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Research Article

The first three largest values of the spectral norm of oriented bicyclic graphs

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Abstract

Let D be a digraph with n vertices and let $\sigma_1(D), \sigma_2(D), \ldots, \sigma_n(D)$ be the singular values of the adjacency matrix of D, where $\sigma_1(D) \geq \sigma_2(D) \geq \cdots \geq \sigma_n(D)$. The spectral norm of D is $\sigma_1(D)$. In this paper, we determine the orientations of graphs with the first three largest values of the spectral norm over the family of all orientations of bicyclic graphs with at least 12 vertices.

Keywords: spectral norm; spectral radius; oriented graph.

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1. Introduction

We consider digraphs without loops or multiple arcs. Let D be a digraph with vertex set $V(D)$ and arc set $E(D)$. The notations and terminologies used but not defined here can be found in [4,5]. Denote by uv the arc from vertex u to vertex v (i.e. the arc with tail u and head v). The out-degree (in-degree, respectively) of a vertex u of D, denoted by $d_D^+(u)$ ($d_D^-(u)$, respectively), is the number of arcs of the form uv (vu, respectively) in D. A vertex u with $d_D^+(u) = 0$ ($d_D^-(u) = 0$, respectively) is called a sink (source, respectively) of D. The transpose $D^\top$ of a digraph D is obtained from D by reversing all arcs.

The adjacency matrix of an n-vertex digraph D is the $n \times n$ matrix $A(D) = (a_{uv})_{u,v \in V(G)}$, where $a_{uv} = 1$ if $uv \in E(D)$ and 0 otherwise. Let $\lambda_1(D), \lambda_2(D), \ldots, \lambda_n(D)$ denote the eigenvalues of $A(D)$. Since $A(D)$ is not necessarily symmetric, its eigenvalues are not necessarily real numbers. The spectral radius of D is defined as $\rho(D) = \max\{|\lambda_i| : i = 1, 2, \ldots, n\}$.

We mention that a (simple undirected) graph G corresponds naturally to a digraph $D(G)$ with the same vertex set such that if there is an edge connecting vertices u and v in G, then there are arcs uv and vu in $D(G)$. The adjacency matrix of G is $A(G) = A(D(G))$. The spectral radius $\rho(G)$ is the largest eigenvalue of its adjacency matrix.

For an $n \times n$ real matrix M, the singular values, $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_n(M)$ of M are the nonnegative square roots of the eigenvalues of $M^T M$ or, equivalently, of MM^T. The largest singular value, $\sigma_1(M)$, is called the spectral norm of M. For a digraph D, the spectral norm $\sigma_1(D)$ is the spectral norm of $A(D)$.

An orientation of a graph G is a digraph D obtained by choosing a direction for each edge of G. In this case, we say that D is an orientation of G and G is the underlying graph of D. A source-sink orientation (SS-orientation for short) of a graph G is an orientation such that each vertex is either a source or a sink. Monsalve and Rada [14] obtained that G has a SS-orientation if and only if G is bipartite.

A connected graph G is a bicyclic graph if $|E(G)| = |V(G)| + 1$. Let B_n^1 (B_n^2, respectively) be the n-vertex bicyclic graph obtained by adding two adjacent (nonadjacent, respectively) edges to the star S_n. Let $D_{n,1}$, $D_{n,2}$ and $D_{n,3}$ be the orientations of B_n^1 as shown in Figure 1. Let $D'_{n,3}$ be the orientation of B_n^2 as shown in Figure 1.

Some extremal problems in spectral digraph theory have attracted a great deal of research; some specific results on extremal problems for digraphs can be found in [1–3,10,11,13]. Gregory and Kirkland [7] obtained lower and upper bounds on the spectral norm of a tournament and determined the tournament with maximum spectral norm. In other words, they found the orientation of K_n attaining maximum spectral norm over the set of all orientations of K_n. Hoppen, Monsalve and Trevisan [9] obtained the extremal values of the spectral norm over the set of oriented trees, oriented unicyclic graphs and connected digraphs with n vertices and n arcs. García, Monsalve and Rada [6] obtained lower bounds for the spectral norm of a digraph in terms of the structure of the digraph. In this paper, we found the orientations of bicyclic graphs attaining the first three largest values of the spectral norm over the family of all orientations of bicyclic graphs.

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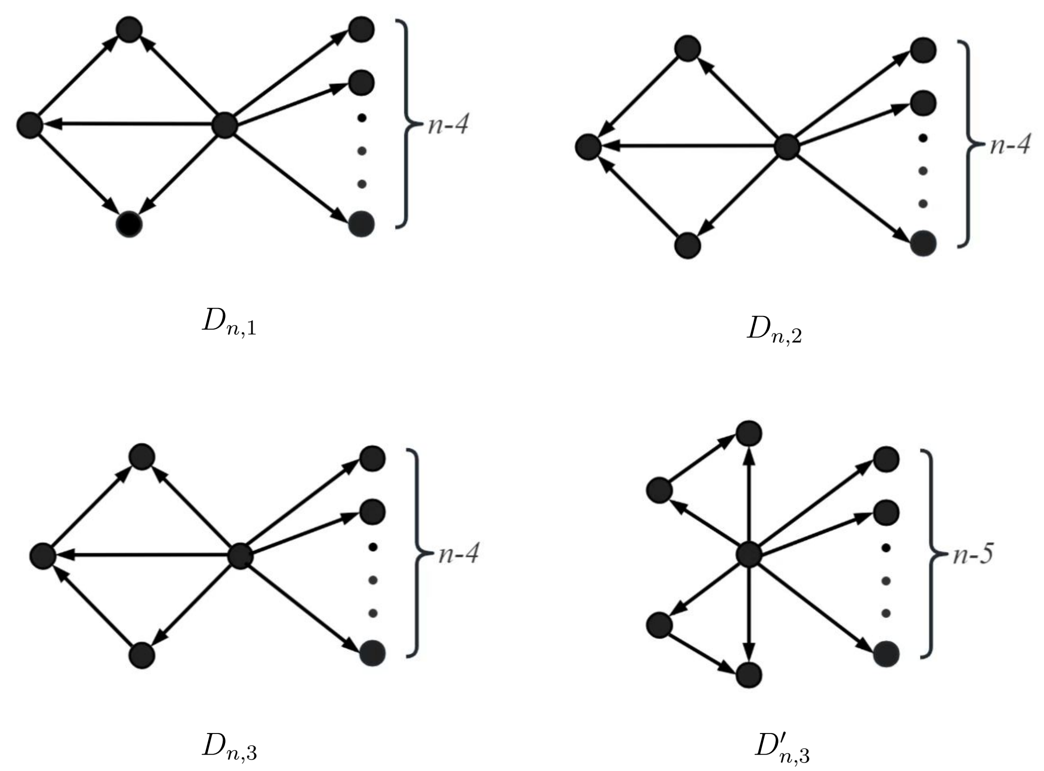

Figure 1: The digraphs $D_{n,1}$, $D_{n,2}$, $D_{n,3}$, and $D'_{n,3}$.

2. Preliminaries

Consider a digraph D. For a vertex u of D that is neither source nor sink, let $N^-(u) = \{w \in V(D) : wu \in E(D)\}$, and let $D' = D(u)$ be the digraph with vertex set $V(D) \cup \{u'\}$ and arc set $(E(D) \setminus \{wu : w \in N^-(u)\}) \cup \{wu' : w \in N^-(u)\}$. Then u is a source and u' is a sink in D'. We say that D' is obtained from D by stretching vertex u. If there exists a vertex in D' that is neither a source nor a sink, then repeating this process, we may finally obtain a digraph $\widetilde{D}$, in which all vertices are either sources or sinks. The digraph $\widetilde{D}$ is called the D-stretched digraph. Let $\widetilde{D} = D$ if all vertices of D are either sources or sinks.

Note that if D is an n-vertex digraph with ℓ arcs such that exactly k vertices are neither sinks nor sources, then the D-stretched digraph $\widetilde{D}$ has $n + k$ vertices and ℓ arcs. Moreover, all vertices of $\widetilde{D}$ are sinks or sources. Hence, $\widetilde{D}$ is a SS-orientation of a bipartite graph that will be denoted by H_D. Obviously, H_D is the underlying graph of the digraph $\widetilde{D}$.

Obviously, $\widetilde{D}$ doesn't depend on the order in which the vertices of D are stretched. In the sense of isomorphism, these maximal vertex-disjoint digraphs whose vertices are either sources or sinks of $\widetilde{D}$ may be viewed as maximal arc-disjoint subdigraphs whose vertices are either sources or sinks of D. Thus we call these maximal vertex-disjoint digraphs whose vertices are either sources or sinks of $\widetilde{D}$ the maximal SS-subdigraphs of $\widetilde{D}$ or D.

Lemma 2.1 (see [9]). *If D is an orientation of a graph G, then $\sigma_1(D) \leq \rho(G)$, with equality if and only if D is an SS-orientation of a bipartite graph G.*

Lemma 2.2 (see [9]). *Let D be a digraph and let H_D be the underlying graph of $\widetilde{D}$. Then $\sigma_1(D) = \sigma_1(\widetilde{D}) = \rho(H_D)$.*

Theorem 2.1. *Let D be an orientation of a graph on n vertices, and $D_1, \ldots, D_k$ be the maximal SS-subdigraphs of D. Then*

$$\sigma_1(D) = \max_{1 \leq i \leq k} \sigma_1(D_i).$$

Proof. Note that $D_1, \ldots, D_k$ are the maximal vertex-disjoint SS-subdigraphs of $\widetilde{D}$. By labelling the vertices of $\widetilde{D}$ properly, $A(\widetilde{D})$ is a diagonal block matrix with diagonal blocks $A(D_1), \ldots, A(D_k)$. Obviously, the singular values of $A(\widetilde{D})$ consist of the singular values of $A(D_1), \ldots, A(D_k)$. Thus

$$\sigma_1(\widetilde{D}) = \max_{1 \leq i \leq k} \sigma_1(D_i).$$

By Lemma 2.2, we have $\sigma_1(D) = \sigma_1(\widetilde{D})$. □

For integers a and b with $b \geq a$, the tree obtained by adding an edge between the centers of two vertex-disjoint stars S_{a+1} and S_{b+1} is denoted by $S_{n,a}$. Obviously, the star $S_n \cong S_{n,0}$. Let $S'_{n,3}$ be the tree obtained by attaching two pendent edge to two pendent vertices of S_{n-2}.

Lemma 2.3 (see [8]). *Let T be a n-vertex tree and $T \ncong S_n, S_{n,1}, S_{n,2}, S'_{n,3}$. Then*

$$\rho(T) < \rho(S'_{n,3}) < \rho(S_{n,2}) < \rho(S_{n,1}) < \rho(S_n).$$

Let $C_4 = v_1v_2v_3v_4$. Let $U_{n,1}$ be the unicyclic graph obtained from C_4 by attaching $n-4$ pendent vertices to v_1 and $U_{n,2}$ be the unicyclic graph obtained from C_4 by attaching $n-5$ pendent vertices to v_1 and a pendent vertex to v_2.

Lemma 2.4 (see [12]). *Let G be a n-vertex unicyclic bipartite graph different from $U_{n,1}$ and $U_{n,2}$. Then*

$$\rho(G) < \rho(U_{n,2}) < \rho(U_{n,1}),$$

where

$$\rho(U_{n,1}) = \sqrt{\frac{n + \sqrt{n^2 - 8n + 32}}{2}}.$$

Let B_n be the graph obtained by joining $n-5$ pendent vertices to a vertex of degree three of the complete bipartite graph $K_{2,3}$.

Lemma 2.5 (see [16]). *Let G be a bicyclic bipartite graphs with $n \geq 5$ vertices, then*

$$\rho(G) \leq \rho(B_n) = \sqrt{\frac{(n+1) + \sqrt{n^2 - 10n + 61}}{2}},$$

with equality if and only if $G \cong B_n$.

Lemma 2.6 (see [15]). *Let G be a connected graph. If H is a proper subgraph of G, then $\rho(H) < \rho(G)$.*

3. Main results

Theorem 3.1. *Let D be an orientation of a bicyclic graph with $n \geq 10$ vertices. Then*

$$\sigma_1(D) \leq \sqrt{\frac{(n+1) + \sqrt{n^2 - 6n + 25}}{2}}$$

with equality if and only if $D \cong D_{n,1}$ or $D^{\top}_{n,1}$.

Proof. Note that $U_{n+1,1}$ is the underlying graph of $\widetilde{D}_{n,1}$ and $\widetilde{D}^{\top}_{n,1}$. By Lemma 2.2, we have

$$\sigma_1(D_{n,1}) = \sigma_1(D^{\top}_{n,1}) = \rho(U_{n+1,1}) = \sqrt{\frac{(n+1) + \sqrt{n^2 - 6n + 25}}{2}}.$$

Let $D_1, \ldots, D_k$ be maximal vertex-disjoint SS-subdigraphs of $\widetilde{D}$ and H_i be the underlying graph of D_i for $1 \leq i \leq k$. Without loss of generality, we assume that

$$\sigma_1(D_1) = \max_{1 \leq i \leq k} \sigma_1(D_i).$$

By Theorem 2.1 and Lemma 2.2, we have $\sigma_1(D) = \sigma_1(D_1) = \rho(H_1)$. Note that H_1 is a tree, a bipartite unicyclic graph or a bipartite bicyclic graph and $\Delta(H_1) \leq n-1$.

If H_1 is a tree with s vertices, then $s \leq n+2$. Recall that $\Delta(H_1) \leq n-1$. Thus, $H_1 \ncong S_{n+2}$ and $H_1 \ncong S_{n+2,1}$ for $s = n+2$, and $H_1 \ncong S_{n+1}$ for $s = n+1$. By Lemma 2.3, we have $\rho(H_1) \leq \rho(S_{n+2,2})$ for $s = n+2$ and $\rho(H_1) \leq \rho(S_{n+1,1})$ for $s = n+1$. If $s \leq n$, then $\rho(H_1) \leq \rho(S_s) \leq \rho(S_n)$. Obviously, S_n and $S_{n+1,1}$ are the proper subgraphs of $S_{n+2,2}$. By Lemma 2.6, we have $\rho(H_1) \leq \rho(S_{n+2,2})$. By Sachs theorem,

$$\phi(S_{n+2,2}, x) = x^{n+2} - (n+1)x^n + (2n-4)x^{n-2}.$$

Thus,

$$\sigma_1(D) \leq \rho(S_{n+2,2}) = \sqrt{\frac{(n+1) + \sqrt{n^2 - 6n + 17}}{2}} < \sigma_1(D_{n,1}).$$

Suppose that H_1 is a bipartite unicyclic graph with s vertices. Then $s \le n+1$. If $s \le n$, then by Lemma 2.4,

$$\rho(H_1) \le \rho(U_{n,1}) < \rho(U_{n+1,1}) = \sigma_1(D_{n,1}).$$

Suppose that $s = n+1$. Then by Lemma 2.4, $\rho(H_1) \le \rho(U_{n+1,1})$. If the equality holds, then $H_1 \cong U_{n+1,1}$. Thus $\widetilde{D} = D(v)$ and $\widetilde{D}$ is the SS-orientation of $U_{n+1,1}$, where v is the unique vertex nither sink nor source. Therefore, D can be obtained from $\widetilde{D}$ by identifying a sink u_1 with a source u_2. Thus $D \cong D_{n,1}$ or $D_{n,1}^{\top}$.

Suppose now that H_1 is a bipartite bicyclic graph with s vertices. Then $s \le n$. By Lemma 2.5,

$$\rho(H_1) \le \rho(B_s) \le \rho(B_n) = \sqrt{\frac{(n+1)+\sqrt{n^2-10n+61}}{2}} < \sigma_1(D_{n,1})$$

for $n \ge 10$. □

Lemma 3.1. *The inequalities $\rho(S'_{n+2,3}) > \rho(U_{n+1,2})$ and $\rho(S'_{n+2,3}) > \rho(B_n)$ hold for $n \ge 12$.*

Proof. By Sachs theorem, we have

$$\phi(S'_{n+2,3},x) = x^{n+2} - (n+1)x^n + (2n-3)x^{n-2} - (n-3)x^{n-4},$$

$$\phi(U_{n+1,2},x) = x^{n+1} - (n+1)x^{n-1} + (3n-10)x^{n-3} - (n-4)x^{n-5},$$

and

$$\phi(B_n,x) = x^n - (n+1)x^{n-2} + (3n-15)x^{n-4}.$$

Let $\rho_1 = \rho(U_{n+1,2})$ and $\rho_2 = \rho(B_n)$. By direct calculation, we have

$$\phi(S'_{n+2,3},\rho_1) = \phi(S'_{n+2,3},\rho_1) - \rho_1\phi(U_{n+1,2},\rho_1) = \rho_1^{n-4}(-(n-7)\rho_1^2 - 1) < 0$$

and

$$\phi(S'_{n+2,3},\rho_2) = \phi(S'_{n+2,3},\rho_2) - \rho_2^2\phi(B_n,\rho_2) = \rho_2^{n-2}((-n+12)\rho_2^2 - (n-3)) < 0.$$

□

Theorem 3.2. *Let D be an oriented bicyclic graph with $n \ge 12$ vertices different from $D_{n,1}$ and $D_{n,1}^{\top}$. Then*

$$\sigma_1(D) \le \sqrt{\frac{(n+1)+\sqrt{n^2-6n+17}}{2}}$$

with equality if and only if $D \cong D_{n,2}$ or $D_{n,2}^{\top}$.

Proof. Note that $S_{n+2,2}$ is the underlying graph of $\widetilde{D}_{n,2}$ and $\widetilde{D}_{n,2}^{\top}$. By Lemma 2.2, we have

$$\sigma_1(D_{n,2}) = \sigma_1(D_{n,2}^{\top}) = \rho(S_{n+2,2}) = \sqrt{\frac{(n+1)+\sqrt{n^2-6n+17}}{2}}.$$

Let $\sigma_1(D) = \sigma_1(D_1) = \rho(H_1)$, where D_1 and H_1 are defined as Theorem 3.1. As above, we may get that $\rho(H_1) \le \rho(S_{n+2,2})$ with equality if and only if $H_1 \cong S_{n+2,2}$ for H is a tree, and $\rho(H_1) \le \rho(S_{n+2,2})$ for H_1 is a bipartite bicyclic graph. Note that D is different from $D_{n,1}$ and $D_{n,1}^{\top}$. If H_1 is a bipartite unicyclic graph, then $H_1 \ncong U_{n+1,1}$. By Lemma 2.4, we have $\rho(H_1) \le \rho(U_{n+1,2})$. By Lemmas 2.3 and 3.1, $\rho(H_1) \le \rho(S_{n+2,2})$ with equality if and only if $H_1 \cong S_{n+2,2}$. If the equality holds, then $H_1 \cong S_{n+2,2}$. Thus $\widetilde{D} = D(u)(v)$ and $\widetilde{D}$ is the SS-orientation of $S_{n+2,2}$, where u and v are the vertices nither sink nor source. Therefore, D can be obtained from $\widetilde{D}$ by identifying a sink u_1 with a source u_2 and identifying a sink u_3 with a source u_4, respectively. Thus $D \cong D_{n,2}$ or $D_{n,2}^{\top}$. □

Since the proof of the next result is similar to the proof of Theorem 3.2, we omit it.

Theorem 3.3. *Let D be an oriented bicyclic graph with $n \ge 12$ vertices different from $D_{n,1}$, $D_{n,1}^{\top}$, $D_{n,2}$ and $D_{n,2}^{\top}$. Then*

$$\sigma_1(D) \le \sqrt{\frac{n+\sqrt{n^2-4n+12}}{2}}$$

with equality if and only if $D \cong D_{n,3}$, $D_{n,3}^{\top}$, $D'_{n,3}$ or $D'^{\top}_{n,3}$.

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