Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **13** (2024) 1–5
DOI: 10.47443/dml.2023.211

Research Article

Combinatorial identities and hypergeometric functions, II

Horst Alzer*

Morsbacher Straße 10, 51545 Waldbröl, Germany

(Received: 6 November 2023. Received in revised form: 17 January 2024. Accepted: 22 January 2024. Published online: 3 April 2024.)

Abstract

Properties of the classical Gaussian hypergeometric function are applied to prove some combinatorial identities. Among others, a corrected and simplified version of a formula of D. Lim [*Notes Number Theory Discrete Math.* **29** (2023) 421–425] is offered.

Keywords: combinatorial identity; hypergeometric function; Knuth's old sums.

2020 Mathematics Subject Classification: 05A19, 33C05.

1. Introduction and statement of the results

Two classical results in the theory of combinatorial identities state that

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}2^{-k}\binom{2k}{k}=2^{-2n}\binom{2n}{n}, \tag{1}$$

$$\sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}2^{-k}\binom{2k}{k}=0. \tag{2}$$

These formulas are known as Knuth's old sums and also as the Reed Dawson identities, see [9, p. 71]. The following related identities are due to Riordan [9, p. 72]:

$$\sum_{k=0}^{2n-1}(-1)^k\binom{2n}{k+1}2^{-k}\binom{2k}{k}=2^{-2n+1}n\binom{2n}{n}, \tag{3}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n+1}{k+1}2^{-k}\binom{2k}{k}=2^{-2n}(2n+1)\binom{2n}{n}. \tag{4}$$

We remark that (3) corrects a misprint given in [9, p. 72], where 2^{-2n+1} is replaced by 2^{-2n-1}.

The work on the present article was inspired by the paper [5] of Lim. He used properties of the Gaussian hypergeometric function

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},\quad (a)_k=a(a+1)\cdots(a+k-1),$$

to prove a generalization of (3) and (4):

$$\sum_{k=0}^{n}(-1)^k\binom{n+j}{k+j}2^{-k}\binom{2k}{k}\frac{(j+1)_k}{(2)_k}=\begin{cases}\displaystyle\binom{2n+j}{j}\frac{(1/2)_n}{(1)_n}, & \text{if } n \text{ is even},\\[2ex] \displaystyle\frac{1}{2}\binom{2n+j+1}{j}\frac{(3/2)_n}{(2)_n}, & \text{if } n \text{ is odd}.\end{cases} \tag{5}$$

Unfortunately, this result is stated incorrectly. Here, a correct version of (5), written in a simpler and slightly more elegant form, is offered. Moreover, four closely related identities are presented. In particular, extensions of (1), (2), (3), and (4) are obtained. Just like Lim [5], the results in this paper are proved by making use of the function ${}_2F_1$. Additional combinatorial identities that were deduced by using properties of hypergeometric functions can be found, for example, in the recently published papers [1,3,7,8]. Readers are also refered to Bailey's fundamental book "Generalized Hypergeometric Series" [2].

*E-mail address: h.alzer@gmx.de

As usual, $[x]$ denotes the greatest integer less than or equal to x.

Theorem 1.1. *Let $n \ge 0$ and $j \ge 0$ be integers. Then*

$$\sum_{k=0}^{n} \frac{(-1/2)^k}{k+1} \binom{2k}{k} \binom{k+j}{k} \binom{n+j}{n-k} = \frac{1}{2^n} \binom{n}{[n/2]} \binom{n+j}{n}, \tag{6}$$

$$\sum_{k=0}^{n} \frac{(-1/2)^k}{(k+1)(k+2)} \binom{2k}{k} \binom{k+j}{k} \binom{n+j}{n-k} = \begin{cases} \dfrac{1}{3\cdot 2^n} \dfrac{2n+3}{n+2} \dbinom{n}{n/2} \dbinom{n+j}{n}, & \text{if } n \text{ is even,} \\[2ex] \dfrac{1}{3\cdot 2^{n-1}} \dbinom{n}{(n-1)/2} \dbinom{n+j}{n}, & \text{if } n \text{ is odd,} \end{cases} \tag{7}$$

$$\sum_{k=0}^{n} \left(\frac{-1}{2}\right)^k \binom{2k}{k} \binom{k+j}{k} \binom{n+j}{n-k} = \begin{cases} \dfrac{1}{2^n} \dbinom{n}{n/2} \dbinom{n+j}{n}, & \text{if } n \text{ is even,} \\[2ex] 0, & \text{if } n \text{ is odd,} \end{cases} \tag{8}$$

$$\sum_{k=0}^{n} \frac{1}{2^k \binom{2n}{k}} \binom{2k}{k} \binom{k+j}{k} \binom{n+j}{n-k} = 2^{2n} \frac{\binom{n+j}{n}\binom{n-1/4}{n}}{\binom{2n}{n}}, \tag{9}$$

$$\sum_{k=0}^{n} \frac{1}{2^k \binom{2n+1}{k}} \binom{2k}{k} \binom{k+j}{k} \binom{n+j}{n-k} = 2^{2n+1} \frac{\binom{n+j}{n}}{\binom{2n+1}{n}} \left\{ \binom{n+3/4}{n+1} - \binom{n+1/4}{n+1} \right\}. \tag{10}$$

Remark 1.1. (i) *Identity (6) is a corrected version of (5).*

(ii) *The special cases $j = 1$ in (6) and $j = 0$ in (8) lead to (3), (4) and (1), (2), respectively.*

(iii) *From (6) and (7) we obtain, for odd n, the following identity:*

$$\sum_{k=0}^{n} \frac{(-1/2)^k}{k+1} \binom{2k}{k} \binom{k+j}{k} \binom{n+j}{n-k} = \frac{3}{2} \sum_{k=0}^{n} \frac{(-1/2)^k}{(k+1)(k+2)} \binom{2k}{k} \binom{k+j}{k} \binom{n+j}{n-k}.$$

(iv) *Applying (6) and (8) gives, for even n, the following identity:*

$$\sum_{k=0}^{n} \frac{(-1/2)^k}{k+1} \binom{2k}{k} \binom{k+j}{k} \binom{n+j}{n-k} = \sum_{k=0}^{n} \left(\frac{-1}{2}\right)^k \binom{2k}{k} \binom{k+j}{k} \binom{n+j}{n-k}.$$

This identity is equivalent to

$${}_2F_1(-n, 1/2; 2; 2) = {}_2F_1(-n, 1/2; 1; 2),$$

which is a special case of a formula given in [4, (18)].

An application of (8) leads to a relative of (8).

Corollary 1.1. *Let $n \ge 0$ and $j \ge 0$ be integers. Then*

$$\sum_{k=0}^{n+1} \left(\frac{-1}{2}\right)^k \binom{2k}{k} \binom{k+j}{k} \binom{n+j}{n+1-k} = \begin{cases} \dfrac{-1}{2^n} \dbinom{n}{n/2} \dbinom{n+j}{n}, & \text{if } n \text{ is even,} \\[2ex] \dfrac{1}{2^{n+1}} \dbinom{n+1}{(n+1)/2} \dbinom{n+1+j}{n+1}, & \text{if } n \text{ is odd.} \end{cases} \tag{11}$$

Remark 1.2. *Identity (11) with $j = 0$ gives striking companions to (1), (2) and (3), (4):*

$$\sum_{k=1}^{2n+1} (-1)^k \binom{2n}{k-1} 2^{-k} \binom{2k}{k} = -2^{-2n} \binom{2n}{n},$$

$$\sum_{k=1}^{2n+2} (-1)^k \binom{2n+1}{k-1} 2^{-k} \binom{2k}{k} = 2^{-2n-2} \binom{2n+2}{n+1}.$$

2. Proofs

Proof of Theorem 1.1. We need the following formulas (see [6, p. 493]):

$$_2F_1(-n,a;2a+1;2)=\frac{1}{\sqrt{\pi}}\Gamma(a+1/2)\Big(\frac{1+(-1)^n}{2}\frac{\Gamma((n+1)/2)}{\Gamma(a+(n+1)/2)}+\frac{1-(-1)^n}{2}\frac{\Gamma(n/2+1)}{\Gamma(a+n/2+1}\Big), \tag{12}$$

$$_2F_1(-n,a;2a;2)=(1+(-1)^n)\frac{n!\Gamma(a+1/2)}{2^{n+1}\Gamma(n/2+1)\Gamma(a+(n+1)/2)}, \tag{13}$$

$$_2F_1(-n,a;-2n;2)=\frac{2^{2n}n!}{(2n)!}\Big(\frac{a+1}{2}\Big)_n, \tag{14}$$

$$_2F_1(-n,a;-2n-1;2)=\frac{2^{2n+1}n!}{(2n+1)!}\Big\{\Big(\frac{a+1}{2}\Big)_{n+1}-\Big(\frac{a}{2}\Big)_{n+1}\Big\}. \tag{15}$$

Moreover, we make use of the formulas

$$_2F_1(-2n,a;2a+2;2)=\Big(1+\frac{2n}{a+1}\Big)\frac{(1/2)_n}{(a+3/2)_n} \tag{16}$$

and

$$_2F_1(-2n-1,a;2a+2;2)=\frac{1}{a+1}\frac{(3/2)_n}{(a+3/2)_n}, \tag{17}$$

which are given in [4, (19), (30)].

We define

$$T_n(j,c)=\sum_{k=0}^{n}\Big(\frac{-1}{2}\Big)^k\binom{2k}{k}\binom{n+j}{n-k}\frac{(j+1)_k}{(c)_k}. \tag{18}$$

Using

$$\binom{2k}{k}=2^{2k}\frac{(1/2)_k}{k!},\quad \binom{n+j}{n-k}=(-1)^k\binom{n+j}{n}\frac{(-n)_k}{(j+1)_k}\quad\text{and}\quad (-n)_k=0\quad(k>n)$$

we obtain the representation

$$T_n(j,c)=\binom{n+j}{n}\sum_{k=0}^{\infty}\frac{(-n)_k(1/2)_k}{(c)_k}\frac{2^k}{k!}=\binom{n+j}{n}{}_2F_1(-n,1/2;c;2). \tag{19}$$

(i). We apply (12) with $a=1/2$ and the formula

$$\Gamma\Big(n+\frac{1}{2}\Big)=\sqrt{\pi}\frac{(2n)!}{n!2^{2n}}.$$

Then

$$_2F_1(-n,1/2;2;2)=\frac{1}{\sqrt{\pi}}\Big(\frac{1+(-1)^n}{2}\frac{\Gamma((n+1)/2)}{\Gamma(n/2+1)}+\frac{1-(-1)^n}{2}\frac{\Gamma(n/2+1)}{\Gamma((n+3)/2)}\Big)=\frac{1}{2^n}\binom{n}{[n/2]}, \tag{20}$$

so that (19) with $c=2$ and (20) lead to

$$T_n(j,2)=\frac{1}{2^n}\binom{n}{[n/2]}\binom{n+j}{n}. \tag{21}$$

Since

$$\frac{(j+1)_k}{(2)_k}=\frac{1}{k+1}\binom{k+j}{k},$$

we conclude from (18) that

$$T_n(j,2)=\sum_{k=0}^{n}\frac{(-1/2)^k}{k+1}\binom{2k}{k}\binom{k+j}{k}\binom{n+j}{n-k}. \tag{22}$$

From (21) and (22) we obtain (6).

(ii). From (19) with $c=1$ and (13) with $a=1/2$, we obtain

$$\begin{aligned} T_n(j,1)=\binom{n+j}{n}{}_2F_1(-n,1/2;1;2) &= \binom{n+j}{n}(1+(-1)^n)\frac{n!}{2^{n+1}\Gamma(n/2+1)^2} \\ &= \begin{cases}\frac{1}{2^n}\binom{n}{n/2}\binom{n+j}{n}, & \text{if } n \text{ is even,}\\ 0, & \text{if } n \text{ is odd.}\end{cases}\end{aligned} \tag{23}$$

We have

$$\frac{(j+1)_k}{(1)_k}=\binom{k+j}{k},$$

so that (18) yields

$$T_n(j,1)=\sum_{k=0}^{n}\Big(\frac{-1}{2}\Big)^k\binom{2k}{k}\binom{k+j}{k}\binom{n+j}{n-k}. \tag{24}$$

From (23) and (24), we conclude that (8) is valid.

(iii). Formula (14) with $a=1/2$ gives

$$_2F_1(-n,1/2;-2n;2)=\frac{2^{2n}n!}{(2n)!}(3/4)_n=2^{2n}\frac{\binom{n-1/4}{n}}{\binom{2n}{n}}. \tag{25}$$

We now use (18), (19) with $c=-2n$, (25) and

$$\frac{(j+1)_k}{(-2n)_k}=(-1)^k\frac{\binom{k+j}{k}}{\binom{2n}{k}}.$$

This leads to (9).

(iv). We apply (15) with $a=1/2$. Then

$$_2F_1(-n,1/2;-2n-1;2)=\frac{2^{2n+1}n!}{(2n+1)!}\big\{(3/4)_{n+1}-(1/4)_{n+1}\big\}=\frac{2^{2n+1}}{\binom{2n+1}{n}}\left\{\binom{n+3/4}{n+1}-\binom{n+1/4}{n+1}\right\}. \tag{26}$$

We have

$$\frac{(j+1)_k}{(-2n-1)_k}=(-1)^k\frac{\binom{k+j}{k}}{\binom{2n+1}{k}}. \tag{27}$$

From (18), (19) with $c=-2n-1$ and (26), (27), we obtain (10).

(v). We have

$$\frac{(j+1)_k}{(3)_k}=\frac{2}{(k+1)(k+2)}\binom{k+j}{k}. \tag{28}$$

Applying (16) with $a=1/2$, (18), (19), (28) and

$$\frac{(1/2)_n}{(2)_n}=\frac{1}{2^{2n}(n+1)}\binom{2n}{n},$$

we obtain

$$T_{2n}(j,3)=2\sum_{k=0}^{2n}\frac{(-1/2)^k}{(k+1)(k+2)}\binom{2k}{k}\binom{k+j}{k}\binom{2n+j}{2n-k} \tag{29}$$

and

$$T_{2n}(j,3)=\binom{2n+j}{2n}\Big(1+\frac{4n}{3}\Big)\frac{(1/2)_n}{(2)_n}=\frac{4n+3}{3(n+1)2^{2n}}\binom{2n+j}{2n}\binom{2n}{n}. \tag{30}$$

Next, we apply (17) with $a=1/2$, (18), (19), (28) and

$$\frac{(3/2)_n}{(2)_n}=\frac{1}{2^{2n}}\binom{2n+1}{n}.$$

This leads to

$$T_{2n+1}(j,3)=2\sum_{k=0}^{2n+1}\frac{(-1/2)^k}{(k+1)(k+2)}\binom{2k}{k}\binom{k+j}{k}\binom{2n+1+j}{2n+1-k} \tag{31}$$

and

$$T_{2n+1}(j,3)=\binom{2n+1+j}{2n+1}\frac{2\cdot(3/2)_n}{3\cdot(2)_n}=\frac{2}{3\cdot 2^{2n}}\binom{2n+1+j}{2n+1}\binom{2n+1}{n}. \tag{32}$$

From (29), (30), (31), and (32), we conclude that (7) holds. □

Proof of Corollary 1.1. Let

$$S_n(j)=\sum_{k=0}^{n}a_k(j)\binom{n+j}{k+j}\quad\text{and}\quad a_k(j)=\Big(\frac{-1}{2}\Big)^k\binom{2k}{k}\binom{k+j}{k}.$$

Using the recurrence relation

$$\binom{n+1+j}{k+j} = \binom{n+j}{k+j} + \binom{n+j}{k+j-1},$$

we have

$$S_{n+1}(j) - S_n(j) = \sum_{k=0}^{n+1} a_k(j) \binom{n+j}{k+j-1}. \tag{33}$$

From (8), it follows that

$$S_{n+1}(j) - S_n(j) = -S_n(j) = -\frac{1}{2^n}\binom{n}{n/2}\binom{n+j}{n}, \quad \text{if } n \text{ is even}, \tag{34}$$

and

$$S_{n+1}(j) - S_n(j) = S_{n+1}(j) = \frac{1}{2^{n+1}}\binom{n+1}{(n+1)/2}\binom{n+1+j}{n+1}, \quad \text{if } n \text{ is odd}. \tag{35}$$

Consequently, (11) is obtained from (33), (34), and (35). □

Acknowledgement

The author thanks K. C. Richards and the two referees for their helpful comments.

References

[1] H. Alzer, K. C. Richards, Combinatorial identities and hypergeometric functions, *Rocky Mountain J. Math.* **52** (2022) 1921–1928.
[2] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University, Cambridge, 1935.
[3] W. Chu, E. Kiliç, Binomial sums involving Catalan numbers, *Rocky Mountain J. Math.* **51** (2021) 1221–1225.
[4] Y. Kim, A. K. Rathie, R. B. Paris, Evaluations of some terminating hypergeometric ${}_2F_1(2)$ series with applications, *Turkish J. Math.* **42** (2018) 2563–2575.
[5] D. Lim, A note on a generalization of Riordan's combinatorial identity via a hypergeometric approach, *Notes Number Theory Discrete Math.* **29** (2023) 421–425.
[6] A. P. Prudnikov, Y. A. Brychkov, O. I. Marichev, *Integrals and Series, Vol. 3: More Special Functions*, Gordon and Breach, New York, 1986.
[7] A. K. Rathie, I. Kim, R. B. Paris, A note on a generalization of two well-known combinatorial identities via a hypergeometric approach, *Integers* **22** (2022) #A28.
[8] A. K. Rathie, D. Lim, A note on generalization of combinatorial identities due to Gould and Touchard, *Axioms* **12** (2023) #268.
[9] J. Riordan, *Combinatorial Identities*, Robert E. Krieger, Huntington, 1979.