, in general the adjacency among A_i's and B_i's can be represented by the relation: each vertex of A_i is adjacent to every vertex of $\bigcup_{j=1}^{n_1-(i-1)} B_j$, for $i = 1, 2, \ldots, n_1$. Thus, the relations of adjacency between A_i's and B_j's are completely known. Next, we find the independent dominating sets of $\Gamma'(\mathbb{Z}_n)$.

The first possibility for an independent domination set is $D = \bigcup_{i=1}^{n_1} A_i$. In this case vertices of A_1 dominates vertices of all B_i's and so D is an independent domination set of cardinality $p^{n_1-1}(q-1)$.

The second possibility is $D = \bigcup_{i=2}^{n_1} A_i \cup B_{n_1}$, since A_1 dominates all vertices of B_i's and A_2 dominates all B_i's except B_{n_1}. So, it follows that each vertex of $V(\Gamma'(\mathbb{Z}_n)) \setminus D$ is adjacent to at least one vertex of D. Thus D is the another independent dominating set of cardinality $\phi(q)(\phi(p) + \cdots + \phi(p^{n_1-1})) + \phi(p) = \phi(q)(p^{n_1-1} - 1) + \phi(p)$. Next we claim that if we remove any of set from $\bigcup_{i=1}^{n_1} A_i$ other than A_1 and add a suitable subset among B_i's, then the resulting set cannot be an independent dominating set. If we remove any set among A_i's other than A_1, say A_j, $j \neq 1$. Then $D' = \bigcup_{i=1}^{n_1} A_i \setminus A_j$ cannot be a dominating set, since A_i remains missing in such a set. We must add some B_k, so that the resulting set $D = D' \cup B_k$ is an independent dominating set. But we cannot add any of the B_k as A_1 is adjacent to all B_i's and that violates the condition of independence in the independent domination set. Thus, it follows that $\bigcup_{i=2}^{n_1} A_i \cup B_{n_1}$ is the only independent dominating set missing exactly one set among A_i's.

Next, we drop two sets among A_i's and find all possible independent dominating sets. Consider $D = \bigcup_{i=3}^{n_1} A_i \cup B_{n_1-1} \cup B_{n_1}$, then by adjacency relations $\bigcup_{i=3}^{n_1} A_i$ dominates all B_i's except $i = n_1 - 1, n_1$. So, D is an independent dominating set. We claim that D is the only independent dominating set missing exactly two sets among A_i's. Suppose that D' is another dominating set missing any two sets among A_i's except $i = 1, 2$. We assume that A_ℓ and $A_\jmath$ are two such sets, then D' cannot be an independent domination set as A_1 dominates all B_i's and A_2 dominates all B_i's except $i = n_1$. Thus selecting any set among B_i's violates independence property and missing of A_ℓ and $A_\jmath$ breaks the domination condition. So, in this case D is the only independent dominating set missing exactly two sets among A_i's and cardinality of such a set is $\phi(q)(\phi(p^2) + \ldots \phi(p^{n_1-1})) + \phi(p^2) + \phi(p) = \phi(q)(p^{n_1-1} - p) + p^2 - 1$.

Similarly, $D = \bigcup_{i=4}^{n_1} A_i \cup B_{n_1-2} \cup B_{n_1-1} \cup B_{n_1}$ is the unique independent dominating set missing exactly three sets among A_i's. The cardinality of D is $\phi(q)(\phi(p^3) + \ldots \phi(p^{n_1-1})) + \phi(p^3) + \phi(p^2) + \phi(p) = \phi(q)(p^{n_1-1} - p^2) + p^3 - 1$.

Proceeding in a similar fashion at the i-th stage, we must remove the first i sets among A_j's and add the last i sets among B_j's so that the resulting set is the unique independent dominating set. That is, $D = \bigcup_{j=i+1}^{n_1} A_j \cup \bigcup_{j=1}^{i} B_{n_1-(j-1)}$ is the only independent set missing i sets from A_j's and containing i sets from B_i's. The cardinality of this independent domination set is $\phi(q)(\phi(p^i) + \cdots + \phi(p^{n_1-1})) + \sum_{j=1}^{i} \phi(p^j) = \phi(q)(p^{n_1-1} - p^{i-1}) + p^i - 1$.

Continuing in this manner, at the end we have the following cases.

The $(n_1 - 2)$-th case is $D = A_{n_1-2} \cup A_{n_1-1} \cup A_{n_1} \cup \bigcup_{i=4}^{n_1} B_i$ and D is an independent dominating set of cardinality $\phi(q)(\phi(p^{n_1-3}) + \phi(p^{n_1-2}) + \phi(p^{n_1-1})) + \sum_{i=1}^{n_1-3} \phi(p^i) = \phi(q)(p^{n_1-1} - p_{n_1-4}) + p^{n_1-3} - 1$.

As in the third case, $D = A_{n_1-1} \cup A_{n_1} \cup \bigcup_{i=3}^{n_1} B_i$ is the only independent dominating set at (n_1-1)-the stage missing exactly two sets among B_i's. The cardinality of this set is $\phi(q)(\phi(p^{n_1-2}) + \phi(p^{n_1-1})) + \sum_{i=1}^{n_1-2} \phi(p^i) = \phi(q)(p^{n_1-1} - p_{n_1-3}) + p^{n_1-2} - 1$.

Lastly at the n_1-th stage, $D = A_{n_1} \cup \bigcup_{i=2}^{n_1} B_i$ is the unique independent dominating set missing exactly one set among B_i's as in the second case. The order of this set is $\phi(q)(\phi(p^{n_1-1})) + \sum_{i=1}^{n_1-1} \phi(p^i) = \phi(q)(p^{n_1-1} - p_{n_1-2}) + p^{n_1-1} - 1$.

Finally, $D = \bigcup_{i=1}^{n_1} B_i$ is another independent set. Since the vertices of B_1 dominates the vertices of all A_i's and it follows that D is an independent dominating set of cardinality $\sum_{i=1}^{n_1} \phi(p^i) = p^{n_1} - 1$.

With these cases and calculations, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is

$$\begin{aligned} D_i(\Gamma'(\mathbb{Z}_n), z) &= z^{p^{n_1-1}(q-1)} + z^{(q-1)(p^{n_1-1}-1)+p-1} + z^{(q-1)(p^{n_1-1}-p)+p^2-1} + z^{(q-1)(p^{n_1-1}-p^2)+p^3-1} \\ &\quad + \cdots + z^{(p-1)(p^{n_1-1}-p^{i-1})+p^i-1} + \cdots + z^{(q-1)(p^{n_1-1}-p^{n_1-4})+p^{n_1-3}-1} \\ &\quad + z^{(q-1)(p^{n_1-1}-p^{n_1-3})+p^{n_1-2}-1} + z^{(q-1)(p^{n_1-1}-p^{n_1-2})+p^{n_1-1}-1} + z^{p^{n_1}-1}. \end{aligned}$$

Thus, the proof is completed. □

3. Unimodal and log-concave properties of $D_i(\Gamma'(\mathbb{Z}_n), z)$

In this section, we will discuss the unimodal and log-concave properties of $D_i(\Gamma'(\mathbb{Z}_n), z)$ for $n \in \{p_1p_2, p_1p_2p_3, p_1^{n_1}p_2\}$.

A polynomial $p(z) = \sum_{i=0}^{n} a_i z^i$ is said to be *unimodal* if the sequence of its coefficients $\{a_1, a_2, \ldots, a_n\}$ is unimodal, that is, there exists a positive integer p $(0 \le p \le n)$, known as the *mode*, such that $a_0 \le a_1 \le \cdots \le a_p \ge a_{p+1} \ge \cdots \ge a_n$. The count of changes of directions (increasing or decreasing) in the sequence of coefficients of $p(z)$ is known as oscillations, denoted by $\mu(p(x))$. By definition, the oscillations of unimodal polynomial is at most one. The polynomial $p(x) = 1 + 8x + 21x^2 + 8x^3 + x^4 + 9x^5$ is not unimodal, since $\mu(p(x)) = 2$ as there are two increasing oscillations. The polynomial $p(x)$ is symmetric (self reciprocal) if $a_i = a_{n-i}$, for $0 \le i \le \lfloor \frac{n}{2} \rfloor$ and *log-concave* if

$$a_i^2 \ge a_{i-1}a_{i+1}, \text{ for all } 1 \le i \le n-1. \tag{1}$$

The following result shows that the independent domination polynomial of $\mathbb{Z}_{p_1p_2}$ has only one real root.

Proposition 3.1. *Let $D_i(\Gamma'(R, z)$ be an independent domination polynomial of $\Gamma'(R)$. Then $D_i(\Gamma'(R, z)$ has only one real root if $R \cong \mathbb{Z}_{p_1p_2}$, where $2 < p_1 < p_2$ are primes.*

Proof. For $R \cong \mathbb{Z}_n$ with $n = p_1p_2$ and by Proposition 2.1, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^{p_1-1} + z^{p_2-1}.$$

As $2 < p_1 < p_2$, it is easy to see that $p_2 - p_1$ is even and from this it follows that $z^{p_1-1} + z^{p_2-1}$ has only real zero as 0. □

From Proposition 2.1 with $2 \le p_1 < p_2$, the independent domination polynomial $p(z) = z^{p_1-1} + z^{p_2-1}$ satisfies the unimodal property if and only if the exponents of $p(z)$ differ by one, that is, $p_2 - p_1 = 1$. In this case, the coefficients from an increasing sequence and hence $p(z)$ is log-concave. Conversely, it is easy to see that $p(z)$ is unimodal if $p_1 = 2$ and $p_2 = 3$. The log-concavity trivially holds true for $p(z)$.

We make this observation precise in the next result.

Proposition 3.2. *The independent domination polynomial of $\mathbb{Z}_{p_1p_2}$ is log-concave and it is unimodal if and only if $p_2 - p_1 = 1$.*

The next proposition shows that $D_i(\Gamma'(\mathbb{Z}_{p_1p_2p_3}), z)$ is not unimodal.

Proposition 3.3. *For $n = p_1p_2p_3$ with $p_1 < p_3 < p_3$, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is not unimodal.*

Proof. By Theorem 2.1, the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ is

$$D_i(\Gamma'(\mathbb{Z}_n), z) = z^{p_1p_2-p_1} + z^{p_1p_2-p_2} + z^{p_2p_3-p_2} + z^{p_2p_3-p_3} + z^{p_1p_3-p_1} + z^{p_1p_3-p_3}.$$

Since the coefficients of the above polynomial are unity, so we are concerned only with the exponents. In order to violate the unimodal condition, we need to show that there is missing at least one term between any two terms of $D_i(\Gamma'(\mathbb{Z}_n), z)$, as all its exponents are different. Without loss of generality, consider $z^{p_2p_3-p_2}$ and $z^{p_2p_3-p_3}$. If their exponents differ by one, then we get $p_3 - p_2 = 1$, which is always true since $p_1 < p_2 < p_3$. Thus there is at least one term missing between the terms of $D_i(\Gamma'(\mathbb{Z}_n), z)$, which implies that $\mu(D_i(\Gamma'(\mathbb{Z}_n), z)) > 1$. Thus, $D_i(\Gamma'(\mathbb{Z}_n), z)$ cannot be unimodal. □

Remark 3.1. *The independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ for $n = p_1p_2p_3$ with $p_1 < p_3 < p_3$ cannot be always log-concave. We can prove it if we show that there is a missing term with its exponent lying between two non-consecutive exponents of $D_i(\Gamma'(\mathbb{Z}_n), z)$. Consider terms $z^{p_2p_3-p_2}$ and $z^{p_2p_3-p_3}$ and suppose that there difference is two. Then $p_2p_3 - p_2 - (z^{p_2p_3-p_3}) = 2$ implies that $p_3 - p_2 = 2$ and in this case $a^2_{p_2p_3-p_2+1} = 0 \ngeq 1 = a_{p_2p_3-p_2}a_{p_2p_3-p_3}$. Thus, $\Gamma'(\mathbb{Z}_n)$ is not always log-concave. For $G \cong \Gamma'(\mathbb{Z}_n)$ with $n = 2 \cdot 2 \cdot 5$, we have $D_i(G, z) = z^3 + z^4 + z^5 + z^8 + z^{10} + z^{12}$, we see that $a_9^2 \ngeq a_8a_9$ and $a_{11}^2 \ngeq a_{10}a_{12}$. Also, note that $\mu(D_i(G, z)) = 3$, as there are three increasing oscillations.*

The next result is related to log-concave and unimodal properties of the independent domination polynomial of $\Gamma'(\mathbb{Z}_{p_1^2p_2})$.

Proposition 3.4. *The independent domination polynomial of* $D_i(\Gamma'(\mathbb{Z}_n), z)$ *for* $n = p_1^2p_2$ *is not unimodal and it is log-concave if and only if* $p_1 - p_2 \neq 2$.

Proof. From Theorem 2.2, the independent domination polynomial of $G \cong \Gamma'(\mathbb{Z}_n)$ with $n = p_1^2p_2$ and $p_1 > p_2$ is

$$D_i(G), z) = z^{p_1p_2-p_1} + z^{p_1p_2-p_2} + z^{p_1^2-1}.$$

Since $p_1 > p_2$, so $p_1p_2 - p_1 \leq p_1p_2 - p_2 \leq p_1^2 - 1$. The above polynomial is not unimodal if its oscillations are more than one. To prove it, we need to show between two of its terms there exists a zero term. Next, if $p_1^2 - 1 - (p_1p_2 - p_2) \geq 2$, then the solutions of this inequality are $p_1 \geq 3$ and $2 \leq p_2 \leq \frac{p_1^2-3}{p_1-1}$. Since the minimum value of p_1 is 3, so this is true and there always lies a missing term between $z^{p_1p_2-p_2}$ and $z^{p_1^2-1}$, which shows that $\mu(D_i(G), z)) \geq 2$ and unimodal property is violated. For log-concave, we must have $p_1p_2 - p_2 - (p_1p_2 - p_1) \neq 2$ and $p_1^2 - 1 - (p_1p_2 - p_2) \neq 2$, since there exists a missing between two terms with consecutive powers and that contradicts the log-concavity. From this, we obtain $p_1 - p_2 \neq 2$ and $p_1 = 3, p_2 = 2$ or $p_1 \geq 5$ and $2 \leq p_2 < p_1$. The second condition hardly matters as there is always a gap between $z^{p_1p_2-p_2}$ and $z^{p_1^2-1}$. Thus, it follows that $D_i(G), z)$ is log-concave if $p_1 - p_2 \neq 2$. Conversely, if $p_1 > p_2$ are primes and $p_1 - p_2 = 2$, then there is $a_{p_1p_2-p_1+1} = 0$ such that $a^2_{p_1p_2-p_1+1} \not\geq a_{p_1p_2-p_1}a_{p_1p_2-p_2}$ and log-concave property fails. □

By Example 2.2, the independent domination polynomial of $\Gamma'(\mathbb{Z}_{48})$ is $4z^{15} + z^{16}$, which is both log-concave and unimodal. Based on this fact, it remains open to discuss the unimodal and log-concave properties of $\Gamma'(\mathbb{Z}_n)$ for $n = p_1^{n_1}p_2$. Also, further study on the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$ for other general values of n along with their unimodal and log-concave property (especially, their zeros) can be discussed.

4. Conclusion

The independent domination polynomial of cozero divisor graphs of $\mathbb{Z}_n$ has been determined for some special values of n. For general n, it seems to be a hard problem and hence, in the future, it would be interesting to establish more results related to the independent domination polynomial of $\Gamma'(\mathbb{Z}_n)$. By Proposition 3.1, $D_i(\Gamma'(\mathbb{Z}_{p_1p_2}))$ has only one real root for $2 < p_1 < p_2$. Hence, it would be interesting to investigate the zeros of $D_i(\Gamma'(\mathbb{Z}_n), z)$ and the bounds for these zeros.

Acknowledgment

The author is thankful to anonymous referees for their useful comments, which have improved the quality of the paper.

References

[1] M. Afkhami, K. Khashyarmanesh, The cozero divisor graph of a commutative ring, *Southeast Asian Bull. Math.* **35** (2011) 753–762.
[2] M. Afkhami, K. Khashyarmanesh, On the cozero divisor graphs of commutative rings and their complements, *Bull. Malays. Math. Sci. Soc.* **35** (2012) 935–944.
[3] M. Afkhami, K. Khashyarmanesh, Planar, outerplanar, and ring graph of the cozero-divisor graph of a finite commutative ring, *J. Algebra Appl.* **11** (2012) #1250103.
[4] M. Afkhami, K. Khashyarmanesh, On the cozero divisor graphs and comaximal graphs of commutative rings, *J. Algebra Appl.* **12** (2013) #1250173.
[5] S. Alikhani, The domination polynomial of a graph at -1, *Graphs Combin.* **29** (2013) 1175–1181.
[6] S. Alikhani, J. Brown, S. Jahari, On the domination polynomials of friendship graphs, *Filomat* **30** (2016) 169–178.
[7] S. Alikhani, Y. H. Peng, Introduction to domination polynomial of a graph, *Ars Combin.* **114** (2014) 257–266.
[8] A. Alwardi, P.M. Shivaswamy, N. D. Soner, Independent dominating polynomial in graphs, *Int. J. Sci. Inn. Math. Research*, **2** (2014) 757–763.
[9] S. Akbari, F. Alizadeh, S. Khojasteh, Some results on cozero divisor graph of a commutative ring, *J. Algebra Appl.* **13** (2014) #1350113.
[10] S. Akbari, S. Khojasteh, Commutative rings whose cozero divisor graphs are unicyclic or of bounded degree, *Commun. Algebra.* **42** (2014) 1594–1605.
[11] M. Bakhtyiari, R. Nikandish, M. Nikmehr, Coloring of cozero-divisor graphs of commutative von Neumann regular rings, *Proc. Math. Sci.* **130** (2020) 1–7.
[12] S. Chattopadhyay, K. L. Patra, B. K. Sahoo, Laplacian eigenvalues of the zero divisor graph of the ring $\mathbb{Z}_n$, *Linear Algebra Appl.* **584** (2020) 267–286.
[13] M. Dod, The independent domination polynomial, *arXiv*:1602.08250 [math.CO], (2016).
[14] W. Goddard, M. A. Henning, Independent domination in graphs: a survey and recent results, *Discrete Math.* **313 2013** 839–854.
[15] N. K. Gürsoy, A. Ülker, A. Gürsoy, Independent domination polynomial of zero-divisor graphs of commutative rings, *Soft Comp.* **26** (2022) 6989–6997.
[16] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.
[17] S. Jahari, S. Alikhani, On the independent domination polynomial of a graph, *Discrete Appl. Math.* **289** (2021) 416–426.
[18] G. P. Lonzaga, Independent dominating polynomial of corona of some special graphs, *Int. J. Sci. Eng. Res.* **9** (2018) 262–267.
[19] P. Mathil, B. Baloda, J. Kumar, On the cozero divisor graphs associated to rings, *AKCE Int. J. Graphs Comb.* **19** (2022) 38–248.
[20] Z. Mehranian, A. Gholami, A. R. Ashrafi, The spectra of power graphs of certain finite groups, *Linear Multilinear Algebra* **65** (2016) 1003–1010.
[21] R. Nikandish, M. Nikmehr, M. Bakhtyiari, Metric and strong metric dimension in cozero-divisor graphs, *Mediter. J. Math.* **18** (2021) 1–12.
[22] B. A. Rather, A note on the independent domination polynomial of zero divisor graph of rings, *arXiv*:2401.02559 [math.CO], (2024).
[23] B. A. Rather, S. Pirzada, T. A. Chishti, A. M. A. Alghamdi, On normalized Laplacian eigenvalues of power graphs associated to finite cyclic groups, *Discrete Math. Algorithms Appl.* **15** (2022) #2250070.
[24] B. A. Rather, S. Pirzada, T. A. Naikoo, Y. Shang, On Laplacian eigenvalues of the zero-divisor graph associated to the ring of integers modulo n, *Math.* **9** (2021) #482.
[25] M. Young, Adjacency matrices of zero divisor graphs of integer modulo n, *Involve* **8** (2015) 753–761.