Research Article

Some necessary conditions for graphs with extremal connected 2-domination number

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Abstract

Let $G$ be a graph with no multiple edges and loops. A subset $S$ of the vertex set of $G$ is a dominating set of $G$ if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex of $S$. A connected $k$-dominating set of $G$ is a subset $S$ of the vertex set $V(G)$ such that every vertex in $V(G) \setminus S$ has at least $k$ neighbors in $S$ and the subgraph $G[S]$ is connected. The domination number of $G$ is the number of vertices in a minimum dominating set of $G$, denoted by $\gamma(G)$. The connected $k$-domination number of $G$, denoted by $\gamma_k(G)$, is the minimum cardinality of a connected $k$-dominating set of $G$. For $k = 1$, we simply write $\gamma(G)$. It is known that the bounds $\gamma_k(G) \geq \gamma(G) + 1$ and $\gamma_k(G) \geq \gamma(G) + 1$ are sharp. In this research article, we present the necessary condition of the connected graphs $G$ with $\gamma_k(G) = \gamma(G) + 1$ and the necessary condition of the connected graphs $G$ with $\gamma_k(G) = \gamma(G) + 1$. Moreover, we present a graph construction that takes in any connected graph with $r$ vertices and gives a graph $G$ with $\gamma_k(G) = r$, $\gamma_2(G) = r - 1$, and $\gamma(G) \in \{r - 1, r - 2\}$.

Keywords: domination; connected domination; $k$-domination; connected $k$-domination.

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1. Introduction

We refer readers to [3] for notations and graph theory terminology not defined here. In our work, we only consider simple graphs i.e. graphs with no multiple edges and loops. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v$ in $G$, written as $deg_G(v)$, is the number of edges that are incident with $v$. A vertex $v$ of $G$ is said to be a leaf or a pendant if $deg_G(v) = 1$. The vertex that is adjacent to a pendant is its support vertex. A universal vertex in $G$ is a vertex that is adjacent to all other vertices of $G$. For any vertex $v \in V(G)$, the open neighborhood of $v$ in $G$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of $v$ in $G$ is the set $\overline{N}_G[v] = N_G(v) \cup \{v\}$. A graph $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a subset $S$ of $V(G)$, the induced subgraph $G[S]$ is the subgraph of $G$ whose vertex set is $S$ and whose edge set consists of all the edges in $E(G)$ that have both endpoints in $S$. That is, for any two vertices $u, v \in S$, $u$ and $v$ are adjacent in $G[S]$ if and only if they are adjacent in $G$.

A subset $S$ of the vertex set of a graph $G$ is a dominating set if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex of $S$. The domination number of $G$, denoted by $\gamma(G)$, is the number of vertices in a minimum dominating set of $G$. This definition was introduced by Ore [4] in 1962.

Many variations of domination arise from imposing additional conditions on the dominating set. Here, we are interested in connected domination, $k$-domination, and the combination of these two.

A connected dominating set of a connected graph $G$ is a dominating set $S$ of $G$ such that $G[S]$ is connected. The connected domination number of $G$, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of $G$. Any connected dominating set of $G$ of cardinality $\gamma_c(G)$ is called a $\gamma_c$-set of $G$. The concept of connected domination in graphs was introduced by Sampathkumar and Walikar [5] in 1979. Since connected dominating sets are dominating sets, $\gamma(G) \leq \gamma_c(G)$ for any connected graph $G$.

A $k$-dominating set of a graph $G$ is a subset $S$ of the vertex set $V(G)$ such that every vertex in $V(G) \setminus S$ has at least $k$ neighbors in $S$. The $k$-domination number of $G$, denoted by $\gamma_k(G)$, is the minimum cardinality of a $k$-dominating set of $G$. Any $k$-dominating set of $G$ of cardinality $\gamma_k(G)$ is called a $\gamma_k$-set of $G$. The $k$-domination in graphs was introduced by Fink and Jacobson [1] in 1985.

A connected $k$-dominating set of a connected graph $G$ is a subset $S$ of the vertex set $V(G)$ such that every vertex in $V(G) \setminus S$ has at least $k$ neighbors in $S$ and the subgraph $G[S]$ is connected. The connected $k$-domination number of $G$, denoted by $\gamma_k(G)$, is the minimum cardinality of a connected $k$-dominating set of $G$. Any connected $k$-dominating set of $G$ of cardinality $\gamma_k(G)$ is called a $\gamma_k$-set of $G$. In 2009, Volkman [6] introduced the connected $k$-domination in graphs.

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Volkmann [6] characterized connected graphs $G$ with $\gamma^2_2(G) = |V(G)|$. For $\delta(G) \geq k \geq 2$, he also characterized connected graphs $G$ with $\gamma^2_2(G) = |V(G)| - 1$. Moreover, he presented various bounds of $\gamma^2_2(G)$ and proposed some open problems.

The bound $\gamma_k(G) \geq \gamma(G) + k - 2$ for any graph $G$ with $\delta(G) \geq k \geq 2$ was given by Fink and Jacobson in [1]. In 2010, Hansberg [2] presented a bound similar to Fink and Jacobson’s for the connected case, that is $\gamma^2_2(G) \geq \gamma_c(G) + k - 2$ where $\delta(G) \geq k \geq 2$. Moreover, she established various sharp bounds on the connected $k$-domination numbers and the $k$-domination numbers. For $k = 2$, Volkmann [6] established the sharp bound $\gamma^2_2(G) \geq \gamma_c(G) + 1$. This implies that $\gamma^2_2(G) \geq \gamma(G) + 1$.

In this article, we study two of the open problems posed by Volkmann [6] in 2009. In particular, we study graphs with the smallest possible connected $2$-domination numbers with respect to domination numbers and connected domination numbers. We provide a characterization of the connected graphs $G$ with $\gamma(G) = 1$ and $\gamma^2_2(G) = 2$. Moreover, we present a necessary condition of the connected graphs $G$ with $\gamma^2_2(G) = \gamma(G) + 1$ and a necessary condition of the connected graphs $G$ with $\gamma^2_2(G) = \gamma_c(G) + 1$, when $\gamma^2_2(G) \geq 3$. Lastly, we present a graph construction that takes in any connected graph with $k$ vertices and gives a graph $G$ with $\gamma^2_2(G) = k$, $\gamma_c(G) = k - 1$ and $\gamma(G) \in \{k - 1, k - 2\}$.}

### 2. Main results

In this section, we find a necessary condition for a connected graph $G$ to have $\gamma^2_2(G) = \gamma(G) + 1$ and a necessary condition for a connected graph $G$ to have $\gamma^2_2(G) = \gamma_c(G) + 1$. First, we provide a characterization of the connected graphs $G$ with $\gamma(G) = \gamma_c(G) = 1$ and $\gamma^2_2(G) = 2$.

**Observation 2.1.** Let $G$ be a connected graph with $\gamma^2_2(G) = 2$. Let $D$ be a $\gamma^2_2$-set of $G$. Then each vertex in $D$ is a universal vertex. In particular, $\gamma(G) = \gamma_c(G) = 1$.

**Definition 2.1.** The join of disjoint graphs $G$ and $H$, written $G \vee H$, is the graph obtained from the disjoint union of $G$ and $H$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$.

**Theorem 2.1.** Let $G$ be a connected graph of order at least 2. Then the following are equivalent.

(i) $\gamma^2_2(G) = 2$,

(ii) $G \cong K_2 \vee H$ for some graph $H$.

**Proof.** (i) $\Rightarrow$ (ii) Assume that $\gamma^2_2(G) = 2$. Let $\{x, y\}$ be a $\gamma^2_2$-set of $G$. Then $x$ and $y$ are universal vertices of $G$. Hence, $G = G[\{x, y\}] \vee V[G \setminus \{x, y\}]$. Observe that $G[\{x, y\}] \cong K_2$.

(ii) $\Rightarrow$ (i) Assume that $G \cong K_2 \vee H$ for some graph $H$. Then the vertex set of $K_2$ is a $\gamma^2_2$-set of $G$. Hence, $\gamma^2_2(G) = 2$. □

From now on, we only consider connected graphs whose connected $2$-domination numbers are at least 3. The following lemma shows the existence of vertices $x$ and $y$ in a $\gamma^2_2$-set $D$ of a graph $G$ such that $x, y \in N_G(D \setminus \{x, y\})$. This shows that the coming necessary conditions are not null.

**Lemma 2.1.** Let $G$ be a connected graph with $\gamma^2_2(G) \geq 3$. Let $D$ be a $\gamma^2_2$-set of $G$. Then there exist distinct vertices $x, y \in D$ such that $x, y \in N_G(D \setminus \{x, y\})$. Moreover, $x$ and $y$ can be chosen so that $D \setminus \{x, y\}$ is connected.

**Proof.** Since $G[D]$ is connected, there exists a spanning tree $T$ of $G[D]$. Since $T$ is a tree of order greater than 2, it has at least two leaves. Let $x$ and $y$ be two distinct leaves in $T$. Then $x, y \in N_G(D \setminus \{x, y\})$ and $D \setminus \{x, y\}$ is connected. □

The following result provides a necessary condition of the connected graphs $G$ with $\gamma^2_2(G) = \gamma(G) + 1$.

**Theorem 2.2.** Let $G$ be a connected graph with $\gamma^2_2(G) \geq 3$ and $\gamma^2_2(G) = \gamma(G) + 1$. Let $D$ be a $\gamma^2_2$-set of $G$. Then $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$ for every pair of distinct vertices $x$ and $y$ in $D$ such that $x, y \in N_G(D \setminus \{x, y\})$.

**Proof.** Let $x$ and $y$ be two distinct vertices in $D$ such that $x, y \in N_G(D \setminus \{x, y\})$. Suppose that $N_G(x) \cap N_G(y) \subseteq N_G(D \setminus \{x, y\})$. So, the vertices in $N_G(x) \cap N_G(y)$ are dominated by $D \setminus \{x, y\}$. Since $x, y \in N_G(D \setminus \{x, y\})$, the vertices $x$ and $y$ are also dominated by $D \setminus \{x, y\}$. Let $v$ be a vertex of $G$ not in $D \cup (N_G(x) \cap N_G(y))$. Then $v$ is adjacent to at least one vertex in $D \setminus \{x, y\}$. Therefore, $D \setminus \{x, y\}$ is a dominating set of $G$ of size $|D| - 2 = \gamma(G) - 1$, a contradiction. Consequently, $N_G(x) \setminus N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$. □

Similarly, we obtain a necessary condition of the connected graphs $G$ with $\gamma^2_2(G) = \gamma_c(G) + 1$.

**Theorem 2.3.** Let $G$ be a connected graph with $\gamma^2_2(G) \geq 3$ and $\gamma^2_2(G) = \gamma_c(G) + 1$. Let $D$ be a $\gamma^2_2$-set of $G$. Then $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$ for every pair of distinct vertices $x$ and $y$ in $D$ such that $x, y \in N_G(D \setminus \{x, y\})$ and $D \setminus \{x, y\}$ is connected.
After obtaining the necessary conditions, we discover that graphs with such conditions have no universal vertices, as shown in the following propositions.

**Proposition 2.1.** Let \( G \) be a connected graph with \( \gamma'_2(G) \geq 3 \). For every \( \gamma'_2 \)-set \( D \) of \( G \), assume that \( N_G(x) \cap N_G(y) \not\subseteq N_G(D \setminus \{x, y\}) \) for every pair of distinct vertices \( x \) and \( y \) in \( D \) such that \( x, y \in N_G(D \setminus \{x, y\}) \). Then \( G \) has no universal vertices.

**Proof.** Let \( x \) and \( y \) be two distinct vertices in a \( \gamma'_2 \)-set \( D \) of \( G \) such that \( x, y \in N_G(D \setminus \{x, y\}) \). So, \( N_G(x) \cap N_G(y) \not\subseteq N_G(D \setminus \{x, y\}) \). Suppose that \( G \) has a universal vertex \( u \). There are two possibilities.

- **Case 1:** \( u \in D \). Since \( N_G(x) \cap N_G(y) \not\subseteq N_G(D \setminus \{x, y\}) \), there is a vertex \( z \) such that \( z \in N_G(x) \cap N_G(y) \), but \( z \not\in N_G(D \setminus \{x, y\}) \). Suppose that \( u \in D \setminus \{x, y\} \). Since \( u \) is a universal vertex, it is adjacent to \( z \). So, \( z \in N_G(D \setminus \{x, y\}) \), which is a contradiction. Thus, \( u \in \{x, y\} \). Without loss of generality, we assume that \( u = x \). Then \( x \) is adjacent to all vertices in \( D \setminus \{x, y\} \). Since \( |D| \geq 3 \), we have \( D \setminus \{x, y\} \neq \emptyset \). Let \( w \) be a vertex in \( D \setminus \{x, y\} \). Since \( x \) is a universal vertex, the vertices \( w, y \in N_G[x] \subseteq N_G(D \setminus \{w, y\}) \). By the assumption, \( N_G(w) \cap N_G(y) \not\subseteq N_G(D \setminus \{w, y\}) \). However, \( N_G(w) \cap N_G(y) \subseteq N_G[x] \subseteq N_G(D \setminus \{w, y\}) \), a contradiction. Therefore, this case cannot happen.

- **Case 2:** \( u \not\in D \). Then \( u \) is adjacent to every vertex in \( D \). Since \( |D| \geq 3 \), the set \( D \setminus \{x, y\} \neq \emptyset \). Let \( w \) be a neighbor of \( x \) in \( D \setminus \{x, y\} \). Let \( D' = (D \setminus \{w\}) \cup \{u\} \). Since \( u \) is a universal vertex, the set \( D' \) is a connected 2-dominating set of \( G \). Since \( |D'| = |D| \), the set \( D' \) is also a \( \gamma'_2 \)-set of \( G \). However, \( u \in D' \). Just as in Case 1, this cannot happen.

From both cases, we conclude that \( G \) has no universal vertices.

**Proposition 2.2.** Let \( G \) be a connected graph with \( \gamma'_2(G) \geq 3 \). For every \( \gamma'_2 \)-set \( D \) of \( G \), assume that \( N_G(x) \cap N_G(y) \not\subseteq N_G(D \setminus \{x, y\}) \) for every pair of distinct vertices \( x \) and \( y \) in \( D \) such that \( x, y \in N_G(D \setminus \{x, y\}) \) and \( G[D \setminus \{x, y\}] \) is connected. Then \( G \) has no universal vertices.

**Proof.** Similar to the proof of Proposition 2.1.

Next, we use the necessary condition to construct an infinite family of graphs \( G \) that satisfy \( \gamma'_2(G) = \gamma_c(G) + 1 \). Note that the condition \( N_G(x) \cap N_G(y) \not\subseteq N_G(D \setminus \{x, y\}) \) in Theorems 2.2 and 2.3 implies that \( N_G(x) \cap N_G(y) \) must contain a vertex outside of \( N_G(D \setminus \{x, y\}) \).

**Definition 2.2.** For a connected graph \( H \) of order at least 3, we let \( g(H) \) be the connected graph obtained from \( H \) by adding new vertices in the following way. For every pair of distinct vertices \( x \) and \( y \) in \( V(H) \) such that \( x, y \in N_H(V(H) \setminus \{x, y\}) \), we add one new vertex and join it to \( x \) and \( y \).

**Observation 2.2.** For any connected graph \( H \), its vertex set \( V(H) \) is a connected 2-dominating set of \( g(H) \).

![Figure 2.1: Graphs \( P_4 \) and \( g(P_4) \).](image-url)

For example, let \( H \) be a path \( P_4 \) of order 4. The connected graph \( G = g(P_4) \) is obtained from \( P_4 \) by adding the red vertices, as illustrated in Figure 2.1. Note that \( v \not\in N_H(V(H) \setminus \{v, w\}) \) so no new vertex was created for the pair \( v, w \). In this case, we say \( v \) and \( w \) do not create a new vertex in \( G \setminus H \). Similarly, \( x \) and \( y \) do not create a new vertex in \( G \setminus H \). Also, note that each new vertex has degree 2.

The following lemmas discuss some useful properties of graphs \( g(H) \).

**Lemma 2.2.** Let \( H \) be a connected graph of order \( k \) where \( k \geq 3 \) and let \( G = g(H) \). The vertices \( x \) and \( y \) in \( H \) do not create a new vertex in \( G \setminus H \) if and only if \( x \) and \( y \) are adjacent and one of the two vertices has degree 1 in \( H \).

**Proof.** We will prove the forward direction by the contrapositive method. Assume that \( x \) and \( y \) are not adjacent or both \( x \) and \( y \) have degree at least 2 in \( H \). Since \( H \) is a connected graph, it implies that \( x, y \in N_H(V(H) \setminus \{x, y\}) \). By construction, \( x \) and \( y \) create a new vertex in \( G \setminus H \).

Conversely, assume that \( x \) and \( y \) are adjacent and one of the two vertices has degree 1 in \( H \). Without loss of generality, let \( \deg_H(x) = 1 \). Then \( x \in N_H(V(H) \setminus \{x, y\}) \). It follows that \( x \) and \( y \) do not create a new vertex in \( G \setminus H \).
Lemma 2.3. Let $H$ be a connected graph of order $k$ where $k \geq 3$ and let $G = g(H)$. Then among any three vertices of $H$, there exist two vertices that create a new vertex in $G \setminus H$.

Proof. Let $x, y, z \in V(H)$. Suppose there are no pairs of vertices among $x, y$ and $z$ that create a new vertex in $G \setminus H$. By Lemma 2.2 and since $x$ and $y$ do not create a new vertex in $G \setminus H$, the vertices $x$ and $y$ are adjacent and one of the two vertices has degree 1 in $H$, say $y$. Similarly, since $x$ and $z$ do not create a new vertex in $G \setminus H$, the vertices $x$ and $z$ are adjacent and $z$ has degree 1 in $H$. Note that $y$ and $z$ are not adjacent in $H$. By Lemma 2.2, the vertices $y$ and $z$ create a new vertex in $G \setminus H$, a contradiction. Hence, there exist two vertices among $x, y$ and $z$ that create a new vertex in $G \setminus H$. \hfill \Box

Lemma 2.4. Let $H$ be a connected graph of order $k$ where $k \geq 3$ and let $l$ be the number of pendants in $H$. Then

$$|V(g(H))| = k + \binom{k}{2} - l.$$

Proof. Let $G = g(H)$. If every pair of vertices in $H$ creates a new vertex in $G \setminus H$, then the number of new vertices in $G$ is $\binom{k}{2}$. By Lemma 2.2, the number of new vertices in $G$ is $\binom{k}{2} - l$. By Definition 2.2, $|V(G)| = |V(H)| + \binom{k}{2} - l$. \hfill \Box

We proceed to find the connected 2-domination numbers of the graphs $g(H)$. We begin by proving two useful lemmas.

Lemma 2.5. Let $H$ be a connected graph of order $k$ where $k \geq 3$. Let $D$ be a connected 2-dominating set of $g(H)$. If $V(H) \setminus D$ contains a vertex $u$ that does not create new vertices with any vertices in $D \cap V(H)$, then $D \cap V(H)$ is an independent set and $u$ is adjacent to every vertex in $D \cap V(H)$.

Proof. Assume that $V(H) \setminus D$ contains a vertex $u$ that does not create new vertices with any vertices in $D \cap V(H)$. By Lemma 2.2, each vertex in $D \cap V(H)$ is adjacent to the vertex $u$. If $|D \cap V(H)| = 1$, then we are done. Otherwise, we have $deg_H(u) \geq 2$ so each vertex in $D \cap V(H)$ has degree 1 in $H$. Hence, $D \cap V(H)$ is an independent set. \hfill \Box

Lemma 2.6. Let $H$ be a connected graph of order 3 and let $G = g(H)$. Suppose that $D$ is a connected 2-dominating set of $G$ of size 2 such that $D \subseteq V(H)$. If there exist two vertices in $V(H) \setminus D$ that do not create a new vertex in $G$, then $|D \cap V(H)| = 1$.

Proof. Let $V(H) = \{x, y, z\}$. Assume that $x, y \in V(H) \setminus D$ and they do not create a new vertex in $G$. By Lemma 2.2, $x$ and $y$ are adjacent and one of the two has degree 1 in $H$, say $y$. Then $y$ and $z$ create a new vertex $v$ in $G \setminus H$. Next, we will show that $v \in D$. Suppose that $v \notin D$. Since $D$ is a 2-dominating set and $v$ is only adjacent to $y$ and $z$, we have $y, z \in D$. This is a contradiction to $y \in V(H) \setminus D$. It follows that $v \in D$. Suppose that $D \cap V(H) = \emptyset$. Since $|D| = 2$, there exists a vertex $w \in D \setminus \{v\}$. Since $N_G(v) = \{y, z\}$, the vertex $w$ is not adjacent to $v$. This is a contradiction to $G[D]$ being a connected graph. Hence, $|D \cap V(H)| = 1$. \hfill \Box

Theorem 2.4. Let $H$ be a connected graph of order $k$ where $k \geq 3$ and let $G = g(H)$. Then $V(H)$ is a $\gamma_2^G$-set of $G$. In particular, $\gamma_2^G(G) = k$.

Proof. By construction, $V(H)$ is a connected 2-dominating set of $G$ of size $k$. Suppose that there exists a connected 2-dominating set $D$ of $G$ of size $k - 1 \geq 2$. Suppose that $D \subseteq V(H)$. Let $u$ be the single vertex in $V(H) \setminus D$. If $u$ does not create new vertices with any vertices in $D$, then by Lemma 2.5, the set $D$ is independent. This contradicts $G[D]$ being a connected graph. Consequently, $u$ creates a new vertex $v \in G \setminus H$ with some vertex $w$ in $D$. Since $u \notin D$ and $N_G(v) = \{u, w\}$, it follows that $D$ is not a 2-dominating set of $G$, a contradiction. Hence, $D \not\subseteq V(H)$. Then there is at least one vertex in $D$ that does not belong to $V(H)$. So, $|D \cap V(H)| \leq k - 2$. It implies that there exist at least two vertices $x$ and $y$ in $V(H) \setminus D$. There are two possibilities.

Case 1: $x$ and $y$ create a new vertex $z$ in $G \setminus H$. Suppose that $z \in D$. Since $N_G(z) = \{x, y\}$, the graph $G[D]$ is disconnected, a contradiction. Thus, $z \notin D$. Then the new vertex $z$ is not dominated by $D$. This is a contradiction to $D$ being a 2-dominating set of $G$.

Case 2: $x$ and $y$ do not create a new vertex in $G \setminus H$. By Lemma 2.2, the two vertices are adjacent and one of the two has degree 1 in $H$, say $y$. Note that $|V(H) \setminus \{x, y\}| = |V(H)| - 2 = k - 2$. Let $V(H) \setminus \{x, y\} = \{u_1, u_2, \ldots, u_{k-2}\}$. Since $H$ is a connected graph and $y$ is adjacent to $x$ in $V(H) \setminus D$, for each $i \in \{1, \ldots, k-2\}$, we have that $u_i, y \in N_H(V(H) \setminus \{u_i, y\})$ so $u_i$ and $y$ create a new vertex $v_i$ in $G \setminus H$. Let $S = \{v_1, v_2, \ldots, v_{k-2}\}$. Next, we will show that $S \not\subseteq D$. Suppose that there exists an $i \in \{1, \ldots, k-2\}$ such that $v_i \notin D$. Since $D$ is a 2-dominating set and $N_G(v_i) = \{u_i, y\}$, the vertices $u_i$ and $y$ are in $D$. This is a contradiction to $y \in V(H) \setminus D$. It implies that $v_i \in D$ for all $i \in \{1, \ldots, k-2\}$. So, $S \not\subseteq D$. 


If \( k = 3 \), then \(|S| = 1\) and \(|D| = 2\). Thus, \( S = \{v_1\} \). By Lemma 2.6, \(|D \cap V(H)| = 1\). Since \( V(H) = \{x, y, u_1\} \) and \( x, y \notin D \), we have \( D \cap V(H) = \{u_1\} \). Since \( S \subseteq D \), the vertex \( v_1 \) belongs to \( D \setminus V(H) \). Thus, \( D = \{u_1, v_1\} \). Since \( y \) is a pendant with \( x \) as its support, \( y \) is not adjacent to \( u_1 \). It follows that \( D \) is not a 2-dominating set of \( G \), a contradiction. Thus, \( k \neq 3 \).

Now, suppose \( k \geq 4 \) so there exist at least 2 vertices in \( S \). By construction, \( S \) is an independent set. Since each vertex \( v_1 \) in \( S \) is created by joining it to \( y \) and \( u_1 \in V(H) \setminus \{x, y\} \), the vertices in \( S \) have only one common neighbor, namely \( y \). But \( y \) is not in \( D \). Since \( S \subseteq D \) and \(|D \setminus S| = 1\), the induced subgraph \( G[D] \) is disconnected, a contradiction.

We conclude from the above two cases that a connected 2-dominating set of \( G \) has at least \( k \) members. Therefore, \( V(H) \) is a \( \gamma_2^G \)-set of \( G \) and \( \gamma_2^G(G) = k \).

**Theorem 2.5.** Let \( H \) be a connected graph of order \( k \geq 3 \) not isomorphic to a path on 3 vertices and let \( G = g(H) \). Then \( V(H) \) is the unique \( \gamma_2^G \)-set of \( G \).

**Proof.** By Theorem 2.4, we have that \( V(H) \) is a \( \gamma_2^G \)-set of \( G \). If \( k = 3 \), then \( H \) is a cycle on 3 vertices and it is easy to check that \( V(H) \) is the only \( \gamma_2 \)-set of \( G \). It remains to consider \( k \geq 4 \). Suppose that there exists a \( \gamma_2 \)-set \( D \) of \( G \) such that \( D \neq V(H) \). So, \(|D| = |V(H)|\) and \(|V(H) \setminus D| = |D \setminus V(H)|\). Consider the following 3 cases.

- **Case 1:** \(|V(H) \setminus D| = |D \setminus V(H)| = 1\). Let \( u \) be the unique vertex in \( V(H) \setminus D \). Suppose that \( u \) does not create new vertices with any vertices in \( D \cap V(H) \). By Lemma 2.5, the set \( D \cap V(H) \) is independent and \( u \) is adjacent to every vertex in \( D \cap V(H) \). Since \( D \cap V(H) \) is an independent set of size at least 3 and the unique vertex in \( D \setminus V(H) \) has degree 2, the graph \( G[D] \) is disconnected, a contradiction. Therefore, \( u \) creates new vertices with some vertices in \( D \cap V(H) \). Suppose \( u \) creates exactly one new vertex. Let \( a \) be the vertex in \( D \cap V(H) \) that creates the new vertex with \( u \). Since \( k \geq 4 \) and \( |V(H) \setminus D| = 1 \), we have \(|(D \cap V(H)) \setminus \{a\}| \geq 2 \). By Lemma 2.2, every vertex in \( (D \cap V(H)) \setminus \{a\} \) is adjacent to \( u \) and has degree 1 in \( H \). So, \( u \) is not adjacent to any vertex in \( (D \cap V(H)) \setminus \{a\} \). Thus, \( N_H(a) \subseteq \{u\} \). By this and Lemma 2.2, the vertices \( a \) and \( u \) are not adjacent. Therefore, \( a \) is not adjacent to any vertices in \( V(H) \setminus \{a\} \). Consequently, \( H \) is disconnected, a contradiction. Thus, \( u \) creates at least two new vertices with some vertices in \( D \cap V(H) \). Since \(|D \setminus V(H)| = 1\), at least one of the new vertices above is not in \( D \) and is not 2-dominated by \( D \), a contradiction.

- **Case 2:** \(|V(H) \setminus D| = |D \setminus V(H)| = 2\). Let \( V(H) \setminus D = \{x, y\} \). Suppose that \( x \) and \( y \) create a new vertex \( z \) in \( G \setminus H \). Suppose that \( z \in D \). Since \( \deg_G(z) = 2 \), the graph \( G[D] \) is disconnected, a contradiction. So, \( z \notin D \). Thus, \( D \) is not a dominating set of \( G \), a contradiction. Therefore, \( x \) and \( y \) do not create a new vertex in \( G \setminus H \). By Lemma 2.2, the vertices \( x \) and \( y \) are adjacent and one of the two has degree 1 in \( H \), say \( y \).

Now, suppose \( x \) does not create new vertices with any vertices in \( D \cap V(H) \). By Lemma 2.5, the set \( D \cap V(H) \) is independent and \( x \) is adjacent to every vertex in \( D \cap V(H) \). Since \( D \cap V(H) \) is an independent set of size at least 2, the graph \( H \) is a star with at least 3 pendant vertices. By Lemma 2.3, there exist at least \(|D \cap V(H)| - 1\) new vertices in \( G \) that are created by joining them to \( y \) and \( D \cap V(H) \). If \(|D \cap V(H)| > 2\), then at least one of the new vertices above is not in \( D \) and so it is not 2-dominated by \( D \), a contradiction. Thus, \(|D \cap V(H)| = 2 \) and \( H \) is a star of order 4. By Lemma 2.4, the number of new vertices in \( g(H) \) is three. Suppose that two new vertices in \( g(H) \) that are created by joining them to \( y \) and \( D \cap V(H) \) belong to \( D \setminus V(H) \). Since both of the two new vertices have degree two and \( D \cap V(H) \) is an independent set, the graph \( G[D] \) is disconnected, a contradiction. Hence, at least one of the two new vertices in \( g(H) \) that is created by joining them to \( y \) and \( D \cap V(H) \) does not belong to \( D \), and so it is not 2-dominated by \( D \), a contradiction. Therefore, \( x \) creates new vertices with some vertices in \( D \cap V(H) \).

Since \( y \) is a pendant with \( x \) as its support, by Lemma 2.6 the vertex \( y \) creates a new vertex with each vertex in \( D \cap V(H) \). It follows that there exist at least \(|D \cap V(H)| + 1 \geq 3\) new vertices in \( G \) that are adjacent to \( x \) or \( y \). Since \(|D \setminus V(H)| = 2\), at least one of the new vertices above is not in \( D \) and is not 2-dominated by \( D \), a contradiction.

- **Case 3:** \(|V(H) \setminus D| \geq 3\). Let \( x, y, z \in V(H) \setminus D \). By Lemma 2.3, there exist two vertices in \( \{x, y, z\} \) that create a new vertex in \( G \). Without loss of generality, let \( x \) and \( y \) create a new vertex \( v \) in \( G \setminus H \). Suppose that \( v \in D \). Since \( \deg_G(v) = 2 \), the graph \( G[D] \) is disconnected, a contradiction. So, \( v \notin D \). Thus, \( D \) is not a dominating set of \( G \), a contradiction.

From the above three cases, we conclude that \( V(H) \) is the unique \( \gamma_2^G \)-set of \( G \).
Next, we find the connected domination numbers of the graphs \( g(H) \) and show how they relate to the connected 2-domination numbers.

**Theorem 2.6.** Let \( H \) be a connected graph of order \( k \) where \( k \geq 3 \) and let \( G = g(H) \). Then \( \gamma_c(G) = k - 1 \).

**Proof.** Let \( S \) be a subset of \( V(H) \) such that \( |S| = k - 1 \) and \( G[S] \) is connected. Since \( V(H) \) is a 2-dominating set of \( G \), the set \( S \) is a connected dominating set of \( G \). Thus, \( \gamma_c(G) \leq k - 1 \). Suppose that there exists a connected dominating set \( D \) of \( G \) of size \( k - 2 \). Suppose that \( D \subseteq V(H) \). Then there exist \( u, v \in V(H) \setminus D \). We consider the vertices \( u \) and \( v \) in \( V(H) \setminus D \) in two cases.

- **Case 1:** \( u \) and \( v \) create a new vertex in \( G \setminus H \). Then the new vertex is not dominated by \( D \). This is a contradiction to \( D \) being a dominating set.
- **Case 2:** \( u \) and \( v \) do not create a new vertex in \( G \setminus H \). By Lemma 2.7, \( u \) and \( v \) are adjacent and one of the two has degree \( 1 \) in \( V(H) \), say \( v \). Then \( v \) is not dominated by \( D \), a contradiction.

From the above two cases, we conclude that \( D \not\subseteq V(H) \). Then at least one vertex in \( D \) does not belong to \( V(H) \). So, \( |D \cap V(H)| \leq k - 3 \). It implies that there exist at least 3 vertices in \( V(H) \setminus D \). Let \( x, y, z \in V(H) \setminus D \). By Lemma 2.3, there exist two vertices in \( V(H) \setminus D \) that create a new vertex in \( G \setminus H \). Without loss of generality, let \( x \) and \( y \) create a new vertex \( t \) in \( G \setminus H \). Suppose that \( t \in D \). Since \( N_G(t) = \{x, y\} \), we have that \( t \not\in N_G(D) \), a contradiction. So, \( t \not\in D \). It follows that the new vertex \( t \) in \( G \) is not dominated by \( D \), a contradiction. Hence, a connected dominating set of \( G \) has at least \( k - 1 \) members. Therefore, \( \gamma_c(G) = k - 1 \).

**Corollary 2.1.** Let \( H \) be a connected graph of order \( k \) where \( k \geq 3 \) and let \( G = g(H) \). Then \( \gamma_2(G) = \gamma_c(G) + 1 \).

Now, we show that for any connected graph \( H \) of order at least 3, the graph \( g(H) \) satisfies either \( \gamma_2(g(H)) = \gamma(g(H)) + 1 \) or \( \gamma_2(g(H)) = \gamma(g(H)) + 2 \).

**Theorem 2.7.** Let \( H \) be a connected graph of order \( 3 \) and let \( G = g(H) \). Then \( \gamma(G) = 2 \).

**Proof.** Since \( H \) is a connected graph of order 3, it follows that \( H \) is either a path \( P_3 \) or a cycle \( C_3 \) of order 3. Since \( g(P_3) \) is a cycle of order 4, it implies that \( \gamma(g(P_3)) = 2 \). Next, we show that \( \gamma(g(C_3)) = 2 \). By Lemma 2.4, we have that \( |V(g(C_3))| = 6 \). Since the maximum degree of \( g(C_3) \) equals 4, no single vertex in \( g(C_3) \) can dominate all vertices in \( g(C_3) \). Thus, \( \gamma(g(C_3)) \geq 2 \). Clearly, any two vertices in \( V(C_3) \) form a dominating set of \( g(C_3) \). Hence, \( \gamma(g(C_3)) \leq 2 \). Therefore, \( \gamma(g(C_3)) = 2 \).

**Lemma 2.7.** Let \( H \) be a connected graph of order \( k \) where \( k \geq 4 \) and let \( G = g(H) \). Then \( \gamma(G) \geq k - 2 \).

**Proof.** Let \( V(H) = \{v_1, v_2, v_3, \ldots, v_k\} \). Let \( X = V(G) \setminus V(H) \). Then \( X \) consists of the new vertices. Suppose there exists \( D \subseteq V(G) \) such that \( |D| = k - 3 \) and \( D \) dominates \( X \). If \( D \) contains a new vertex \( x \) in \( X \), then \( x \) was created by some vertices \( u \) and \( v \) in \( H \). Since \( N_G[x] \cap X \subseteq N_G[u] \cap X \), we can use the vertex \( u \) in \( H \) to dominate new vertices in \( X \) instead of the vertex \( x \). Hence, it is sufficient to consider that the vertices in \( D \) are from \( V(H) \). Without loss of generality, let \( D = \{v_1, v_2, v_3, \ldots, v_{k-3}\} \). We divide the argument into two cases according to the number of pendants in \( \{v_{k-2}, v_{k-1}, v_k\} \).

- **Case 1:** \( \{v_{k-2}, v_{k-1}, v_k\} \) contains at most one pendant. Without loss of generality, assume \( v_{k-1} \) and \( v_k \) are not pendants. By Lemma 2.2, \( v_{k-1} \) and \( v_k \) create a new vertex in \( G \setminus H \) which is not dominated by \( D \), a contradiction.

- **Case 2:** \( \{v_{k-2}, v_{k-1}, v_k\} \) contains at least two pendants. Without loss of generality, assume \( v_{k-1} \) and \( v_k \) are the two pendants. By Lemma 2.2, \( v_{k-1} \) and \( v_k \) create a new vertex in \( G \setminus H \) which is not dominated by \( D \), a contradiction.

We conclude from the above two cases that at least \( k - 2 \) vertices are required to dominate \( X \). Thus, \( \gamma(G) \geq k - 2 \).

**Theorem 2.8.** Let \( H \) be a connected graph of order \( k \) where \( k \geq 4 \) and let \( G = g(H) \). If \( H \) contains two pendants that share a support vertex in \( H \), then \( \gamma(G) = k - 2 \).

**Proof.** Let \( V(H) = \{v_1, v_2, v_3, \ldots, v_k\} \). Assume that \( H \) contains 2 pendants that share a support vertex in \( H \). For \( i \neq j \), when \( v_i \) and \( v_j \) create a new vertex in \( G \setminus H \), we let \( v_{ij} \) denote the new vertex. Since \( |V(H)| = k \geq 4 \), no two pendants are adjacent. Without loss of generality, let \( v_{k-1} \) and \( v_k \) be two pendants of \( H \) with the common support vertex \( v_{k-2} \). Let \( D = \{v_1, v_2, v_3, \ldots, v_{k-3}\} \cup \{v_{k-1}, v_k\} \). By Lemma 2.2, \( v_{k-2} \) does not create a new vertex with either \( v_{k-1} \) or \( v_k \). Since \( H \) is connected, the vertex \( v_{k-2} \) is adjacent to some vertex in \( \{v_1, v_2, \ldots, v_{k-3}\} \). By construction, all vertices in \( G \) except \( v_{k-1}, v_k \) and \( v_{k-1}, v_k \) are dominated by \( \{v_1, v_2, \ldots, v_{k-3}\} \) but \( v_{k-1}, v_k \) and \( v_{k-1}, v_k \) are dominated by \( v_{k-1}, v_k \). Hence, \( D \) dominates all vertices in \( G \). Since \( |D| = k - 2 \), we have that \( \gamma(G) \leq k - 2 \). By Lemma 2.7, we have \( \gamma(G) = k - 2 \).
Theorem 2.9. Let $H$ be a connected graph of order $k$ such that $k \geq 4$ and no two pendants share a support vertex. Let $G = g(H)$. Then $\gamma(G) = k - 1$.

Proof. Let $V(H) = \{v_1, v_2, v_3, \ldots, v_k\}$. Let $X = V(G) \setminus V(H)$. For $i \neq j$, when $v_i$ and $v_j$ create a new vertex in $G \setminus H$, we let $v_{ij}$ denote the new vertex. Suppose there exists $D \subseteq V(G)$ such that $|D| = k - 2$ and $D$ dominates $X$. Similar to the proof of Theorem 2.7, we can assume that $D \subseteq V(H)$ and let $D = \{v_1, v_2, v_3, \ldots, v_{k-2}\}$. Let $\alpha$ be the number of vertices in $X$ that are dominated by $D$. Let $l$ be the number of pendants in $H$. By Lemma 2.4, we have

$$\alpha = |X| = \binom{k}{2} - l.$$ 

We will also compute $\alpha$ by counting the number of additional vertices that are dominated by each $v_i$ for $1 \leq i \leq k - 2$. By Lemma 2.2, for each $v \in D$, if $v$ is a pendant or a support of a pendant, then $v$ is adjacent to $k - 2$ vertices in $X$; otherwise, $v$ is adjacent to $k - 1$ vertices in $X$.

First, suppose that both $v_{k-1}$ and $v_k$ are not pendants in $H$. Then all $l$ pendants are in $D$, so

$$\alpha = (k - 1) + (k - 2) + \cdots + 2 - l = \binom{k}{2} - 1 - l.$$ 

Thus, $\alpha < |X|$, which is a contradiction.

Suppose that both $v_{k-1}$ and $v_k$ are pendants in $H$. Then the support vertices of $v_{k-1}$ and $v_k$ are distinct and are in $D$. It implies that $\alpha = (k - 1) + (k - 2) + \cdots + 2 - l = \binom{k}{2} - 1 - l$. Thus, $\alpha < |X|$, a contradiction.

Therefore, exactly one vertex in $\{v_{k-1}, v_k\}$ is a pendant in $H$. Then $D$ contains $l - 1$ pendants. Without loss of generality, let $v_k$ be a pendant. First, suppose that the support vertex of $v_k$ is in $D$. It follows that $\alpha = (k - 1) + (k - 2) + \cdots + 2 - l = \binom{k}{2} - 1 - l$. Thus, $\alpha < |X|$, a contradiction. Thus, the support vertex of $v_k$ is not in $D$, i.e. $v_{k-1}$ is the support vertex of $v_k$. Then

$$\alpha = (k - 1) + (k - 2) + \cdots + 2 - (l - 1) = \binom{k}{2} - l.$$ 

It follows that we need at least $k - 2$ vertices to dominate every vertex in $X$. Each vertex $v_i$ in $D$ dominates at least 2 additional vertices $v_{i,k-1}$ and $v_{i,k}$. Each vertex $v_{ij}$ in $X$ can only dominate one vertex (itself) in $X$. So, to use exactly $k - 2$ vertices to dominate $X$, we cannot use any vertex from $X$. Since the pendant $v_k$ and its support vertex $v_{k-1}$ are not in $D$, the vertex $v_k$ is not dominated by $D$. Thus, we must use one more vertex to dominate $v_k$. Then a dominating set of $G$ has at least $k - 1$ members. So, $\gamma(G) \geq k - 1$.

Let $D' = \{v_1, v_2, v_3, \ldots, v_{k-1}\}$. Clearly, $D'$ dominate all vertices in $G$. Since $|D'| = k - 1$, we have that $\gamma(G) \leq k - 1$. Therefore, $\gamma(G) = k - 1$. \hfill \Box

**Figure 2.2:** Graphs $S_5$ and $g(S_5)$.

Remark 2.1. Theorems 2.4 and 2.8 imply that our necessary condition for graphs $G$ with $\gamma'_2(G) = \gamma(G) + 1$ is not a sufficient condition.

Lastly, we apply Theorems 2.4, 2.6, 2.7, 2.8, and 2.9 to stars, paths, and cycles. We let $S_k$, $P_k$ and $C_k$ denote a star, a path and a cycle of order $k$, respectively.

**Corollary 2.2.** For $k \geq 4$, let $G = g(S_k)$. Then $\gamma'_2(G) = k$, $\gamma_c(G) = k - 1$ and $\gamma(G) = k - 2$.

**Corollary 2.3.** For $k \geq 3$, let $G = g(P_k)$. Then $\gamma'_2(G) = k$, $\gamma_c(G) = k - 1$ and $\gamma(G) = k - 1$.

**Corollary 2.4.** For $k \geq 3$, let $G = g(C_k)$. Then $\gamma'_2(G) = k$, $\gamma_c(G) = k - 1$ and $\gamma(G) = k - 1$.

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References


