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Research Article

On colorings and orientations of signed graphs

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Abstract

A classical theorem independently due to Gallai and Roy states that a graph G has a proper k-coloring if and only if G has an orientation without coherent paths of length k. An analogue of this result for signed graphs is proved in this article.

Keywords: signed graph; signed-graph coloring; orientation; bidirection.

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1. Introduction

Theorem 1.1 is a classical result characterizing the existence of a proper k-coloring of a graph G in terms of orientations.

Theorem 1.1 (Gallai [3] and Roy [8]). *For a graph G the following statements are equivalent.*

(1) G has a proper k-coloring.

(2) G has an acyclic orientation without coherent paths of length k.

(3) G has an orientation without coherent paths of length k.

A *signed graph* is a pair (G, σ) in which G is a graph and $\sigma\colon E(G) \to \{+, -\}$. The first paper on an algebraic notion of coloring signed graphs is by Zaslavsky [11].[†] His approach was clarified by Máčajová, Raspaud, and Škoviera [7] as follows. Let $M_{2k+1} = \{-k, \ldots, -1, 0, 1, \ldots, k\}$ and $M_{2k} = \{-k, \ldots, -1, 1, \ldots, k\}$. An *$n$-coloring* of a signed graph (G, σ) is a function $\kappa\colon V(G) \to M_n$. An n-coloring κ is *proper* when for each edge e in (G, σ) with endpoints u and v (possibly equal) $\kappa(u) \neq \sigma(e)\kappa(v)$. Evidently every signed graph (G, σ) without positive loops has a proper $2|V(G)|$-coloring and if (G, σ) has a proper n-coloring, then (G, σ) has a proper $(n + 1)$-coloring. Thus it makes sense to define the *chromatic number* $\chi(G, \sigma)$ as the smallest n such that (G, σ) has a proper n-coloring.

The coloring of (G, σ) in [7, 11] use color sets in $\mathbb{Z}$. There are other algebraic notions of coloring signed graphs which together with integer coloring all have a common generalization in the concept of permutation coloring by Slilaty [9]. We will give a short survey of these notions and their relationships in Section 2. Arguably, however, integer coloring of signed graphs is the most central of these notions. Evidence enough is the rich theory of chromatic polynomials developed by Zaslavsky [12, 14] which broadly generalizes chromatic polynomials of ordinary graphs along with their relationship to matroid theory.

The main result of the present article is Theorem 5.2 which is an analogue of Theorem 1.1 for signed graphs. Theorem 5.1 is also a result of interest.

2. A survey of different coloring notions

Slilaty's notion of coloring permutation-gain graphs [9] specializes to signed graphs as follows. Let K be a set of colors on which the group $\{+, -\}$ has a left multiplicative action. A *K-coloring* of a signed graph (G, σ) is a function $f\colon V(G) \to K$ and the K-coloring is said to be *proper* when for each edge e with endpoints u and v we have that $f(u) \neq \sigma(e)f(v)$. This includes the case for which e is a loop, that is, when $u = v$. Thus a positive loop prevents any K-coloring from being proper while a negative loop simply requires that the color at its vertex is not fixed under the group action.

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†The earlier paper by Cartwright and Harary [1] which states its main concept as "coloring signed graphs" is actually better understood within the topic of clusterability of signed graphs.

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Color sets $K = M_{2t} = \{-t, \ldots, -1, 1, \ldots, t\}$ and $K = M_{2t+1} = \{-t, \ldots, -1, 0, 1, \ldots, t\}$ within $\mathbb{Z}$ were already discussed in the introduction. Evidently, K-coloring for $K = M_{2t}$ or M_{2t+1} falls under the notion of permutation coloring. Using color set K being a finite additive group is also possible, but has received less attention. Let's call this type of coloring *modular coloring*. Again, modular coloring is clearly generalized by permutation coloring. Kang and Steffen have recently made investigations [4, 5] into modular coloring. Kim and Ozeki [6] noted that integer coloring and modular coloring are both generalized by Dvořák and Postle's correspondence coloring (a.k.a. DP-coloring) [2]. Permutation coloring is also generalized by correspondence coloring; however, permutation coloring does provides more structure. In particular, permutation coloring has a well-defined notion of chromatic polynomials and extensions of colorings to covering graphs.

Say that the group $\{+, -\}$ acts on the set K. Hence for each $x \in K$ either x is fixed (i.e., $x = -x$) or $\{x, -x\}$ is an orbit of size two. Propositions 2.1–2.3 contain some basic observations concerning K-coloring of signed graphs.

Proposition 2.1. *Say that $\{+, -\}$ acts on finite sets K_1 and K_2 in which K_i has s_i fixed elements and t_i orbits of order two. If (G, σ) has a proper K_1-coloring, $|K_2| \geq |K_1|$, and $t_2 \geq t_1$, then (G, σ) has a proper K_2-coloring.*

Proof. Because $|K_2| \geq |K_1|$ and $t_2 \geq t_1$ there is an injective function $\iota\colon K_1 \to K_2$ such that for every $x \in K_1$ in an orbit of size two, $-\iota(x) = \iota(-x)$. Now if f is a proper K_1-coloring of (G, σ), then ιf is a proper K_2-coloring of (G, σ). □

Proposition 2.2. *Let A be any additive group with $|A| = 2k+1$. If (G, σ) is a signed graph, then there is a bijection between the proper A-colorings of (G, σ) and the proper $(2k+1)$-colorings of (G, σ).*

Proof. For an additive group of odd order, the action of $\{+, -\}$ on A fixes 0, only. Similarly, the action of $\{+, -\}$ on M_{2k+1} fixes 0, only. Thus the injection defined in the proof of Proposition 2.1 is now a bijection and provides the desired correspondence. □

Proposition 2.3. *If every point of K is fixed under the action of K, then (G, σ) has a proper K-coloring if and only if the underlying graph G has a proper $|K|$-coloring.*

3. Background material

In a graph G, an *incidence* is where an end of an edge meets a vertex. As such every edge (including loops) has two distinct incidences. An incidence can be denoted by a pair (v, e) in which vertex v is an endpoint of edge e. Although this notation does not distinguish between the two distinct incidences of a loop, it can be modified as $(v, e)_1$ and $(v, e)_2$ in order to distinguish the two. A *bidirection* on G is a function $\beta\colon I(G) \to \{1, -1\}$. When $\beta(v, e) = +1$ we think of the incidence as having an arrow pointing at v and when $\beta(v, e) = -1$ the incidence has its arrow pointing away from v. Thus bidirections produce three types of edges: *introverted*, *extroverted*, and *directed* (see Figure 1).

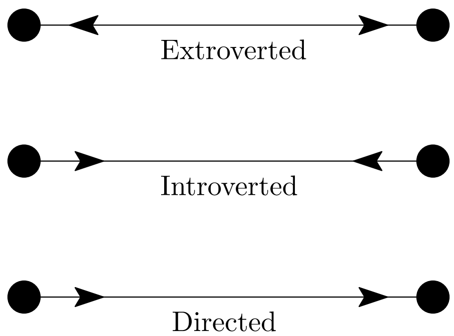

Figure 1: The three types of edges in a bidirected graph.

Again, a *signed graph* is a pair (G, σ) in which G is a graph and $\sigma\colon E(G) \to \{+, -\}$. A path or cycle C in G is called *positive* (or *negative*) when the product of signs on its edges is positive (or negative). A *circuit* in (G, σ) is a subgraph which is either a positive cycle, two negative cycles which intersect in a single vertex (called a *tight handcuff*), or two vertex-disjoint negative cycles along with a minimal connecting path (called a *loose handcuff*).

An *orientation* of a signed graph (G, σ) is a bidirection β satisfying $\beta(v, e)\beta(u, e) = -\sigma(e)$. When e is a loop, this means $\beta((v, e)_1)\beta((u, e)_2) = -\sigma(e)$. As such, each negative edge is either introverted or extroverted and each positive edge has one of two possible directions. An *oriented* signed graph is a triple (G, σ, β) where β is an orientation of (G, σ). A vertex v in (G, σ, β) is a *source* (or *sink*) when all of the bidirectional arrows at v are directed towards (or away) from v. A vertex in v in (G, σ, β) is *singular* when it is either a source or a sink.

Let (G, σ, β) be an oriented signed graph. A path P in (G, σ, β) is *coherent* when every internal vertex of (P, σ, β) is non-singular. A cycle C in (G, σ, β) is *coherent* when every vertex of (C, σ, β) is non-singular. It is worth noting that if a cycle C is negative, then (C, σ, β) has an odd number of singular vertices and so is never coherent. A circuit C in (G, σ) is *coherent* when every vertex in (C, σ, β) is nonsingular. The reader can check that there are exactly two possibilities for a coherent β on a circuit C and if β is one of them, then $-\beta$ is the other. An orientation β of (G, σ) is *acyclic* when it contains no coherent circuit. Zaslavsky [13, Corollary 5.3] proved that if (G, σ, β) is acyclic, then (G, σ, β) has a singular vertex.

Given a signed graph (G, σ), a *switching function* is a function $\eta\colon V(G) \to \{+, -\}$. Define σ^η by $\sigma^\eta(e) = \eta(u)\sigma(e)\eta(v)$ in which u and v are the endpoints of e. (This includes the case for a loop.) If β is a bidirection of G, then $\eta\beta$ is a bidirection on G. One can think of $\eta\beta$ as being obtained from β by reversing the arrows at v when $\eta(v) = -$ and leaving the arrows at v the same when $\eta(v) = +$. Since a vertex is singular in (G, σ, β) if and only if it is singular in $(G, \sigma^\eta, \eta\beta)$, a signed-graphic circuit C is coherent in (G, σ, β) if and only if it is coherent in $(G, \sigma^\eta, \eta\beta)$.

Now if κ is a proper integer coloring of (G, σ), then there is a natural orientation of (G, σ) induced by κ, call it β_κ. For each edge e with ends (u, e) and (v, e) there is exactly one choice for orientation $\beta_\kappa(e)$ so that $\beta_\kappa(u, e)\kappa(u) + \beta_\kappa(v, e)\kappa(v) > 0$. Call β_κ the orientation *induced* by κ. Zaslavsky [11] notes that β_κ is acyclic when κ is proper. He then uses induced orientations to generalize the work on pairings of acyclic orientations and proper colorings of ordinary graphs by Stanley [10]. If η is a switching function for (G, σ) and κ a proper integer coloring, then $\eta\kappa$ is a proper coloring of (G, σ^η). In fact, $\kappa \mapsto \eta\kappa$ is a bijection between the collection of all proper colorings of (G, σ) and those of (G, σ^η). If κ is a proper integer coloring of (G, σ), then $\beta_{\eta\kappa} = \eta\beta_\kappa$.

4. Normalizing colorings and acyclic orientations

If β is an acyclic orientation of (G, σ), then there is partitioning $L_0, L_1, \ldots$ of $V(G)$ which we shall call the *canonical level decomposition* of (G, σ, β) which is defined as follows. Let L_0^+ denote the set of sinks and isolated vertices in (G, σ, β) and let L_0^- denote the set of sources in (G, σ, β). Let $L_0 = L_0^+ \cup L_0^-$. Because (G, σ, β) is acyclic $L_0 \neq \emptyset$. Now let $G_{i+1} = G - (L_0 \cup \cdots \cup L_i)$. If $G_{i+1} \neq \emptyset$, then let L_{i+1}^+ and L_{i+1}^- be respectively the set of sinks along with isolated vertices and the set of sources in acyclic (G_{i+1}, σ, β) and let $L_{i+1} = L_{i+1}^+ \cup L_{i+1}^-$; otherwise, we halt the process. The *normalization* of β is $\eta\beta$ in which η is a switching function for which $\eta(v) = \epsilon$ when $v \in L_i^\epsilon$. Note that the canonical level decompositions of β and $\eta\beta$ are both $L_0, L_1, \ldots$.

Proposition 4.1. *Let β be a normalized acyclic orientation of (G, σ, β) and let $L_0, \ldots, L_{k-1}$ be the canonical level decomposition.*

(1) If e is a negative edge, then e is extroverted.

(2) If e is a positive edge with head end in L_i and tail end in L_j, then $i < j$.

(3) If $v \in L_j$ for $j > 0$, then there is a positive edge e with v as its tail with $w \in L_{j-1}$ as its head.

Proof. (1) If e is an introverted negative edge, then the endpoint(s) of e would be indicated as sources in the construction of the canonical level decomposition. However, since β is normalized, every vertex is indicated as a sink or isolated vertex in the construction of the canonical level decomposition, a contradiction.

(2) If e is a positive edge for which $j \leq i$, then its tail would be indicated as a source during the construction of the canonical level decomposition. This yields the same contradiction as in the proof of (1).

(3) If such a positive edge does not exist, then v would have been removed at an earlier iteration during the construction of the canonical level decomposition. □

An n-coloring κ is *normalized* by switching function η for which $\eta(v) = -$ if and only if $\kappa(v) < 0$. Thus $\eta\kappa(v) \geq 0$ for all v. Note that all negative edges of $(G, \sigma, \beta_{\eta\kappa})$ are extroverted.

Proposition 4.2. *Let κ be a proper n-coloring of (G, σ) chosen among all proper n-colorings so that $\sum_{v \in V(G)} |\kappa(v)|$ is a maximum. If $\eta\kappa$ is the normalization of κ and $L_0, \ldots, L_{k-1}$ is the canonical level decomposition of $\beta_{\eta\kappa}$, then*

(1) $\eta\kappa(v) = i$ if and only if $v \in L_{t-i}$ when $n = 2t$ is even,

(2) $\eta\kappa(v) = i$ if and only if $v \in L_{t-1-i}$ when $n = 2t - 1$ is odd, and

(3) $\beta_{\eta\kappa}$ is normalized.

Proof. (1 and 2) Since $\eta\kappa$ is normalized, each $v \in V(G)$ satisfies $\eta\kappa(v) \in \{1, \ldots, t\}$ when $n = 2t$ and $\eta\kappa(v) \in \{0, 1, \ldots, t-1\}$ when $n = 2t - 1$. We proceed by induction with base case covering L_0. In the base case, if $\eta\kappa(v)$ has the maximum value, then $\beta_{\eta\kappa}$ must be a sink or isolated vertex and so $v \in L_0^+$. Conversely if $v \in L_0^+$ but $\eta\kappa(v)$ does not have the maximum value, then v cannot have a positive link to any vertex having the maximum value under $\eta\kappa$. Thus $\eta\kappa(v)$ may be changed to the maximum value without affecting the propriety of the coloring, a contradiction of the maximality of $\sum_{v\in V(G)} |\kappa(v)|$. Thus $L_0^+ = \{v : \kappa(v) = k\}$ when n is even and $L_0^+ = \{v : \kappa(v) = k-1\}$ when n is odd. Finally we must show that $L_0^- = \emptyset$. By way of contradiction, assume that $v \in L_0^-$. Thus v is not incident to any negative edges; furthermore, every positive neighbor of v must have $\eta\kappa$-value strictly greater than $\eta\kappa(v)$ and v is not isolated as isolated vertices are placed in L_0^+. Thus without affecting the propriety of $\eta\kappa$ we can change the color $\eta\kappa(v)$ to $\eta\kappa(v) = -k$ when n is even or $\eta\kappa(v) = -(k-1)$ when n is odd and then renormalize. This, however, contradicts the maximality of $\sum_{v\in V(G)} |\kappa(v)|$. Thus $L_0^- = \emptyset$, as required.

Assume inductively that $L_i = \{v : \kappa(v) = k - i\}$ when n is even and $L_i = \{v : \kappa(v) = k - 1 - i\}$ when n is odd for every $i \in \{0, \ldots, t\}$. Thus the maximum $\eta\kappa$-value among vertices in $G_{t+1} = G - (L_0 \cup \cdots \cup L_t)$ is $k - (t+1)$ when n is even and $k-1-(t+1)$ when n is odd. Thus the argument for the base case applies to $(G_{t+1}, \sigma, \beta_{\eta\kappa})$ making $L_{t+1} = \{v : \kappa(v) = k-(t+1)\}$ when n is even and $L_{t+1} = \{v : \kappa(v) = k - 1 - (t+1)\}$ when n is odd.

(3) Follows from Parts (1) and (2), the maximality of $\sum_{v\in V(G)} |\kappa(v)|$, and the fact that any vertex whose color is a maximum must be a sink. □

5. Main result

Theorem 5.1. *Let (G, σ) be a signed graph and k a positive integer.*

(1) (G, σ) has a proper $2k$-coloring if and only if (G, σ) has an acyclic orientation whose canonical level decomposition has at most k levels.

(2) (G, σ) has a proper $(2k-1)$-coloring if and only if (G, σ) has an acyclic orientation whose canonical level decomposition has at most k levels where level L_{k-1} is independent or empty.

Proof. Suppose β exists with a canonical level decomposition $L_0, \ldots, L_{t-1}$ in which $t \le k$. Define $2k$-coloring κ by $\kappa(v) = \epsilon(k-i)$ when $v \in L_i^\epsilon$. Evidently κ is a proper $2k$-coloring. If L_{k-1} is independent or empty, then define $(2k-1)$-coloring κ by $\kappa(v) = \epsilon(k-1-i)$ when $v \in L_i^\epsilon$. Similarly, κ is a proper $(2k-1)$-coloring.

Conversely assume that κ is a proper $2k$- or $(2k-1)$-coloring of (G, σ). Choose κ so that $\sum_{v\in V(G)} |\kappa(v)|$ has maximum possible value among all such colorings. The result now follows from Proposition 4.2. □

A *balloon* in a signed graph is either a negative cycle or a negative cycle along with a path intersecting the cycle at exactly one of its endpoints. The *length* of a balloon is the length of the cycle plus twice the length of the path. A balloon is coherently oriented when is has exactly one singular vertex. When the balloon includes a path, this singular vertex must therefore be the endpoint of the path that is not on the cycle. When the balloon does not contain a path, then this singular vertex is just some vertex of the cycle.

Theorem 5.2 (Main Result). *If (G, σ) is a signed graph, then (G, σ) has a proper n-coloring if and only if (G, σ) has an acyclic orientation β whose normalization does not contain*

(1) a positive coherent path of length $\lceil \frac{n}{2} \rceil$,

(2) a negative coherent path of length n, or

(3) a coherent balloon of length n.

Proof. For both directions of the proof, let k be the unique integer satisfying $n \in \{2k, 2k+1\}$; that is, $\lceil \frac{n}{2} \rceil = k+1$ when n is odd and $\lceil \frac{n}{2} \rceil = k$ when n is even.

$(\longrightarrow)$ Let κ be a proper n-coloring of (G, σ) chosen among all proper n-colorings so that $\sum_{v\in V(G)} |\kappa(v)|$ is a maximum. Let $\eta\kappa$ be the normalization of κ. Proposition 4.2 implies that $\beta_{\eta\kappa}$ is a normalized acyclic orientation. Since every negative edge in $(G, \sigma, \beta_{\eta\kappa})$ is extroverted, a coherent path in $(G, \sigma, \beta_{\eta\kappa})$ cannot contain more than one negative edge. Thus a positive coherent path in $(G, \sigma, \beta_{\eta\kappa})$ cannot have length $\lceil \frac{n}{2} \rceil$ because the colors of the vertices along the path are strictly increasing. A negative coherent path contains exactly one negative edge e and so is of the form $P_1 e P_2$ in which e is extroverted and P_i is a positive coherent path where the colors of the vertices along P_i are strictly increasing. When $n = 2k$, this implies that $P_1 e P_2$ has length at most $1 + 2(k-1) = 2k-1$, as required. When $n = 2k+1$, the length of $P_1 e P_2$ is at most $1 + k + (k-1) = 2k$

because the endpoints of e cannot both have color 0. A coherent balloon has exactly one negative edge, call it e, which is in the cycle of the balloon and then has the form P_1eP_2 in which P_1 and P_2 are coherent positive paths from the two endpoints of e and which both end at the same vertex. Again, then length of the balloon is at most $2k-1$ when $n=2k$ and at most $2k$ when $n=2k+1$, as required.

($\longleftarrow$) Let β be a normalized a acyclic orientation of (G,σ) which contains no positive coherent path of length $\lceil \frac{n}{2} \rceil$, no negative coherent path of length n, and no coherent balloon of length n.

If $n=2k$ is even, then let the levels for β be called $L_1,\ldots,L_m$. Proposition 4.1 and the fact that there is no coherent positive path of length $k=\lceil \frac{n}{2} \rceil$ imply that $m \le k$. Define a $2k$-coloring κ by $\kappa(v)=k+1-i$ for $v \in L_i$. Proposition 4.1 now implies that κ is a proper coloring.

If $n=2k+1$, then let the levels for β be called $L_0,\ldots,L_m$. Proposition 4.1 and the fact that there is no coherent positive path of length $k+1=\lceil \frac{n}{2} \rceil$ imply that $m \le k$. Define a $(2k+1)$-coloring κ by $\kappa(v)=k-i$ for $v \in L_i$. Proposition 4.1 now implies that κ is a proper coloring excepting the possibility that there is a negative link or loop on vertices of color 0. If such a negative edge e exists, however, then Proposition 4.1 could be used to construct a coherently bidirected balloon or negative path of length $n=2k+1$, a contradiction. □

We considered the question of whether or not any two of the three necessary conditions (in particular, conditions (2) and (3)) from Theorem 5.2 might themselves be sufficient to imply the existence of a proper n-coloring. We were not able to find a proof or counterexample.

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