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Research Article

Algorithmic study of d_2-transitivity of graphs*

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Abstract

Let $G = (V, E)$ be a graph, where V and E are the vertex and edge sets, respectively. For two disjoint subsets A and B of V, we say A *dominates* B if every vertex of B is adjacent to at least one vertex of A. A vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of G is called a *transitive partition* of size k if V_i dominates V_j for all $1 \leq i < j \leq k$. In this article, we initiate the study of a generalization of transitive partition, namely d_2-*transitive partition*. For two disjoint subsets A and B of V, we say A d_2-*dominates* B if, for every vertex of B, there exists a vertex in A such that the distance between them is at most two. A vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of G is said to be a d_2-*transitive partition* of size k if V_i d_2-dominates V_j for all $1 \leq i < j \leq k$. The maximum integer k for which d_2-transitive partition exists is called d_2-*transitivity* of G, and it is denoted by $Tr_{d_2}(G)$. The Maximum d_2-Transitivity Problem is to find a d_2-transitive partition of a given graph with the maximum number of parts. We show that this problem can be solved in linear time for the complement of bipartite graphs and bipartite chain graphs. On the other side, we prove that the decision version of the Maximum d_2-Transitivity Problem is NP-complete for split graphs and bipartite graphs.

Keywords: d_2-transitivity; linear algorithm; NP-completeness; split graphs; bipartite graphs.

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1. Introduction

Partitioning a graph is one of the fundamental problems in graph theory. In the partitioning problem, the objective is to partition the vertex set (or edge set) into some parts with desired properties, such as independence, minimal edges across partite sets, etc. In literature, partitioning the vertex set into certain parts so that the partite sets follow particular domination relations among themselves has been studied. Let G be a graph with $V(G)$ as its vertex set and $E(G)$ as its edge set. When the context is clear, V and E are used instead of $V(G)$ and $E(G)$. The *neighbourhood* of a vertex $v \in V$ in a graph $G = (V, E)$ is the set of all the vertices adjacent to v and is denoted by $N(v)$. The *closed neighborhood* of a vertex $v \in V$, denoted as $N[v]$, is defined by $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v in G, denoted as $\deg_G(v)$, is the number of edges incident to v. A vertex v is said to *dominate* itself and all its neighbouring vertices. A *dominating set* of $G = (V, E)$ is a subset D of vertices such that every vertex $x \in V \setminus D$ has a neighbour $y \in D$; that is, x is dominated by some vertex y of D. For two disjoint subsets A and B of V, we say A *dominates* B if every vertex of B is adjacent to at least one vertex of the set A.

There has been a lot of research on graph partitioning problems that are based on a domination relationship between the different sets. Cockayne and Hedetniemi introduced the concept of *domatic partition* of a graph $G = (V, E)$ in 1977, in which the vertex set is partitioned into k parts, say $\pi = \{V_1, V_2, \ldots, V_k\}$, such that each V_i is a dominating set of G [3]. The number that represents the highest possible order of a domatic partition is referred to as the *domatic number* of G, and it is denoted by $d(G)$. Another similar type of partitioning problem is the *Grundy partition*. Christen and Selkow introduced a Grundy partition of a graph $G = (V, E)$ in 1979 [2]. In the Grundy partitioning problem, the vertex set is partitioned into k parts, say $\pi = \{V_1, V_2, \ldots, V_k\}$, such that each V_i is an independent set and for all $1 \leq i < j \leq k$, V_i dominates V_j. The maximum order of such a partition is called the *Grundy number* of G, and it is denoted by $\Gamma(G)$. In 2018, J. T. Hedetniemi and S. T. Hedetniemi [8] introduced a transitive partition as a generalization of the Grundy partition. A *transitive partition* of size k is defined as a partition of the vertex set into k parts, say $\pi = \{V_1, V_2, \ldots, V_k\}$, such that for all $1 \leq i < j \leq k$, V_i dominates V_j. The maximum order of such a transitive partition is called *transitivity* of G and is denoted by $Tr(G)$. Recently, in 2020, Haynes et al. generalized the idea of domatic partition as well as transitive partition and introduced the concept of *upper domatic partition* of a graph G, where the vertex set is partitioned into k parts, say $\pi = \{V_1, V_2, \ldots, V_k\}$, such that for every pair i, j, with $1 \leq i < j \leq k$, either V_i dominates V_j or V_j dominates V_i or both [7].

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The maximum order of such an upper domatic partition is called *upper domatic number* of G, and it is denoted by $D(G)$. All these problems, domatic number [1, 17, 18], Grundy number [4, 5, 9, 15, 16], transitivity [6, 8, 12, 13], upper domatic number [7, 14] have been extensively studied both from an algorithmic and structural point of view. Clearly, a Grundy partition is a transitive partition with the additional restriction that each partite set must be independent. In a transitive partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of G, we have domination property in one direction, that is, V_i dominates V_j for $1 \leq i < j \leq k$. However, in a upper domatic partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of G, for all $1 \leq i < j \leq k$, either V_i dominates V_j or V_j dominates V_i or both. The definition of each vertex partitioning problem ensures the following inequalities for any graph G. For any graph G, $1 \leq \Gamma(G) \leq Tr(G) \leq D(G) \leq n$.

In this article, we introduce a similar graph partitioning problem, namely d_2-*transitive partition*, which is a generalization of transitive partition. For two disjoint subsets A and B, we say A d_2-*dominates* B if, for every vertex of B, there exists a vertex in A, such that the distance between them is at most two. A d_2-*transitive partition* of size k is defined as a partition of the vertex set into k parts, say $\pi = \{V_1, V_2, \ldots, V_k\}$, such that for all $1 \leq i < j \leq k$, V_i d_2-dominates V_j. The maximum order of such a d_2-transitive partition is called d_2-*transitivity* of G and is denoted by $Tr_{d_2}(G)$. The Maximum d_2-Transitivity Problem and its corresponding decision version are defined as follows:

Maximum d_2-Transitivity Problem (Md_2TP)
Instance: A graph $G = (V, E)$
Solution: A d_2-transitive partition of G
Measure: Order of the d_2-transitive partition of G

Maximum d_2-Transitivity Decision Problem (Md_2TDP)
Instance: A graph $G = (V, E)$, integer k
Question: Does G have a d_2-transitive partition of order at least k?

Every transitive partition is also a d_2-transitive partition. Therefore, for any graph G, $1 \leq Tr(G) \leq Tr_{d_2}(G) \leq n$. Also, the difference between $Tr_{d_2}(G)$ and $Tr(G)$ can be arbitrarily large. For complete bipartite graphs $K_{t,t}$, $Tr_{d_2}(G) = 2t$ whereas $Tr(G) = t + 1$. From the complexity point of view, there are some graph classes where transitivity can be solved in linear time, but d_2-transitivity is NP-complete. For example, in split graphs, the transitivity problem can be solved in linear time [13], but later in this paper, we show that d_2-transitivity is NP-complete in split graphs. There are some vertex partition parameters where the value of the parameter in a subgraph can be greater than the original graph. The upper domatic number is one such example. But in the case of a d_2-transitive partition, $Tr_{d_2}(H) \leq Tr_{d_2}(G)$, for every subgraph H of G. As a consequence, for a disconnected graph, the d_2-transitivity is equal to the maximum d_2-transitivity among all of its components. Therefore, we focus only on connected graphs in this paper.

In this paper, we study the computational complexity of the d_2-transitivity of graphs. The main contributions are summarized below:

1. We show that the d_2-transitivity can be computed in linear time for the complement of bipartite graphs and bipartite chain graphs.

2. We show that the Md_2TDP is NP-complete for split graphs and bipartite graphs.

The rest of the paper is organized as follows. Section 2 contains basic definitions and notations that are followed throughout the article. Some basic properties of d_2-transitivity of graphs are also discussed in Section 2. Section 3 describes linear-time algorithms for the complement of bipartite graphs and bipartite chain graphs. In Section 4, it is shown that the Md_2TDP is NP-complete in split graphs and bipartite graphs. Finally, Section 5 concludes the article.

2. Preliminaries

Definitions and notations

Let $G = (V, E)$ be a graph with V and E as its vertex and edge sets, respectively. A graph $H = (V', E')$ is said to be a *subgraph* of a graph $G = (V, E)$ if and only if $V' \subseteq V$ and $E' \subseteq E$. For a subset $S \subseteq V$, the *induced subgraph* on S of G is defined as the subgraph of G whose vertex set is S and edge set consists of all of the edges in E that have both endpoints in S, and it is denoted by $G[S]$. The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (\overline{V}, \overline{E})$, such that $\overline{V} = V$ and $\overline{E} = \{uv | uv \notin E\}$. For any $x, y \in V$, the *distance* between x and y is defined as the number of edges in the shortest path starting at x and ending at y in G, and it is denoted by $d(x, y)$. The *diameter* of a graph G is defined as the greatest length of the shortest path between each pair of vertices, and it is denoted by $diam(G)$. Let G be a graph; the *square* of G is a graph with the same set of vertices as G and for any $x, y \in V$, xy is an edge in the square graph if and only if $d(x, y) \leq 2$.

The square graph of a graph G is denoted by G^2. A subset of $S \subseteq V$ is said to be an *independent set* of G if every pair of vertices in S are non-adjacent. A subset of $K \subseteq V$ is said to be a *clique* of G if every pair of vertices in K are adjacent. The cardinality of a clique of maximum size is called *clique number* of G, and it is denoted by $\omega(G)$.

A graph is called *bipartite* if its vertex set can be partitioned into two independent sets. A bipartite graph $G = (X \cup Y, E)$ is called a *bipartite chain graph* if there exists an ordering of vertices of X and Y, say $\sigma_X = (x_1, x_2, \ldots, x_{n_1})$ and $\sigma_Y = (y_1, y_2, \ldots, y_{n_2})$, such that $N(x_{n_1}) \subseteq N(x_{n_1-1}) \subseteq \ldots \subseteq N(x_2) \subseteq N(x_1)$ and $N(y_{n_2}) \subseteq N(y_{n_2-1}) \subseteq \ldots \subseteq N(y_2) \subseteq N(y_1)$. Such an ordering of X and Y is called a *chain ordering*, and it can be computed in linear time [11]. A graph $G = (V, E)$ is said to be a *split graph* if V can be partitioned into an independent set S and a clique K.

Basic properties of d_2-transitivity

In this subsection, we present some basic properties of d_2-transitivity. First, we show the following bounds for d_2-transitivity.

Lemma 2.1. *For any graph G, $\Delta(G) + 1 \leq Tr_{d_2}(G) \leq \min\{n, (\Delta(G))^2 + 1\}$, where $\Delta(G)$ is the maximum degree of G.*

Proof. Let x be a vertex of G with degree $\Delta(G)$. Consider a vertex partition $\pi = \{V_1, V_2, \ldots, V_{\Delta(G)+1}\}$ such that each V_i for $2 \leq i \leq \Delta(G)+1$ contains exactly one vertex from $N_G[x]$ and all the other vertices are in V_1. Clearly, π forms a d_2-transitive partition of G. Therefore, $Tr_{d_2}(G) \geq \Delta(G) + 1$.

Let $Tr_{d_2}(G) = k$. Clearly, $Tr_{d_2}(G) \leq n$, where n is the number of vertices of G. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a d_2-transitive partition of G of size k. Also, let $x \in V_k$ and $N_G(x) = \{x_1, x_2, \ldots, x_l\}$. First, we show that

$$\sum_{i=1}^{l} deg(x_i) \geq k - 1.$$

If $l \geq k-1$, then we are done. Otherwise, let us assume that $l < k-1$. Hence, there are some sets in π that do not contain any vertex from $N_G(x)$. Let V_i be such a set in π and $y \in V_i$ d_2-dominates x. This implies that y is adjacent to some vertex of $N_G(x)$, and hence one vertex from V_i contributes one to the sum $\sum_{i=1}^{l} deg(x_i)$. Also, if V_j is a set in π that contains a vertex from $N_G(x)$, then the vertex x contributes one to the sum $\sum_{i=1}^{l} deg(x_i)$. In either case, we have a contribution of one to the sum corresponding to each set in π, except the last set, V_k. Hence, $\sum_{i=1}^{l} deg(x_i) \geq k - 1$. Since the maximum degree is $\Delta(G)$, we have $(\Delta(G))^2 \geq k - 1$. Therefore, $Tr_{d_2}(G) \leq \min\{n, (\Delta(G))^2 + 1\}$. $\square$

Note that the above bounds are tight. For the graph K_n, both lower and upper bounds are reached and for the graph C_n with $n \geq 5$, the upper bound is reached. The d_2-transitivity of paths and cycles in the following propositions is immediately found by using the above bound.

Proposition 2.1. *If P_n is a path on n vertices, then the d_2-transitivity of P_n is given as follows:*

$$Tr_{d_2}(P_n) = \begin{cases} 1 & n = 1, \\ 2 & n = 2, \\ 3 & n = 3, 4, \\ 4 & n = 5, 6, \\ 5 & n \geq 7. \end{cases}$$

Proposition 2.2. *If C_n is a cycle on n vertices, then the d_2-transitivity of C_n is given as follows:*

$$Tr_{d_2}(C_n) = \begin{cases} 3 & n = 3, \\ 4 & n = 4, \\ 5 & n \geq 5. \end{cases}$$

Next, we characterize graphs with small d_2-transitivity.

Lemma 2.2. *Let G be a connected graph.*

(a) $Tr_{d_2}(G) = 1$ *if and only if* $G = K_1$.

(b) $Tr_{d_2}(G) = 2$ *if and only if* $G = K_2$.

(c) $Tr_{d_2}(G) = 3$ *if and only if* $G \in \{P_3, K_3, P_4\}$.

Proof. The proofs of the statements (a) and (b) are trivial and hence we omit them. In what follows, we prove (c). If $G \in \{P_3, K_3, P_4\}$, then clearly, $Tr_{d_2}(G) = 3$. Conversely, let $Tr_{d_2}(G) = 3$. Now by Lemma 2.1, we have $\Delta(G)+1 \leq 3$, that is, $\Delta(G) \leq 2$. Therefore, G is either a path or a cycle. From Proposition 2.1, we know that P_3 and P_4 are the only two paths for which d_2-transitivity is 3. On the other hand, from Proposition 2.2, we know that C_3 (equivalently K_3) is the only cycle for which d_2-transitivity is 3. Therefore, if $Tr_{d_2}(G) = 3$, then $G \in \{P_3, K_3, P_4\}$. □

Next, we characterize the graphs having d_2-transitivity equal to n, where n is the number of vertices of the graph.

Lemma 2.3. *Let G be a graph with n vertices. Then $Tr_{d_2}(G) = n$ if and only if $diam(G) \leq 2$.*

Proof. If $Tr_{d_2}(G) = n$, then every vertex in G d_2-dominates every other vertex in G. Therefore, the distance between every pair of vertices is at most two. Therefore, $diam(G) \leq 2$.

On the other hand, if $diam(G) \leq 2$, then the distance between every pair of vertices is at most two. Hence, by putting every vertex in different sets, we get a d_2-transitive partition of size n. Therefore, $Tr_{d_2}(G) = n$. □

Remark 2.1. *Many important graph classes, including threshold graphs, $(2K_2, P_4)$-free graphs, connected strongly regular graphs, have a diameter of at most two. Lemma 2.3 implies that for these graph classes, we can solve Md_2TP trivially.*

3. Algorithms for Md_2TP

In this section, we find the d_2-transitivity for the complement of bipartite graphs and bipartite chain graphs.

The complement of bipartite graphs

In this subsection, we find the d_2-transitivity of the complement of bipartite graphs. Let G be the complement of a bipartite graph $\overline{G} = (X \cup Y, \overline{E})$.

Lemma 3.1. *Let G be the complement of a bipartite graph $\overline{G} = (X \cup Y, \overline{E})$ with $|X| = n$ and $|Y| = m$. Also, let*

$$X' = \{x \in X \mid deg_{\overline{G}}(x) = m\} \quad \textit{and} \quad Y' = \{y \in Y \mid deg_{\overline{G}}(y) = n\}.$$

If $\overline{G}[X' \cup Y']$ has a maximum matching of size t, then $Tr_{d_2}(G) = n + m - t$.

Proof. Let $M = \{e_1, e_2, \ldots, e_t\}$ be a maximum matching in $\overline{G}[X' \cup Y']$ of size t and $e_i = x_i y_i$ for all $1 \leq i \leq t$. Let $X_t = \{x_1, x_2, \ldots, x_t\}$ and $Y_t = \{y_1, y_2, \ldots, y_t\}$. Note that since $\overline{G}[X' \cup Y']$ forms a complete bipartite graph, M saturates either X' or Y'. So, without loss of generality, let us assume that $Y_t = Y'$. Consider a vertex partition, say $\pi = \{V_1, V_2, \ldots, V_{n+m-t}\}$, of G of size $n+m-t$ as follows: $V_i = \{x_i, y_i\}$ for all $1 \leq i \leq t$, and every V_j contains exactly one vertex from $(X \setminus X_t) \cup (Y \setminus Y_t)$ for all $t+1 \leq j \leq n+m-t$. We show that π is a d_2-transitive partition of G. Note that every vertex of $Y \setminus Y_t$ is adjacent to at least one vertex of $X \setminus X'$ in G. Therefore, every pair of vertices of $(X \setminus X_t) \cup (Y \setminus Y_t)$ are within distance two from each other. Therefore, V_p d_2-dominates V_q for all $t+1 \leq p < q \leq n+m-t$. Also, since every set in $\{V_1, V_2, \ldots, V_t\}$ contains vertices from X and Y both, V_i d_2-dominates every set in π for all $1 \leq i \leq t$. Therefore, π is a d_2-transitive partition of G. Hence, $Tr_{d_2}(G) \geq n+m-t$.

On the other hand, let us assume that G has a d_2-transitive partition, say π, of size more than $n+m-t$. Since there are $n+m-2t$ vertices in $(X \setminus X_t) \cup (Y \setminus Y_t)$, at most $n+m-2t$ many sets of π contains vertices from $(X \setminus X_t) \cup (Y \setminus Y_t)$. Therefore, there are at least $t+1$ many remaining sets in π that contain vertices from $X_t \cup Y_t$. By the pigeonhole principle, there are at least two sets in π, say V_i and V_j, such that V_i contains vertices only from X_t and V_j contains vertices only from Y_t. But, in that case, neither set d_2-dominates the other because the distance between any vertex of X_t and any vertex of Y_t is more than two. This contradicts the fact that π is a d_2-transitive partition of G. So, we have

$$Tr_{d_2}(G) \leq n+m-t.$$

Therefore, the d_2-transitivity of G is $n+m-t$. □

From Lemma 3.1, we can compute the d_2-transitivity as follows: given the complement of a bipartite graph G, first, we compute the sizes of X' and Y' in linear time. The size t of the maximum matching M is clearly the minimum of these sizes. Then $Tr_{d_2}(G) = n+m-t$. Hence, we have the following theorem.

Theorem 3.1. *The* Maximum d_2-Transitivity Problem *can be solved in linear time for the complement of bipartite graphs.*

Bipartite chain graphs

In this subsection, we find the d_2-transitivity of a given bipartite chain graph G. To find the d_2-transitivity of a given bipartite chain graph G, first we show that the d_2-transitive partition of a graph G is the same as the transitivity of its square graph, namely G^2. Then we show that if G is a connected bipartite chain graph, then G^2 is the complement of another bipartite chain graph H, that is, $G^2 = \overline{H}$.

Lemma 3.2. *For any graph G, $Tr_{d_2}(G) = Tr(G^2)$.*

Proof. Let $Tr_{d_2}(G) = k$ and $\pi = \{V_1, V_2, \ldots, V_k\}$ be a d_2-transitive partition of G. Then, by the definition of d_2-transitive partition, V_i d_2-dominates V_j for all $1 \leq i < j \leq k$. So, for all $y \in V_j$, there exists a $x \in V_i$, such that $d(x, y) \leq 2$ in G. From the definition of G^2, it follows that there is an edge xy in G^2. This implies that V_i dominates V_j in G^2 and hence π is a transitive partition of G^2 of size k. Therefore, $Tr_{d_2}(G) \leq Tr(G^2)$.

On the other hand, let $Tr(G^2) = k$ and $\pi = \{V_1, V_2, \ldots, V_k\}$ be a transitive partition of G^2. By the definition of transitive partition, V_i dominates V_j for all $1 \leq i < j \leq k$. So, for all $y \in V_j$, there exists a $x \in V_i$, such that xy is an edge in G^2. From the definition of G^2, it follows that $d(x, y) \leq 2$ in G. This implies that V_i d_2-dominates V_j in G, and hence π is a d_2-transitive partition of G of size k. Therefore, $Tr_{d_2}(G) \geq Tr(G^2)$. So, we have $Tr_{d_2}(G) = Tr(G^2)$. □

Next, we show that if G is a connected bipartite chain graph, then $G^2 = \overline{H}$, where H is another bipartite chain graph.

Lemma 3.3. *If G is a connected bipartite chain graph, then G^2 is the complement of another bipartite chain graph, H.*

Proof. Let $G = (X \cup Y, E)$ be a connected bipartite chain graph with chain ordering $\sigma_X = (x_1, x_2, \ldots, x_m)$ and $\sigma_Y = (y_1, y_2, \ldots, y_n)$ such that $N(x_m) \subseteq N(x_{m-1}) \subseteq \ldots \subseteq N(x_1)$ and $N(y_n) \subseteq N(y_{n-1}) \subseteq \ldots \subseteq N(y_1)$. Note that every pair of vertices of X are at a distance of two as $N(y_1) = X$. Hence, X forms a clique in G^2. Similarly, Y forms another clique in G^2. Moreover, for any two vertices $x \in X$ and $y \in Y$, the distance between x and y is either 1 or greater or equal to 3. Hence, all the edges of G are present in G^2, and there is no other edge in G^2. Now, the complement of G^2 is a bipartite graph. Let H be the bipartite graph such that $\overline{H} = G^2$. Next, we prove that H is a bipartite chain graph. Since the edges across X and Y are the same in G and G^2, $N(x_m) \subseteq N(x_{m-1}) \subseteq \ldots \subseteq N(x_1)$ and $N(y_n) \subseteq N(y_{n-1}) \subseteq \ldots \subseteq N(y_1)$ in G^2 as well. Therefore, in H, we have $N(x_1) \subseteq N(x_2) \subseteq \ldots \subseteq N(x_m)$ and $N(y_1) \subseteq N(y_2) \subseteq \ldots \subseteq N(y_n)$. Therefore, vertices of H have chain ordering and hence H is a bipartite chain graph. □

From Lemma 3.2 and Lemma 3.3, we know that finding the d_2-transitivity of a bipartite chain graph G is the same as finding the transitivity of the complement of another bipartite chain graph H. From the proof of Lemma 3.3, it follows that we can obtain H by taking the complement of G and then deleting edges inside X and Y. Note that H contains some isolated vertices as $N(x_m) = N(y_n) = \emptyset$. Hence, we can compute H in linear time. Moreover, the transitivity of the complement of a bipartite chain graph can be computed in linear time [13]. Therefore, we have the following theorem:

Theorem 3.2. *The Maximum d_2-Transitivity Problem can be solved in linear time for bipartite chain graphs.*

4. NP-completeness of Md_2TDP

In this section, we present two NP-completeness results for Maximum d_2-Transitivity Decision Problem, namely in split graphs and bipartite graphs.

Split graphs

In this subsection, we show that the Maximum d_2-Transitivity Decision Problem (Md_2TDP) is NP-complete for split graphs, which form an important subclass of the class of chordal graphs.

Theorem 4.1. *The Maximum d_2-Transitivity Decision Problem is NP-complete for split graphs.*

Proof. Given a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of a given split graph, we can verify in polynomial time whether π is a d_2-transitive partition of that graph or not. Hence, the Maximum d_2-Transitivity Decision Problem (Md_2TDP) is in NP. To prove that this problem is NP-hard, we show a polynomial time reduction from the Maximum Transitivity Decision Problem in general graphs, which is known to be NP-complete [10]. The reduction is as follows: given an instance of the Maximum Transitivity Decision Problem, that is, a graph $G = (V, E)$ and an integer k, we first subdivide each edge of G exactly once. Let u_i be the subdivision vertex corresponding to the edge $e_i \in E$ for every $1 \leq i \leq m$. Then we put edges between every pair of subdivision vertices. Let the new graph be $G' = (V', E')$. Clearly, G' is a split graph having $n + m$ vertices and $\frac{m^2+3m}{2}$ edges. The construction of G' is illustrated in Figure 4.1.

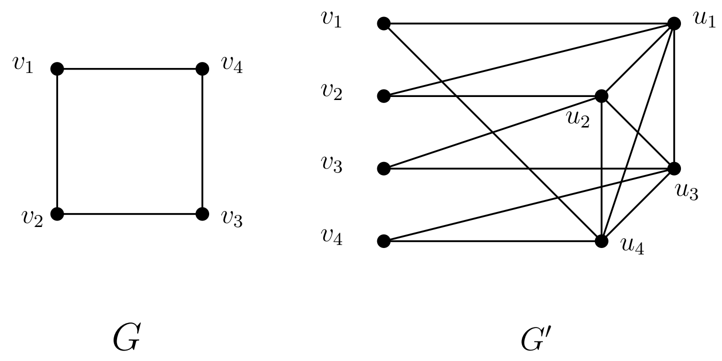

Figure 4.1: Construction of G' in Theorem 4.1.

Claim 4.1. *The graph G has a transitive partition of size k if and only if G' has a d_2-transitive partition of size $k+m$.*

Proof of Claim 4.1. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a transitive partition of G of size k. Let us consider a vertex partition, say $\pi' = \{V_1', V_2', \ldots, V_{k+m}'\}$ of G', as follows: $V_i' = V_i$ for all $1 \leq i \leq k$ and $V_{k+j}' = \{u_j\}$, $1 \leq j \leq m$. We show that π' is a d_2-transitive partition of G'. For any pair of sets V_i' and V_j' with $1 \leq i < j \leq k$, V_i' d_2-dominates V_j' in G' as V_i dominates V_j in G. Also, the set $\{u_1, u_2, \ldots, u_m\}$ induces a complete graph in G'. Hence, V_i' d_2-dominates V_j' in G' for $k+1 \leq i < j \leq k+m$. Finally, every vertex of G', other than the subdivision vertices, is adjacent to at least one subdivision vertex in G'. Hence, V_i' d_2-dominates V_j' in G' for $1 \leq i \leq k$ and $k+1 \leq j \leq k+m$. Therefore, π' is a d_2-transitive partition of G' of size $k+m$.

Conversely, let $\pi = \{V_1, V_2, \ldots, V_{k+m}\}$ be a d_2-transitive partition of G' of size $k+m$. Let $V_{p_1}, V_{p_2}, \ldots, V_{p_t}$ be the sets in π that do not contain any subdivision vertex, where $p_1 < p_2 < \ldots < p_t$. Since there are m subdivision vertices in G', there exist at least k such sets in π. Therefore, $t \geq k$. Let us consider the vertex partition, say $\pi' = \{V_1', V_2', \ldots, V_k'\}$ of G as follows: $V_i' = V_{p_i}$ for $2 \leq i \leq k$, and V_1' contains the rest of the vertices of G. Since V_{p_i} d_2-dominates V_{p_j} in G', every vertex of V_{p_j} must be adjacent to some vertex of V_{p_i} in G. Therefore, π' is a transitive partition of G of size k. □

From Claim 4.1, it follows that the Maximum d_2-Transitivity Decision Problem is NP-complete for split graphs. This completes the proof of Theorem 4.1 □

Bipartite graphs

In this subsection, we show that the Maximum d_2-Transitivity Decision Problem is NP-complete for bipartite graphs as well.

Theorem 4.2. *The Maximum d_2-Transitivity Decision Problem is NP-complete for bipartite graphs.*

Proof. Given a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of a bipartite graph, we can verify in polynomial time whether π is a d_2-transitive partition of that graph or not. Hence, the Maximum d_2-Transitivity Decision Problem is in NP. To prove that this problem is NP-hard, we show a polynomial time reduction from the Maximum Transitivity Decision Problem in general graphs, which is known to be NP-complete [10]. The reduction is as follows: given an instance of the Maximum Transitivity Decision Problem, that is, a graph $G = (V, E)$ and an integer k, we construct another graph $G' = (V', E')$. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$ vertices and edges of G, respectively. For each vertex $v_i \in V$, we take two vertices v_i^1 and v_i^2 in V', and for each edge $e_j \in E$, we take two vertices u_j^1 and u_j^2 in V'. Hence, $V' = V_1 \cup V_2 \cup U_1 \cup U_2$, where $V_1 = \{v_1^1, v_2^1, \ldots, v_n^1\}$, $V_2 = \{v_1^2, v_2^2, \ldots, v_n^2\}$, $U_1 = \{u_1^1, u_2^1, \ldots, u_m^1\}$ and $U_2 = \{u_1^2, u_2^2, \ldots, u_m^2\}$. Next, we add edges between v_i^1 and u_j^1 if e_j is incident on v_i in G. Similarly, we add edges between v_i^2 and u_j^2 if e_j is incident on v_i in G. Finally, we add edges between every vertex of U_1 and every vertex of U_2; that is, U_1 and U_2 induce a complete bipartite graph in G'. Clearly, G' is a bipartite graph with $V_1 \cup U_2$ and $V_2 \cup U_1$ forming the bipartition, and G' has $2(m+n)$ vertices and $m^2 + 4m$ edges. The construction is illustrated in Figure 4.2.

Claim 4.2. *The graph G has a transitive partition of size k if and only if G' has a d_2-transitive partition of size $k+2m$.*

Proof of Claim 4.2. Let $\pi = \{W_1, W_2, \ldots, W_k\}$ be a transitive partition of G of size k. Let us consider the following vertex partition, say $\pi' = \{W_1', W_2', \ldots, W_{k+2m}'\}$ of G' as follows: for each $1 \leq i \leq k$, W_i' is defined as $W_i' = \{v_j^1, v_j^2 | v_j \in W_i\}$ and for each $k+1 \leq i \leq 2m$, W_i' contains exactly one vertex from $U_1 \cup U_2$. Clearly, π' is a vertex partition of G' of size $k+2m$. We show that π' is a d_2-transitive partition of G'. Note that, by the construction of G', it follows that if $e_t = v_p v_q$ is an edge in G, then $d(v_p^1, v_q^1) = d(v_p^2, v_q^2) = 2$ in G'. Hence, since W_i dominates W_j in G, W_i' d_2-dominates W_j' in G' for all $1 \leq i < j \leq k$.

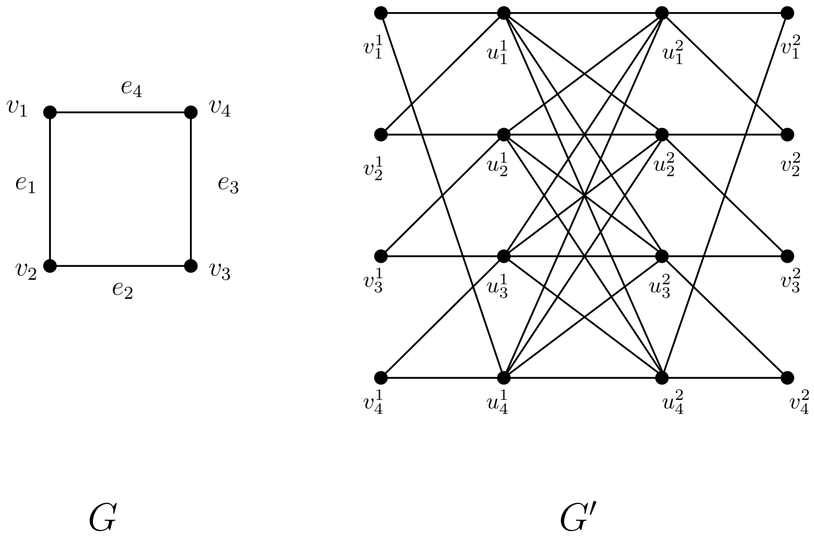

Figure 4.2: Construction of G' in Theorem 4.2.

Also, since $U_1 \cup U_2$ induces a complete bipartite graph in G', W_i' d_2-dominates W_j' for all $k+1 \leq i < j \leq k+2m$. Note that every vertex of U_1 is at a distance of two from every vertex of V_2, and every vertex of U_2 is at a distance of two from every vertex of V_1. Therefore, every W_j' for $k+1 \leq j \leq k+2m$, is d_2-dominated by every W_i', where $1 \leq i \leq k$. Therefore, π' is a d_2-transitive partition of G' of size $k+2m$.

Conversely, let $\pi = \{W_1, W_2, \ldots, W_{k+2m}\}$ be a d_2-transitive partition of G' of size $k+2m$. Let $W_{p_1}, W_{p_2}, \ldots, W_{p_t}$ be the sets in π that do not contain any vertex from $U_1 \cup U_2$, where $p_1 < p_2 < \ldots < p_t$. Since there are $2m$ vertices in $U_1 \cup U_2$, there exist at least k such sets in π. Therefore, $t \geq k$. Note that since the distance between any vertex of V_1 and any vertex of V_2 is more than two, if W_{p_t} contains a vertex from V_1 (or V_2), then W_{p_i} for all $i < t$ contains at least one vertex from V_1 (respectively from V_2). Let us assume that W_{p_t} contains vertices from V_1. Consider a vertex partition, say $\pi' = \{W_1', W_2', \ldots, W_k'\}$, of G as follows: $W_i' = \{v_j | v_j^1 \in W_{p_i}\}$ for each $2 \leq i \leq k$, and W_1' contains every other vertex of G. Note that if $v_r^1 \in W_{p_j}$ for some $2 \leq j \leq k$, then every W_{p_i}, with $i < j$, contains at least one vertex, say v_s^1, such that $d(v_s^1, v_r^1) = 2$ in G'. By the construction of G', it follows that v_s and v_r are adjacent in G. Hence, W_i' dominates W_j' for all $1 \leq i < j \leq k$. Therefore, π' is a transitive partition of G of size k. For the case where W_{p_t} does not contain any vertex from V_1, that is, it contains vertices from V_2 only, we can argue in a similar way by considering vertices from V_2 and show that π' is a transitive partition of G of size k. $\square$

Claim 4.2 shows that the Maximum d_2-Transitivity Decision Problem is NP-complete for bipartite graphs. This completes the proof of Theorem 4.2 $\square$

5. Conclusion

In this paper, we have introduced the notion of d_2-transitivity in graphs, which is a generalization of transitivity. First, we have shown some basic properties for d_2-transitivity. We have also proved that the d_2-transitivity can be computed in linear time for the complement of bipartite graphs and bipartite chain graphs. On the other side, we have shown that the Maximum d_2-Transitivity Decision Problem is NP-complete for split graphs and bipartite graphs. It would be interesting to investigate the complexity status of this problem in other graph classes. Designing an approximation algorithm for this problem would be another challenging open problem. We know that for a graph G, the upper bound of transitivity is equal to the lower bound of d_2-transitivity of G, namely $\Delta(G)+1$. Naturally, for the complete graph on n vertices K_n, these two parameters are equal. It would be an interesting open problem to characterize the graphs where these two parameters are the same.

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