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*Research Article*

# Some necessary conditions for graphs with extremal connected 2-domination number

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**Abstract**

Let $G$ be a graph with no multiple edges and loops. A subset $S$ of the vertex set of $G$ is a dominating set of $G$ if every vertex in $V(G)\setminus S$ is adjacent to at least one vertex of $S$. A connected $k$-dominating set of $G$ is a subset $S$ of the vertex set $V(G)$ such that every vertex in $V(G)\setminus S$ has at least $k$ neighbors in $S$ and the subgraph $G[S]$ is connected. The domination number of $G$ is the number of vertices in a minimum dominating set of $G$, denoted by $\gamma(G)$. The connected $k$-domination number of $G$, denoted by $\gamma_k^c(G)$, is the minimum cardinality of a connected $k$-dominating set of $G$. For $k=1$, we simply write $\gamma_c(G)$. It is known that the bounds $\gamma_2^c(G)\geqslant\gamma(G)+1$ and $\gamma_2^c(G)\geqslant\gamma_c(G)+1$ are sharp. In this research article, we present the necessary condition of the connected graphs $G$ with $\gamma_2^c(G)=\gamma(G)+1$ and the necessary condition of the connected graphs $G$ with $\gamma_2^c(G)=\gamma_c(G)+1$. Moreover, we present a graph construction that takes in any connected graph with $r$ vertices and gives a graph $G$ with $\gamma_2^c(G)=r$, $\gamma_c(G)=r-1$, and $\gamma(G)\in\{r-1,r-2\}$.

**Keywords:** domination; connected domination; $k$-domination; connected $k$-domination.

**2020 Mathematics Subject Classification:** 05C69.

## 1. Introduction

We refer readers to [3] for notations and graph theory terminology not defined here. In our work, we only consider simple graphs i.e. graphs with no multiple edges and loops. Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *degree* of a vertex $v$ in $G$, written as $deg_G(v)$, is the number of edges that are incident with $v$. A vertex $v$ of $G$ is said to be *a leaf* or *a pendant* if $deg_G(v)=1$. The vertex that is adjacent to a pendant is its *support vertex*. A *universal vertex* in $G$ is a vertex that is adjacent to all other vertices of $G$. For any vertex $v\in V(G)$, the *open neighborhood* of $v$ in $G$ is the set $N_G(v)=\{u\in V(G):uv\in E(G)\}$ and the *closed neighborhood* of $v$ in $G$ is the set $N_G[v]=N_G(v)\cup\{v\}$. A graph $H$ is a *subgraph* of $G$ if $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$. For a subset $S$ of $V(G)$, the *induced subgraph* $G[S]$ is the subgraph of $G$ whose vertex set is $S$ and whose edge set consists of all the edges in $E(G)$ that have both endpoints in $S$. That is, for any two vertices $u,v\in S$, $u$ and $v$ are adjacent in $G[S]$ if and only if they are adjacent in $G$.

A subset $S$ of the vertex set of a graph $G$ is a *dominating set* if every vertex in $V(G)\setminus S$ is adjacent to at least one vertex of $S$. The *domination number* of $G$, denoted by $\gamma(G)$, is the number of vertices in a minimum dominating set of $G$. This definition was introduced by Ore [4] in 1962.

Many variations of domination arise from imposing additional conditions on the dominating set. Here, we are interested in connected domination, $k$-domination, and the combination of these two.

A *connected dominating set* of a connected graph $G$ is a dominating set $S$ of $G$ such that $G[S]$ is connected. The *connected domination number* of $G$, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of $G$. Any connected dominating set of $G$ of cardinality $\gamma_c(G)$ is called a $\gamma_c$*-set* of $G$. The concept of connected domination in graphs was introduced by Sampathkumar and Walikar [5] in 1979. Since connected dominating sets are dominating sets, $\gamma(G)\leqslant\gamma_c(G)$ for any connected graph $G$.

A $k$*-dominating set* of a graph $G$ is a subset $S$ of the vertex set $V(G)$ such that every vertex in $V(G)\setminus S$ has at least $k$ neighbors in $S$. The $k$*-domination number* of $G$, denoted by $\gamma_k(G)$, is the minimum cardinality of a $k$-dominating set of $G$. Any $k$-dominating set of $G$ of cardinality $\gamma_k(G)$ is called a $\gamma_k$*-set* of $G$. The $k$-domination in graphs was introduced by Fink and Jacobson [1] in 1985.

A *connected* $k$*-dominating set* of a connected graph $G$ is a subset $S$ of the vertex set $V(G)$ such that every vertex in $V(G)\setminus S$ has at least $k$ neighbors in $S$ and the subgraph $G[S]$ is connected. The *connected* $k$*-domination number* of $G$, denoted by $\gamma_k^c(G)$, is the minimum cardinality of a connected $k$-dominating set of $G$. Any connected $k$-dominating set of $G$ of cardinality $\gamma_k^c(G)$ is called a $\gamma_k^c$*-set* of $G$. In 2009, Volkmann [6] introduced the connected $k$-domination in graphs.

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Volkmann [6] characterized connected graphs $G$ with $\gamma_k^c(G) = |V(G)|$. For $\delta(G) \geqslant k \geqslant 2$, he also characterized connected graphs $G$ with $\gamma_k^c(G) = |V(G)| - 1$. Moreover, he presented various bounds of $\gamma_k^c(G)$ and proposed some open problems.

The bound $\gamma_k(G) \geqslant \gamma(G) + k - 2$ for any graph $G$ with $\delta(G) \geqslant k \geqslant 2$ was given by Fink and Jacobson in [1]. In 2010, Hansberg [2] presented a bound similar to Fink and Jacobson's for the connected case, that is $\gamma_k^c(G) \geqslant \gamma_c(G) + k - 2$ where $\delta(G) \geqslant k \geqslant 2$. Moreover, she established various sharp bounds on the connected $k$-domination numbers and the $k$-domination numbers. For $k = 2$, Volkmann [6] established the sharp bound $\gamma_2^c(G) \geqslant \gamma_c(G) + 1$. This implies that $\gamma_2^c(G) \geqslant \gamma(G) + 1$.

In this article, we study two of the open problems posted by Volkmann [6] in 2009. In particular, we study graphs with the smallest possible connected 2-domination numbers with respect to domination numbers and connected domination numbers. We provide a characterization of the connected graphs $G$ with $\gamma(G) = 1$ and $\gamma_2^c(G) = 2$. Moreover, we present a necessary condition of the connected graphs $G$ with $\gamma_2^c(G) = \gamma(G) + 1$ and a necessary condition of the connected graphs $G$ with $\gamma_2^c(G) = \gamma_c(G) + 1$, when $\gamma_2^c(G) \geqslant 3$. Lastly, we present a graph construction that takes in any connected graph with $k$ vertices and gives a graph $G$ with $\gamma_2^c(G) = k$, $\gamma_c(G) = k - 1$ and $\gamma(G) \in \{k - 1, k - 2\}$.

## 2. Main results

In this section, we find a necessary condition for a connected graph $G$ to have $\gamma_2^c(G) = \gamma(G) + 1$ and a necessary condition for a connected graph $G$ to have $\gamma_2^c(G) = \gamma_c(G) + 1$. First, we provide a characterization of the connected graphs $G$ with $\gamma(G) = \gamma_c(G) = 1$ and $\gamma_2^c(G) = 2$.

**Observation 2.1.** *Let $G$ be a connected graph with $\gamma_2^c(G) = 2$. Let $D$ be a $\gamma_2^c$-set of $G$. Then each vertex in $D$ is a universal vertex. In particular, $\gamma(G) = \gamma_c(G) = 1$.*

**Definition 2.1.** *The join of disjoint graphs $G$ and $H$, written $G \vee H$, is the graph obtained from the disjoint union of $G$ and $H$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$.*

**Theorem 2.1.** *Let $G$ be a connected graph of order at least 2. Then the following are equivalent.*

*(i)* $\gamma_2^c(G) = 2$,

*(ii)* $G \cong K_2 \vee H$ *for some graph $H$.*

**Proof.** $(i) \Rightarrow (ii)$ Assume that $\gamma_2^c(G) = 2$. Let $\{x, y\}$ be a $\gamma_2^c$-set of $G$. Then $x$ and $y$ are universal vertices of $G$. Hence, $G = G[\{x, y\}] \vee G[V(G) \setminus \{x, y\}]$. Observe that $G[\{x, y\}] \cong K_2$.

$(ii) \Rightarrow (i)$ Assume that $G \cong K_2 \vee H$ for some graph $H$. Then the vertex set of $K_2$ is a $\gamma_2^c$-set of $G$. Hence, $\gamma_2^c(G) = 2$. □

From now on, we only consider connected graphs whose connected 2-domination numbers are at least 3. The following lemma shows the existence of vertices $x$ and $y$ in a $\gamma_2^c$-set $D$ of a graph $G$ such that $x, y \in N_G(D \setminus \{x, y\})$. This shows that the coming necessary conditions are not null.

**Lemma 2.1.** *Let $G$ be a connected graph with $\gamma_2^c(G) \geqslant 3$. Let $D$ be a $\gamma_2^c$-set of $G$. Then there exist distinct vertices $x, y \in D$ such that $x, y \in N_G(D \setminus \{x, y\})$. Moreover, $x$ and $y$ can be chosen so that $G[D \setminus \{x, y\}]$ is connected.*

**Proof.** Since $G[D]$ is connected, there exists a spanning tree $T$ of $G[D]$. Since $T$ is a tree of order greater than 2, it has at least two leaves. Let $x$ and $y$ be two distinct leaves in $T$. Then $x, y \in N_G(D \setminus \{x, y\})$ and $G[D \setminus \{x, y\}]$ is connected. □

The following result provides a necessary condition of the connected graphs $G$ with $\gamma_2^c(G) = \gamma(G) + 1$.

**Theorem 2.2.** *Let $G$ be a connected graph with $\gamma_2^c(G) \geqslant 3$ and $\gamma_2^c(G) = \gamma(G) + 1$. Let $D$ be a $\gamma_2^c$-set of $G$. Then $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$ for every pair of distinct vertices $x$ and $y$ in $D$ such that $x, y \in N_G(D \setminus \{x, y\})$.*

**Proof.** Let $x$ and $y$ be two distinct vertices in $D$ such that $x, y \in N_G(D \setminus \{x, y\})$. Suppose that $N_G(x) \cap N_G(y) \subseteq N_G(D \setminus \{x, y\})$. So, the vertices in $N_G(x) \cap N_G(y)$ are dominated by $D \setminus \{x, y\}$. Since $x, y \in N_G(D \setminus \{x, y\})$, the vertices $x$ and $y$ are also dominated by $D \setminus \{x, y\}$. Let $v$ be a vertex of $G$ not in $D \cup (N_G(x) \cap N_G(y))$. Then $v$ is adjacent to at least one vertex in $D \setminus \{x, y\}$. Therefore, $D \setminus \{x, y\}$ is a dominating set of $G$ of size $|D| - 2 = \gamma(G) - 1$, a contradiction. Consequently, $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$. □

Similarly, we obtain a necessary condition of the connected graphs $G$ with $\gamma_2^c(G) = \gamma_c(G) + 1$.

**Theorem 2.3.** *Let $G$ be a connected graph with $\gamma_2^c(G) \geqslant 3$ and $\gamma_2^c(G) = \gamma_c(G) + 1$. Let $D$ be a $\gamma_2^c$-set of $G$. Then $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x, y\})$ for every pair of distinct vertices $x$ and $y$ in $D$ such that $x, y \in N_G(D \setminus \{x, y\})$ and $G[D \setminus \{x, y\}]$ is connected.*

After obtaining the necessary conditions, we discover that graphs with such conditions have no universal vertices, as shown in the following propositions.

**Proposition 2.1.** *Let $G$ be a connected graph with $\gamma_2^c(G) \geqslant 3$. For every $\gamma_2^c$-set $D$ of $G$, assume that $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x,y\})$ for every pair of distinct vertices $x$ and $y$ in $D$ such that $x, y \in N_G(D \setminus \{x,y\})$. Then $G$ has no universal vertices.*

**Proof.** Let $x$ and $y$ be two distinct vertices in a $\gamma_2^c$-set $D$ of $G$ such that $x, y \in N_G(D \setminus \{x,y\})$. So, $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x,y\})$. Suppose that $G$ has a universal vertex $u$. There are two possibilities.

- ▷ **Case 1**: $u \in D$. Since $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x,y\})$, there is a vertex $z$ such that $z \in N_G(x) \cap N_G(y)$, but $z \notin N_G(D \setminus \{x,y\})$. Suppose that $u \in D \setminus \{x,y\}$. Since $u$ is a universal vertex, it is adjacent to $z$. So, $z \in N_G(D \setminus \{x,y\})$, which is a contradiction. Thus, $u \in \{x,y\}$. Without loss of generality, we assume that $u = x$. Then $x$ is adjacent to all vertices in $D \setminus \{x,y\}$. Since $|D| \geqslant 3$, we have $D \setminus \{x,y\} \neq \phi$. Let $w$ be a vertex in $D \setminus \{x,y\}$. Since $x$ is a universal vertex, the vertices $w, y \in N_G[x] \subseteq N_G(D \setminus \{w,y\})$. By the assumption, $N_G(w) \cap N_G(y) \nsubseteq N_G(D \setminus \{w,y\})$. However, $N_G(w) \cap N_G(y) \subseteq N_G[x] \subseteq N_G(D \setminus \{w,y\})$, a contradiction. Therefore, this case cannot happen.
- ▷ **Case 2**: $u \notin D$. Then $u$ is adjacent to every vertex in $D$. Since $|D| \geqslant 3$, the set $D \setminus \{x,y\} \neq \phi$. Let $w$ be a neighbor of $x$ in $D \setminus \{x,y\}$. Let $D' = (D \setminus \{w\}) \cup \{u\}$. Since $u$ is a universal vertex, the set $D'$ is a connected 2-dominating set of $G$. Since $|D'| = |D|$, the set $D'$ is also a $\gamma_2^c$-set of $G$. However, $u \in D'$. Just as in Case 1, this cannot happen.

From both cases, we conclude that $G$ has no universal vertices. □

**Proposition 2.2.** *Let $G$ be a connected graph with $\gamma_2^c(G) \geqslant 3$. For every $\gamma_2^c$-set $D$ of $G$, assume that $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x,y\})$ for every pair of distinct vertices $x$ and $y$ in $D$ such that $x, y \in N_G(D \setminus \{x,y\})$ and $G[D \setminus \{x,y\}]$ is connected. Then $G$ has no universal vertices.*

**Proof.** Similar to the proof of Proposition 2.1. □

Next, we use the necessary condition to construct an infinite family of graphs $G$ that satisfy $\gamma_2^c(G) = \gamma_c(G) + 1$. Note that the condition $N_G(x) \cap N_G(y) \nsubseteq N_G(D \setminus \{x,y\})$ in Theorems 2.2 and 2.3 implies that $N_G(x) \cap N_G(y)$ must contain a vertex outside of $N_G(D \setminus \{x,y\})$.

**Definition 2.2.** *For a connected graph $H$ of order at least 3, we let $g(H)$ be the connected graph obtained from $H$ by adding new vertices in the following way. For every pair of distinct vertices $x$ and $y$ in $V(H)$ such that $x, y \in N_H(V(H) \setminus \{x,y\})$, we add one new vertex and join it to $x$ and $y$.*

**Observation 2.2.** *For any connected graph $H$, its vertex set $V(H)$ is a connected 2-dominating set of $g(H)$.*

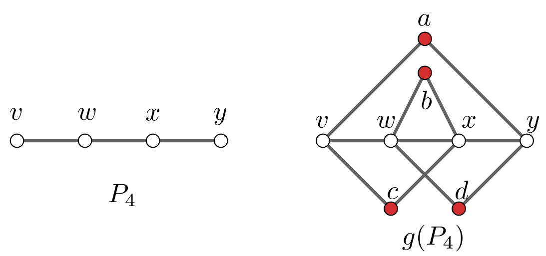


**Figure 2.1:** Graphs $P_4$ and $g(P_4)$.

For example, let $H$ be a path $P_4$ of order 4. The connected graph $G = g(P_4)$ is obtained from $P_4$ by adding the red vertices, as illustrated in Figure 2.1. Note that $v \notin N_H(V(H) \setminus \{v,w\})$ so no new vertex was created for the pair $v, w$. In this case, we say *$v$ and $w$ do not create a new vertex in $G \setminus H$*. Similarly, $x$ and $y$ do not create a new vertex in $G \setminus H$. Also, note that each new vertex has degree 2.

The following lemmas discuss some useful properties of graphs $g(H)$.

**Lemma 2.2.** *Let $H$ be a connected graph of order $k$ where $k \geqslant 3$ and let $G = g(H)$. The vertices $x$ and $y$ in $H$ do not create a new vertex in $G \setminus H$ if and only if $x$ and $y$ are adjacent and one of the two vertices has degree 1 in $H$.*

**Proof.** We will prove the forward direction by the contrapositive method. Assume that $x$ and $y$ are not adjacent or both $x$ and $y$ have degree at least 2 in $H$. Since $H$ is a connected graph, it implies that $x, y \in N_H(V(H) \setminus \{x,y\})$. By construction, $x$ and $y$ create a new vertex in $G \setminus H$.

Conversely, assume that $x$ and $y$ are adjacent and one of the two vertices has degree 1 in $H$. Without loss of generality, let $deg_H(x) = 1$. Then $x \notin N_H(V(H) \setminus \{x,y\})$. It follows that $x$ and $y$ do not create a new vertex in $G \setminus H$. □

**Lemma 2.3.** *Let $H$ be a connected graph of order $k$ where $k \geqslant 3$ and let $G = g(H)$. Then among any three vertices of $H$, there exist two vertices that create a new vertex in $G \setminus H$.*

**Proof.** Let $x, y, z \in V(H)$. Suppose there are no pairs of vertices among $x, y$ and $z$ that create a new vertex in $G \setminus H$. By Lemma 2.2 and since $x$ and $y$ do not create a new vertex in $G \setminus H$, the vertices $x$ and $y$ are adjacent and one of the two vertices has degree 1 in $H$, say $y$. Similarly, since $x$ and $z$ do not create a new vertex in $G \setminus H$, the vertices $x$ and $z$ are adjacent and $z$ has degree 1 in $H$. Note that $y$ and $z$ are not adjacent in $H$. By Lemma 2.2, the vertices $y$ and $z$ create a new vertex in $G \setminus H$, a contradiction. Hence, there exist two vertices among $x, y$ and $z$ that create a new vertex in $G \setminus H$. □

**Lemma 2.4.** *Let $H$ be a connected graph of order $k$ where $k \geqslant 3$ and let $l$ be the number of pendants in $H$. Then*

$$|V(g(H))| = k + \binom{k}{2} - l.$$

**Proof.** Let $G = g(H)$. If every pair of vertices in $H$ creates a new vertex in $G \setminus H$, then the number of new vertices in $G$ is $\binom{k}{2}$. By Lemma 2.2, the number of new vertices in $G$ is $\binom{k}{2} - l$. By Definition 2.2, $|V(G)| = |V(H)| + \binom{k}{2} - l$. □

We proceed to find the connected 2-domination numbers of the graphs $g(H)$. We begin by proving two useful lemmas.

**Lemma 2.5.** *Let $H$ be a connected graph of order $k$ where $k \geqslant 3$. Let $D$ be a connected 2-dominating set of $g(H)$. If $V(H) \setminus D$ contains a vertex $u$ that does not create new vertices with any vertices in $D \cap V(H)$, then $D \cap V(H)$ is an independent set and $u$ is adjacent to every vertex in $D \cap V(H)$.*

**Proof.** Assume that $V(H) \setminus D$ contains a vertex $u$ that does not create new vertices with any vertices in $D \cap V(H)$. By Lemma 2.2, each vertex in $D \cap V(H)$ is adjacent to the vertex $u$. If $|D \cap V(H)| = 1$, then we are done. Otherwise, we have $deg_H(u) \geqslant 2$ so each vertex in $D \cap V(H)$ has degree 1 in $H$. Hence, $D \cap V(H)$ is an independent set. □

**Lemma 2.6.** *Let $H$ be a connected graph of order 3 and let $G = g(H)$. Suppose that $D$ is a connected 2-dominating set of $G$ of size 2 such that $D \nsubseteq V(H)$. If there exist two vertices in $V(H) \setminus D$ that do not create a new vertex in $G$, then $|D \cap V(H)| = 1$.*

**Proof.** Let $V(H) = \{x, y, z\}$. Assume that $x, y \in V(H) \setminus D$ and they do not create a new vertex in $G$. By Lemma 2.2, $x$ and $y$ are adjacent and one of the two has degree 1 in $H$, say $y$. Then $y$ and $z$ create a new vertex $v$ in $G \setminus H$. Next, we will show that $v \in D$. Suppose that $v \notin D$. Since $D$ is a 2-dominating set and $v$ is only adjacent to $z$ and $y$, we have $y, z \in D$. This is a contradiction to $y \in V(H) \setminus D$. It follows that $v \in D$. Suppose that $D \cap V(H) = \phi$. Since $|D| = 2$, there exists a vertex $w \in D \setminus \{v\}$. Since $N_G(v) = \{y, z\}$, the vertex $w$ is not adjacent to $v$. This is a contradiction to $G[D]$ being a connected graph. Hence, $|D \cap V(H)| = 1$. □

**Theorem 2.4.** *Let $H$ be a connected graph of order $k$ where $k \geqslant 3$ and let $G = g(H)$. Then $V(H)$ is a $\gamma_2^c$-set of $G$. In particular, $\gamma_2^c(G) = k$.*

**Proof.** By construction, $V(H)$ is a connected 2-dominating set of $G$ of size $k$. Suppose that there exists a connected 2-dominating set $D$ of $G$ of size $k - 1 \geqslant 2$. Suppose that $D \subseteq V(H)$. Let $u$ be the single vertex in $V(H) \setminus D$. If $u$ does not create new vertices with any vertices in $D$, then by Lemma 2.5, the set $D$ is independent. This contradicts $G[D]$ being a connected graph. Consequently, $u$ creates a new vertex $v \in G \setminus H$ with some vertex $w$ in $D$. Since $u \notin D$ and $N_G(v) = \{u, w\}$, it follows that $D$ is not a 2-dominating set of $G$, a contradiction. Hence, $D \nsubseteq V(H)$. Then there is at least one vertex in $D$ that does not belong to $V(H)$. So, $|D \cap V(H)| \leqslant k - 2$. It implies that there exist at least two vertices $x$ and $y$ in $V(H) \setminus D$. There are two possibilities.

- ▷ **Case 1**: $x$ and $y$ create a new vertex $z$ in $G \setminus H$. Suppose that $z \in D$. Since $N_G(z) = \{x, y\}$, the graph $G[D]$ is disconnected, a contradiction. Thus, $z \notin D$. Then the new vertex $z$ is not dominated by $D$. This is a contradiction to $D$ being a 2-dominating set of $G$.

- ▷ **Case 2**: $x$ and $y$ do not create a new vertex in $G \setminus H$. By Lemma 2.2, the two vertices are adjacent and one of the two has degree 1 in $H$, say $y$. Note that $|V(H) \setminus \{x, y\}| = |V(H)| - 2 = k - 2$. Let $V(H) \setminus \{x, y\} = \{u_1, u_2, \ldots, u_{k-2}\}$. Since $H$ is a connected graph and $y$ is adjacent to $x$ in $V(H) \setminus D$, for each $i \in \{1, ..., k-2\}$, we have that $u_i, y \in N_H(V(H) \setminus \{u_i, y\})$ so $u_i$ and $y$ create a new vertex $v_i$ in $G \setminus H$. Let $S = \{v_1, v_2, \ldots, v_{k-2}\}$. Next, we will show that $S \subseteq D$. Suppose that there exists an $i \in \{1, ..., k-2\}$ such that $v_i \notin D$. Since $D$ is a 2-dominating set and $N_G(v_i) = \{u_i, y\}$, the vertices $u_i$ and $y$ are in $D$. This is a contradiction to $y \in V(H) \setminus D$. It implies that $v_i \in D$ for all $i \in \{1, ..., k-2\}$. So, $S \subseteq D$.

If $k = 3$, then $|S| = 1$ and $|D| = 2$. Thus, $S = \{v_1\}$. By Lemma 2.6, $|D \cap V(H)| = 1$. Since $V(H) = \{x, y, u_1\}$ and $x, y \notin D$, we have $D \cap V(H) = \{u_1\}$. Since $S \subseteq D$, the vertex $v_1$ belongs to $D \setminus V(H)$. Thus, $D = \{u_1, v_1\}$. Since $y$ is a pendant with $x$ as its support, $y$ is not adjacent to $u_1$. It follows that $D$ is not a 2-dominating set of $G$, a contradiction. Thus, $k \neq 3$.

Now, suppose $k \geqslant 4$ so there exist at least 2 vertices in $S$. By construction, $S$ is an independent set. Since each vertex $v_i$ in $S$ is created by joining it to $y$ and $u_i \in V(H) \setminus \{x, y\}$, the vertices in $S$ have only one common neighbor, namely $y$. But $y$ is not in $D$. Since $S \subseteq D$ and $|D \setminus S| = 1$, the induced subgraph $G[D]$ is disconnected, a contradiction.

We conclude from the above two cases that a connected 2-dominating set of $G$ has at least $k$ members. Therefore, $V(H)$ is a $\gamma_2^c$-set of $G$ and $\gamma_2^c(G) = k$. □

**Theorem 2.5.** *Let $H$ be a connected graph of order $k \geqslant 3$ not isomorphic to a path on 3 vertices and let $G = g(H)$. Then $V(H)$ is the unique $\gamma_2^c$-set of $G$.*

**Proof.** By Theorem 2.4, we have that $V(H)$ is a $\gamma_2^c$-set of $G$. If $k = 3$, then $H$ is a cycle on 3 vertices and it is easy to check that $V(H)$ is the only $\gamma_2^c$-set of $G$. It remains to consider $k \geqslant 4$. Suppose that there exists a $\gamma_2^c$-set $D$ of $G$ such that $D \neq V(H)$. So, $|D| = |V(H)|$ and $|V(H) \setminus D| = |D \setminus V(H)|$. Consider the following 3 cases.

- **Case 1**: $|V(H) \setminus D| = |D \setminus V(H)| = 1$. Let $u$ be the unique vertex in $V(H) \setminus D$. Suppose that $u$ does not create new vertices with any vertices in $D \cap V(H)$. By Lemma 2.5, the set $D \cap V(H)$ is independent and $u$ is adjacent to every vertex in $D \cap V(H)$. Since $D \cap V(H)$ is an independent set of size at least 3 and the unique vertex in $D \setminus V(H)$ has degree 2, the graph $G[D]$ is disconnected, a contradiction. Therefore, $u$ creates new vertices with some vertices in $D \cap V(H)$. Suppose $u$ creates exactly one new vertex. Let $a$ be the vertex in $D \cap V(H)$ that creates the new vertex with $u$. Since $k \geqslant 4$ and $|V(H) \setminus D| = 1$, we have $|(D \cap V(H)) \setminus \{a\}| \geqslant 2$. By Lemma 2.2, every vertex in $(D \cap V(H)) \setminus \{a\}$ is adjacent to $u$ and has degree 1 in $H$. Then $a$ is not adjacent to any vertex in $(D \cap V(H)) \setminus \{a\}$. Thus, $N_H(a) \subseteq \{u\}$. By this and Lemma 2.2, the vertices $u$ and $a$ are not adjacent. Therefore, $a$ is not adjacent to any vertices in $V(H) \setminus \{a\}$. Consequently, $H$ is disconnected, a contradiction. Thus, $u$ creates at least two new vertices with some vertices in $D \cap V(H)$. Since $|D \setminus V(H)| = 1$, at least one of the new vertices above is not in $D$ and is not 2-dominated by $D$, a contradiction.

- **Case 2**: $|V(H) \setminus D| = |D \setminus V(H)| = 2$. Let $V(H) \setminus D = \{x, y\}$. Suppose that $x$ and $y$ create a new vertex $z$ in $G \setminus H$. Suppose that $z \in D$. Since $deg_G(z) = 2$, the graph $G[D]$ is disconnected, a contradiction. So, $z \notin D$. Thus, $D$ is not a dominating set of $G$, a contradiction. Therefore, $x$ and $y$ do not create a new vertex in $G \setminus H$. By Lemma 2.2, the vertices $x$ and $y$ are adjacent and one of the two has degree 1 in $H$, say $y$.

  Now, suppose $x$ does not create new vertices with any vertices in $D \cap V(H)$. By Lemma 2.5, the set $D \cap V(H)$ is independent and $x$ is adjacent to every vertex in $D \cap V(H)$. Since $D \cap V(H)$ is an independent set of size at least 2, the graph $H$ is a star with at least 3 pendants. By Lemma 2.3, there exist at least $|D \cap V(H)|$ new vertices in $G$ that are created by joining them to $y$ and $D \cap V(H)$. If $|D \cap V(H)| > 2$, then at least one of the new vertices above is not in $D$ and so it is not 2-dominated by $D$, a contradiction. Thus, $|D \cap V(H)| = 2$ and $H$ is a star of order 4. By Lemma 2.4, the number of new vertices in $g(H)$ is three. Suppose that two new vertices in $g(H)$ that are created by joining them to $y$ and $D \cap V(H)$ belong to $D \setminus V(H)$. Since both of the two new vertices have degree two and $D \cap V(H)$ is an independent set, the graph $G[D]$ is disconnected, a contradiction. Hence, at least one of the two new vertices in $g(H)$ that is created by joining them to $y$ and $D \cap V(H)$ does not belong to $D$, and so it is not 2-dominated by $D$, a contradiction. Therefore, $x$ creates new vertices with some vertices in $D \cap V(H)$.

  Since $y$ is a pendant with $x$ as its support, by Lemma 2.6 the vertex $y$ creates a new vertex with each vertex in $D \cap V(H)$. It follows that there exist at least $|D \cap V(H)| + 1 \geqslant 3$ new vertices in $G$ that are adjacent to $x$ or $y$. Since $|D \setminus V(H)| = 2$, at least one of the new vertices above is not in $D$ and is not 2-dominated by $D$, a contradiction.

- **Case 3**: $|V(H) \setminus D| \geqslant 3$. Let $x, y, z \in V(H) \setminus D$. By Lemma 2.3, there exist two vertices in $\{x, y, z\}$ that create a new vertex in $G$. Without loss of generality, let $x$ and $y$ create a new vertex $v$ in $G \setminus H$. Suppose that $v \in D$. Since $deg_G(v) = 2$, the graph $G[D]$ is disconnected, a contradiction. So, $v \notin D$. Thus, $D$ is not a dominating set of $G$, a contradiction.

From the above three cases, we conclude that $V(H)$ is the unique $\gamma_2^c$-set of $G$. □

Next, we find the connected domination numbers of the graphs $g(H)$ and show how they relate to the connected 2-domination numbers.

**Theorem 2.6.** *Let $H$ be a connected graph of order $k$ where $k \geqslant 3$ and let $G = g(H)$. Then $\gamma_c(G) = k - 1$.*

**Proof.** Let $S$ be a subset of $V(H)$ such that $|S| = k - 1$ and $G[S]$ is connected. Since $V(H)$ is a 2-dominating set of $G$, the set $S$ is a connected dominating set of $G$. Thus, $\gamma_c(G) \leqslant k - 1$. Suppose that there exists a connected dominating set $D$ of $G$ of size $k - 2$. Suppose that $D \subseteq V(H)$. Then there exist $u, v \in V(H) \setminus D$. We consider the vertices $u$ and $v$ in $V(H) \setminus D$ in two cases.

- **Case 1**: $u$ and $v$ create a new vertex in $G \setminus H$. Then the new vertex is not dominated by $D$. This is a contradiction to $D$ being a dominating set.
- **Case 2**: $u$ and $v$ do not create a new vertex in $G \setminus H$. By Lemma 2.2, $u$ and $v$ are adjacent and one of the two has degree 1 in $V(H)$, say $v$. Then $v$ is not dominated by $D$, a contradiction.

From the above two cases, we conclude that $D \nsubseteq V(H)$. Then at least one vertex in $D$ does not belong to $V(H)$. So, $|D \cap V(H)| \leqslant k - 3$. It implies that there exist at least 3 vertices in $V(H) \setminus D$. Let $x, y, z \in V(H) \setminus D$. By Lemma 2.3, there exist two vertices in $V(H) \setminus D$ that create a new vertex in $G \setminus H$. Without loss of generality, let $x$ and $y$ create a new vertex $t$ in $G \setminus H$. Suppose that $t \in D$. Since $N_G(t) = \{x, y\}$, we have that $t \notin N_G(D)$, a contradiction. So, $t \notin D$. It follows that the new vertex $t$ in $G$ is not dominated by $D$, a contradiction. Hence, a connected dominating set of $G$ has at least $k - 1$ members. Therefore, $\gamma_c(G) = k - 1$. □

**Corollary 2.1.** *Let $H$ be a connected graph of order $k$ where $k \geqslant 3$ and let $G = g(H)$. Then $\gamma_2^c(G) = \gamma_c(G) + 1$.*

Now, we show that for any connected graph $H$ of order at least 3, the graph $g(H)$ satisfies either $\gamma_2^c(g(H)) = \gamma(g(H)) + 1$ or $\gamma_2^c(g(H)) = \gamma(g(H)) + 2$.

**Theorem 2.7.** *Let $H$ be a connected graph of order 3 and let $G = g(H)$. Then $\gamma(G) = 2$.*

**Proof.** Since $H$ is a connected graph of order 3, it follows that $H$ is either a path $P_3$ or a cycle $C_3$ of order 3. Since $g(P_3)$ is a cycle of order 4, it implies that $\gamma(g(P_3)) = 2$. Next, we show that $\gamma(g(C_3)) = 2$. By Lemma 2.4, we have that $|V(g(C_3))| = 6$. Since the maximum degree of $g(C_3)$ equals 4, no single vertex in $g(C_3)$ can dominate all vertices in $g(C_3)$. Thus, $\gamma(g(C_3)) \geqslant 2$. Clearly, any two vertices in $V(C_3)$ form a dominating set of $g(C_3)$. Hence, $\gamma(g(C_3)) \leqslant 2$. Therefore, $\gamma(g(C_3)) = 2$. □

**Lemma 2.7.** *Let $H$ be a connected graph of order $k$ where $k \geqslant 4$ and let $G = g(H)$. Then $\gamma(G) \geqslant k - 2$.*

**Proof.** Let $V(H) = \{v_1, v_2, v_3, \ldots, v_k\}$. Let $X = V(G) \setminus V(H)$. Then $X$ consists of the new vertices. Suppose there exists $D \subseteq V(G)$ such that $|D| = k - 3$ and $D$ dominates $X$. If $D$ contains a new vertex $x$ in $X$, then $x$ was created by some vertices $u$ and $v$ in $H$. Since $N_G[x] \cap X \subseteq N_G[u] \cap X$, we can use the vertex $u$ in $H$ to dominate new vertices in $X$ instead of the vertex $x$. Hence, it is sufficient to consider that the vertices in $D$ are from $V(H)$. Without loss of generality, let $D = \{v_1, v_2, v_3, \ldots, v_{k-3}\}$. We divide the argument into two cases according to the number of pendants in $\{v_{k-2}, v_{k-1}, v_k\}$.

- **Case 1**: $\{v_{k-2}, v_{k-1}, v_k\}$ contains at most one pendant. Without loss of generality, assume $v_{k-1}$ and $v_k$ are not pendants. By Lemma 2.2, $v_{k-1}$ and $v_k$ create a new vertex in $G \setminus H$ which is not dominated by $D$, a contradiction.
- **Case 2**: $\{v_{k-2}, v_{k-1}, v_k\}$ contains at least two pendants. Without loss of generality, assume $v_{k-1}$ and $v_k$ are the two pendants. By Lemma 2.2, $v_{k-1}$ and $v_k$ create a new vertex in $G \setminus H$ which is not dominated by $D$, a contradiction.

We conclude from the above two cases that at least $k - 2$ vertices are required to dominate $X$. Thus, $\gamma(G) \geqslant k - 2$. □

**Theorem 2.8.** *Let $H$ be a connected graph of order $k$ where $k \geqslant 4$ and let $G = g(H)$. If $H$ contains two pendants that share a support vertex in $H$, then $\gamma(G) = k - 2$.*

**Proof.** Let $V(H) = \{v_1, v_2, v_3, \ldots, v_k\}$. Assume that $H$ contains 2 pendants that share a support vertex in $H$. For $i \neq j$, when $v_i$ and $v_j$ create a new vertex in $G \setminus H$, we let $v_{ij}$ denote the new vertex. Since $|V(H)| = k \geqslant 4$, no two pendants are adjacent. Without loss of generality, let $v_{k-1}$ and $v_k$ be two pendants of $H$ with the common support vertex $v_{k-2}$. Let $D = \{v_1, v_2, v_3, \ldots, v_{k-3}\} \cup \{v_{k-1,k}\}$. By Lemma 2.2, $v_{k-2}$ does not create a new vertex with either $v_{k-1}$ or $v_k$. Since $H$ is connected, the vertex $v_{k-2}$ is adjacent to some vertex in $\{v_1, v_2, \ldots, v_{k-3}\}$. By construction, all vertices in $G$ except $v_{k-1}, v_k$ and $v_{k-1,k}$ are dominated by $\{v_1, v_2, v_3, \ldots, v_{k-3}\}$ but $v_{k-1}, v_k$ and $v_{k-1,k}$ are dominated by $v_{k-1,k}$. Hence, $D$ dominates all vertices in $G$. Since $|D| = k - 2$, we have that $\gamma(G) \leqslant k - 2$. By Lemma 2.7, we have $\gamma(G) = k - 2$. □

**Theorem 2.9.** *Let $H$ be a connected graph of order $k$ such that $k \geqslant 4$ and no two pendants share a support vertex. Let $G = g(H)$. Then $\gamma(G) = k - 1$.*

**Proof.** Let $V(H) = \{v_1, v_2, v_3, \ldots, v_k\}$. Let $X = V(G) \setminus V(H)$. For $i \neq j$, when $v_i$ and $v_j$ create a new vertex in $G \setminus H$, we let $v_{ij}$ denote the new vertex. Suppose there exists $D \subseteq V(G)$ such that $|D| = k - 2$ and $D$ dominates $X$. Similar to the proof of Theorem 2.7, we can assume that $D \subseteq V(H)$ and let $D = \{v_1, v_2, v_3, \ldots, v_{k-2}\}$. Let $\alpha$ be the number of vertices in $X$ that are dominated by $D$. Let $l$ be the number of pendants in $H$. By Lemma 2.4, we have

$$\alpha = |X| = \binom{k}{2} - l.$$

We will also compute $\alpha$ by counting the number of additional vertices that are dominated by each $v_i$ for $1 \leqslant i \leqslant k-2$. By Lemma 2.2, for each $v \in D$, if $v$ is a pendant or a support of a pendant, then $v$ is adjacent to $k-2$ vertices in $X$; otherwise, $v$ is adjacent to $k - 1$ vertices in $X$.

First, suppose that both $v_{k-1}$ and $v_k$ are not pendants in $H$. Then all $l$ pendants are in $D$, so

$$\alpha = (k-1) + (k-2) + \cdots + 2 - l = \binom{k}{2} - 1 - l.$$

Thus, $\alpha < |X|$, which is a contradiction.

Suppose that both $v_{k-1}$ and $v_k$ are pendants in $H$. Then the support vertices of $v_{k-1}$ and $v_k$ are distinct and are in $D$. It implies that $\alpha = (k-1) + (k-2) + \cdots + 2 - l = \binom{k}{2} - 1 - l$. Thus, $\alpha < |X|$, a contradiction.

Therefore, exactly one vertex in $\{v_{k-1}, v_k\}$ is a pendant in $H$. Then $D$ contains $l-1$ pendants. Without loss of generality, let $v_k$ be a pendant. First, suppose that the support vertex of $v_k$ is in $D$. It follows that $\alpha = (k-1) + (k-2) + \cdots + 2 - l = \binom{k}{2} - 1 - l$. Thus, $\alpha < |X|$, a contradiction. Thus, the support vertex of $v_k$ is not in $D$, i.e. $v_{k-1}$ is the support vertex of $v_k$. Then

$$\alpha = (k-1) + (k-2) + \cdots + 2 - (l-1) = \binom{k}{2} - l.$$

It follows that we need at least $k-2$ vertices to dominate every vertex in $X$. Each vertex $v_i$ in $D$ dominates at least 2 additional vertices $v_{i,k-1}$ and $v_{ik}$. Each vertex $v_{ij}$ in $X$ can only dominate one vertex (itself) in $X$. So, to use exactly $k-2$ vertices to dominate $X$, we cannot use any vertex from $X$. Since the pendant $v_k$ and its support vertex $v_{k-1}$ are not in $D$, the vertex $v_k$ is not dominated by $D$. Thus, we must use one more vertex to dominate $v_k$. Then a dominating set of $G$ has at least $k-1$ members. So, $\gamma(G) \geqslant k-1$.

Let $D' = \{v_1, v_2, v_3, \ldots, v_{k-1}\}$. Clearly, $D'$ dominate all vertices in $G$. Since $|D'| = k-1$, we have that $\gamma(G) \leqslant k-1$. Therefore, $\gamma(G) = k-1$. □

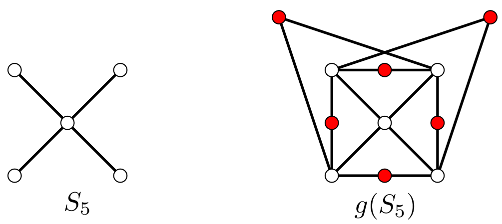


**Figure 2.2:** Graphs $S_5$ and $g(S_5)$.

**Remark 2.1.** *Theorems 2.4 and 2.8 imply that our necessary condition for graphs $G$ with $\gamma_2^c(G) = \gamma(G)+1$ is not a sufficient condition.*

Lastly, we apply Theorems 2.4, 2.6, 2.7, 2.8, and 2.9 to stars, paths, and cycles. We let $S_k$, $P_k$ and $C_k$ denote a star, a path and a cycle of order $k$, respectively.

**Corollary 2.2.** *For $k \geqslant 4$, let $G = g(S_k)$. Then $\gamma_2^c(G) = k$, $\gamma_c(G) = k-1$ and $\gamma(G) = k-2$.*

**Corollary 2.3.** *For $k \geqslant 3$, let $G = g(P_k)$. Then $\gamma_2^c(G) = k$, $\gamma_c(G) = k-1$ and $\gamma(G) = k-1$.*

**Corollary 2.4.** *For $k \geqslant 3$, let $G = g(C_k)$. Then $\gamma_2^c(G) = k$, $\gamma_c(G) = k-1$ and $\gamma(G) = k-1$.*

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