Optimal $t$-rubbling on complete graphs and paths

Nándor Sieben*

Department of Mathematics and Statistics, Northern Arizona University, Flagstaff, Arizona, USA

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Abstract

Given a distribution of pebbles on the vertices of a graph, a rubbing move places one pebble at a vertex and removes a pebble each at two not necessarily distinct adjacent vertices. One pebble is the cost of transportation. A vertex is $t$-reachable if at least $t$ pebbles can be moved to the vertex using rubbing moves. The optimal $t$-rubbling number of a graph is the minimum number of pebbles in a pebble distribution that makes every vertex $t$-reachable. The optimal $t$-rubbling numbers of complete graphs and paths are determined.

Keywords: optimal $t$-rubbling; pebbling.

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1. Introduction

Graph pebbling is a simple model for the transportation of perishable resources. Let $G$ be a connected simple graph with vertex set $V$. A pebble distribution $p : V \to \{0, 1, 2, \ldots\}$ on $G$ is a placement of some pebbles at the vertices of $G$. A pebbling move $(v \to u)$ removes two pebbles from $v$ and places one pebble at the adjacent vertex $u$. We think of the lost pebble as the cost of transportation along the edge $vu$. A vertex $r$ is $t$-reachable from a pebble distribution if at least $t$ pebbles can be moved to $r$ by a sequence of moves. A pebble distribution is $t$-solvable if every vertex is $t$-reachable. A recent guide to the extensive literature of graph pebbling can be found in [13]. Another useful reference is [12].

The $t$-pebbling number of $G$ is the smallest number $\pi_t(G)$ of pebbles in a pebble distribution that forces the pebble distribution to be $t$-solvable. The optimal $t$-pebbling number of $G$ is the least number $\pi_t^*(G)$ of pebbles we need to create a $t$-solvable pebble distribution. Deciding whether $\pi_t^*(G) \leq k$ is an NP-complete problem [18]. The $t$-pebbling number of some graph families has been found [6, 8, 11, 16, 17]. Some optimal $t$-pebbling numbers were determined in [10, 19–21].

Graph rubbing allows for an extra move. A strict rubbing move $(v, w \to u)$ removes one pebble each from the distinct vertices $v$ and $w$ and places one pebble at the common neighbor vertex $u$. This time the pebbles are moved along the edges $vw$ and $wu$ and this transportation costs one pebble. A rubbing move is a pebbling or a strict rubbing move. Graph rubbing was introduced in [4] and further developed in [1–3, 7, 9, 14, 15].

The $t$-rubbling number of $G$ is the smallest number $\rho_t(G)$ of pebbles in a pebble distribution that forces the pebble distribution to be $t$-solvable. The optimal $t$-rubbling number is the least number $\rho_t^*(G)$ of pebbles we need to create a $t$-solvable pebble distribution. In this paper we determine the optimal $t$-rubbling numbers of complete graphs and paths.

2. Preliminaries

We start with some basic results about graph rubbing. If the total number of pebbles on the vertices that are adjacent to a vertex $v$ is $a$, then the maximum number of pebbles we can transfer to $v$ using only these pebbles is $\left\lfloor \frac{a}{2} \right\rfloor$. This is because the pebbles can be paired up and used in rubbing moves until we run out of pebbles. Transferring pebbles between the vertices adjacent to $v$, instead of directly moving them to $v$, has no benefit.

Since the expression $\left\lfloor \frac{a}{2} \right\rfloor$ plays an important role in our calculations, we collect some tools that help handling it. Let $pty(k)$ be the parity of the integer $k$. That is, $pty(k) := 0$ if $k$ is even and $pty(a) := 1$ if $a$ is odd. Then $\left\lfloor \frac{a}{2} \right\rfloor = \frac{a}{2} - pty(a)$. For $x \in \mathbb{R}$ and $a \in \mathbb{Z}$ we often use the identities

\[-x = -\lfloor x \rfloor, \quad [a + x] = a + \lfloor x \rfloor, \quad [a + x] = a + \lfloor x \rfloor.

*E-mail address: nandor.sieben@nau.edu
We will denote the vertex set of the graph $G$ by $V = \{v_1, \ldots, v_n\}$. Let $m_i$ be the maximum number of pebbles that we can move to vertex $v_i$ using rubbing moves. A pebble distribution is $t$-solvable if $m_i \geq t$ for all $i$. A pebble distribution is called $t$-optimal if it is $t$-solvable and contains $\rho^*_t(G)$ pebbles.

The transition digraph of a sequence of rubbing moves on $G$ is a directed multigraph with vertex set $V$. Every rubbing move in the rubbing sequence contributes two arrows to the transition digraph. The move $(v, w \rightarrow u)$ adds the arrows $(v, u)$ and $(w, u)$. The No-Cycle Lemma of [4] essentially states that if a vertex is $t$-reachable, then it is also $t$-reachable with a rubbing sequence whose transition digraph has no directed cycles. In particular, we can avoid moving pebbles back and forth along an edge.

If a pebble distribution is $s$-solvable and another pebble distribution is $t$-solvable, then the sum of these pebble distributions is clearly $(s + t)$-solvable. Hence $\rho^*_s(G) \leq \rho^*_s(G) + \rho^*_t(G)$.

### 3. Optimal $t$-rubbling on the complete graph

In this section we find the optimal $t$-rubbling number $\rho^*_t(K_n)$ of the complete graph $K_n$ with $n$ vertices. It was shown in [4] that $\rho^*(K_n) = 2$ for $n \geq 2$.

**Proposition 3.1.** If $n$ and $t$ are positive integers, then $\rho^*_t(K_n) = \left\lceil \frac{2nt}{n+1} \right\rceil$.

**Proof:** Consider a $t$-optimal pebble distribution with $a_i$ pebbles on vertex $v_i$ for all $i$. Let $\alpha := \sum_{i=1}^{n} a_i$. Then

$$t \leq m_i = a_i + \frac{a - a_i}{2} \leq \frac{1}{2} a_i + \frac{1}{2} a.$$

Adding these inequalities for all $i$ gives

$$nt \leq \frac{1}{2} a + \frac{1}{2} n a = \frac{1}{2} (n+1)a.$$

This implies $\frac{2nt}{n+1} \leq a$. Since $a$ is an integer, we must have $\left\lceil \frac{2nt}{n+1} \right\rceil \leq a = \rho^*_t(K_n)$.

Now we show that $\left\lceil \frac{2nt}{n+1} \right\rceil$ pebbles are sufficient. Let $s := \left\lceil \frac{2t}{n+1} \right\rceil$ and $2t = s(n + 1) + r$ with $0 \leq r \leq n$. Then

$$s + t = s(n + 1) + r = sn + r.$$

We verify that the pebble distribution

$$a_i := \begin{cases} s, & i \in \{1, \ldots, n-1\} \\ s + r, & i = n \end{cases}$$

containing $\left\lceil \frac{2nt}{n+1} \right\rceil$ pebbles is $t$-optimal. If $i \in \{1, \ldots, n-1\}$, then

$$m_i = a_i + \left\lfloor \frac{(n - 1)s + r}{2} \right\rfloor = s + \left\lfloor \frac{2t - 2s + r}{2} \right\rfloor = s + t - s = t.$$

We also have

$$m_n = a_n + \left\lfloor \frac{(n - 1)s}{2} \right\rfloor = s + r + \left\lfloor \frac{2t - 2s - r}{2} \right\rfloor = s + r + t - s + \left\lfloor \frac{r}{2} \right\rfloor = t + r + \left\lfloor \frac{r}{2} \right\rfloor \geq t.$$

\[\square\]

### 4. Optimal $t$-rubbling on the path

In this section we find the optimal $t$-rubbling number $\rho^*_t(P_n)$ of the path $P_n$ with $n$ vertices. It was shown in [4] that $\rho^*(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$ for $n \geq 1$.

**Preliminary results**

We start with developing some tools. Consider a pebble distribution with $a_i$ pebbles on vertex $v_i$ of $P_n$ for $i \in \{1, \ldots, n\}$. For $i \in \{2, \ldots, n+1\}$ let $l_i$ be the maximum number of pebbles we can move to vertex $v_{i-1}$ using only the pebbles on $v_{i-1}$. For $i \in \{0, \ldots, n \}$ let $r_i$ be the maximum number of pebbles we can move to vertex $v_{i+1}$ using only the pebbles on $v_i, v_{i+1}, \ldots, v_n$. We define $l_1 := 0$ and $r_n := 0$ to simplify some formulas. Note that $m_i = r_0$ and $m_n = l_{n+1}$. We define

$$\lambda_i := \text{pty}(l_i), \quad \mu_i := \text{pty}(l_i + r_i), \quad \nu_i := \text{pty}(r_i).$$
The No-Cycle Lemma implies the following result.

**Corollary 4.1.** For \( i \in \{1, \ldots, n\} \) we have
(1) \( m_i = a_i + \left\lfloor \frac{1}{2}(l_i + r_i) \right\rfloor = a_i + \frac{1}{2}(l_i + r_i - \mu_i) \);
(2) \( l_{i+1} = a_i + \left\lfloor \frac{1}{2}l_i \right\rfloor = a_i + \frac{1}{2}(l_i - \lambda_i) \);
(3) \( r_{i-1} = a_i + \left\lfloor \frac{1}{2}r_i \right\rfloor = a_i + \frac{1}{2}(r_i - \nu_i) \).

We can now express \( m_i \) without \( a_i \).

**Proposition 4.1.** For \( i \in \{1, \ldots, n\} \) we have
(1) \( m_i = l_{i+1} + \frac{1}{2}(r_i + \lambda_i - \mu_i) \);
(2) \( m_i = r_{i-1} + \frac{1}{2}(l_i + \nu_i - \mu_i) \).

**Proof.** Formulas (2) and (3) of Corollary 4.1 imply \( a_i = l_{i+1} - \frac{1}{2}(l_i - \lambda_i) \) and \( a_i = r_{i-1} - \frac{1}{2}(r_i - \nu_i) \). Substituting these into Corollary 4.1(1) give the desired results.

We prove an identity.

**Proposition 4.2.** If \( k \in \{1, \ldots, n\} \) then
\[
\sum_{i=1}^{k} l_i = 2 \sum_{i=1}^{k} a_i - 2l_{k+1} - \sum_{i=1}^{k} \lambda_i.
\]

**Proof.** We use induction on \( k \). The statement is true for \( k = 1 \) since
\[
l_1 = 0 = 2a_1 - 2a_1 - 0 = 2a_1 - 2l_2 - \lambda_1.
\]
The inductive step uses Corollary 4.1(2):
\[
\sum_{i=1}^{k+1} l_i = 2 \sum_{i=1}^{k} a_i - 2l_{k+1} - \sum_{i=1}^{k} \lambda_i + l_{k+1} = 2 \sum_{i=1}^{k} a_i - l_{k+1} - \sum_{i=1}^{k} \lambda_i
\]
\[
= 2 \sum_{i=1}^{k+1} a_i - 2a_{k+1} - l_{k+1} + \lambda_{k+1} - \sum_{i=1}^{k+1} \lambda_i = 2 \sum_{i=1}^{k+1} a_i - 2l_{k+2} - \sum_{i=1}^{k+1} \lambda_i.
\]

The following result is an important special case.

**Corollary 4.2.** If \( n \) is a positive integer, then
\[
\sum_{i=1}^{n} l_i = 2 \sum_{i=1}^{n} a_i - 2l_{n+1} - \sum_{i=1}^{n} \lambda_i.
\]

Reversing the path gives the following result.

**Corollary 4.3.** If \( n \) is a positive integer, then
\[
\sum_{i=1}^{n} r_i = 2 \sum_{i=1}^{n} a_i - 2r_0 - \sum_{i=1}^{n} \nu_i.
\]

Now we prove a formula connecting the sum of the \( m_i \) with the sum of the \( a_i \).

**Proposition 4.3.** If \( n \) is a positive integer, then
\[
\sum_{i=1}^{n} a_i = \frac{1}{3}(m_1 + m_n + \sum_{i=1}^{n} m_i + \frac{1}{2} \sum_{i=1}^{n} (\lambda_i + \nu_i + \mu_i)).
\]

**Proof.** Applying the previous two corollaries to the sum of the formulas in Corollary 4.1(1) gives
\[
\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} a_i + \frac{1}{2} \sum_{i=1}^{n} l_i + \frac{1}{2} \sum_{i=1}^{n} r_i - \frac{1}{2} \sum_{i=1}^{n} \mu_i
\]
\[
= \sum_{i=1}^{n} a_i + (\sum_{i=1}^{n} a_i - l_{n+1} - \frac{1}{2} \sum_{i=1}^{n} \lambda_i) + (\sum_{i=1}^{n} a_i - r_0 - \frac{1}{2} \sum_{i=1}^{n} \nu_i) - \frac{1}{2} \sum_{i=1}^{n} \mu_i
\]
\[
= 3 \sum_{i=1}^{n} a_i - m_1 - m_n - \frac{1}{2} \sum_{i=1}^{n} (\lambda_i + \nu_i + \mu_i).
\]
The case $t = 3s$

**Proposition 4.4.** If $s$ and $n$ are positive integers, then $\rho^*_3(P_n) \leq s(n + 2)$.

**Proof.** It is clear that the pebble distribution with 2 pebbles each on vertices $v_1$ and $v_n$ and 1 pebble each on vertices $v_2, \ldots, v_{n-1}$ is 3-solvable and contains $n + 2$ pebbles. Multiplying the number of pebbles on every vertex by $s$ creates a 3s-solvable distribution with $s(n + 2)$ pebbles.

**Proposition 4.5.** If $s$ and $n$ are positive integers, then $\rho^*_3(P_n) \geq s(n + 2)$.

**Proof.** Consider a 3s-solvable pebble distribution with $a_i$ pebbles on vertex $v_i$ of $P_n$ for all $i$. Since $m_i \geq 3s$ for all $i \in \{1, \ldots, n\}$, Proposition 4.3 implies

$$\sum_{i=1}^{n} a_i \geq \frac{1}{3} (3s + 3s + n3s) = s(n + 2).$$

**Corollary 4.4.** If $s$ and $n$ are positive integers, then $\rho^*_3(P_n) = s(n + 2)$.

The case $t = 3s + 1$

**Proposition 4.6.** If $s$ is a non-negative and $n$ is a positive integer, then $\rho^*_{3s+1}(P_n) \leq \left\lceil \frac{n+1}{2} \right\rceil + s(n + 2)$.

**Proof.** Since $\rho^*(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$ and $\rho^*_3(P_n) = s(n + 2)$, the result follows from the inequality $\rho^*_{3s+1}(G) \leq \rho^*_3(G) + \rho^*_1(G)$.

To find a lower bound for $\rho^*_{3s+1}(P_n)$, we need a preliminary result.

**Lemma 4.1.** For $i \in \{1, \ldots, n-1\}$ we have

$$m_i + m_{i+1} = \frac{3}{2} (r_i + l_{i+1}) + \frac{1}{2} (\lambda_i + \nu_{i+1} - \mu_i - \mu_{i+1}).$$

**Proof.** Proposition 4.1 implies

$$m_i + m_{i+1} = l_{i+1} + \frac{1}{2} (r_i + \lambda_i - \mu_i) + r_i + \frac{1}{2} (l_{i+1} + \nu_{i+1} - \mu_{i+1}),$$

which simplifies to the desired formula.

**Proposition 4.7.** If $s$ is a non-negative and $n$ is a positive integer, then $\rho^*_{3s+1}(P_n) \geq \left\lceil \frac{n+1}{2} \right\rceil + s(n + 2)$.

**Proof.** Consider a $(3s + 1)$-solvable pebble distribution with $a_i$ pebbles on vertex $v_i$ of $P_n$ for all $i$. First we show that the inequality

$$m_i + m_{i+1} + \lambda_{i+1} + n_i \geq 6s + 3$$

holds for all $i \in \{1, \ldots, n - 1\}$. Note that $m_i + m_{i+1} \geq 6s + 2$ by $(3s + 1)$-solvability. If $l_{i+1}$ or $r_i$ is odd, then $\lambda_{i+1} + n_i \geq 1$ and the inequality holds.

Next assume that $l_{i+1}$ and $r_i$ are both even. Then $\mu_i = \lambda_i$ and $\mu_{i+1} = \nu_{i+1}$, which implies $\lambda_{i+1} + \nu_{i+1} - \mu_i - \mu_{i+1} = 0$. Hence $m_i + m_{i+1} = \frac{3}{2} (r_i + l_{i+1})$ by Lemma 4.1. This implies that $m_i + m_{i+1}$ is divisible by 3. Since we also have $m_i + m_{i+1} \geq 6s + 2$, we must have $m_i + m_{i+1} \geq 6s + 3$ as desired.

Using Proposition 4.3 and our inequality, now we have

$$\sum_{i=1}^{n} a_i = \frac{1}{3} (m_1 + m_n + \sum_{i=1}^{n} (m_i + m_{i+1} + \lambda_i + \nu_i + \mu_i))$$

$$= \frac{1}{3} (m_1 + m_n) + \frac{1}{6} \sum_{i=1}^{n} (2m_i + \lambda_i + \nu_i + \mu_i)$$

$$= \frac{1}{3} (m_1 + m_n) + \frac{1}{6} (m_1 + m_n + \lambda_1 + \nu_1 + \mu_1) + \frac{1}{6} \sum_{i=1}^{n-1} (m_i + m_{i+1} + \lambda_{i+1} + \nu_{i+1} + \mu_{i+1})$$

$$\geq \frac{1}{3} (m_1 + m_n) + \frac{1}{6} (m_1 + m_n) + \frac{1}{6} \sum_{i=1}^{n-1} (6s + 3)$$

$$\geq \frac{2}{3} (3s + 1) + \frac{2}{6} (3s + 1) + \frac{1}{6} (n-1)(6s + 3)$$

$$= 3s + 1 + (n-1)s + \frac{1}{2} (n-1)$$

$$= (n+2)s + \frac{1}{2} (n+1).$$
Corollary 4.5. If \( s \) is a non-negative and \( n \) is a positive integer, then \( \rho^*_{3s+1}(P_n) = \left\lfloor \frac{n+1}{2} \right\rfloor + s(n+2) \).

The case \( t = 3s + 2 \)

Proposition 4.8. If \( s \) is a non-negative and \( n \) is a positive integer, then \( \rho^*_{3s+2}(P_n) \leq n + 1 + s(n+2) \).

Proof. It is clear that the pebble distribution with 2 pebbles on vertex \( v_1 \) and 1 pebble each on vertices \( v_2, \ldots, v_n \) is 2-solvable and contains \( n + 1 \) pebbles. Since \( \rho^*_{3s}(P_n) = s(n+2) \), the result follows from the inequality \( \rho^*_{3s+1}(G) \leq \rho^*_{3s}(G) + \rho^*_s(G) \).

Finding a lower bound is a bit harder than it was in the previous two cases. We need a tool often used in optimal pebbling. A smoothing move removes two pebbles at a vertex of degree two and places one pebble each on the two neighboring vertices. The proof of the following is essentially the same as that of [5, Lemma 6].

Proposition 4.9. Let \( v \) be a vertex of degree two with at least two pebbles and \( u \) be a vertex different from \( v \). If \( u \) is \( t \)-reachable from a pebble distribution, then \( u \) is also \( t \)-reachable from the pebble distribution created by a smoothing move at \( v \).

Proposition 4.10. If the pebble distribution on \( P_n \) with \( a_i \) pebbles on vertex \( v_i \) is \( t \)-solvable, then

\[
\frac{4}{5} t - \frac{2}{5} a_i - \frac{2}{5} + \frac{1}{10} (\lambda_i + \nu_i) \leq \frac{1}{2} (d_i + r_i)
\]

for all \( i \in \{2, \ldots, n-1\} \).

Proof. Proposition 4.1(2) and Corollary 4.1(2) imply

\[
t \leq m_{i+1} = r_i + \frac{1}{2} (l_{i+1} + \nu_{i+1} - \mu_{i+1})
\]

\[
= r_i + \frac{1}{2} (a_i + \frac{1}{2} (l_i - \lambda_i) + \nu_{i+1} - \mu_{i+1})
\]

\[
= r_i + \frac{1}{2} a_i + \frac{1}{4} (l_i - \lambda_i) + \frac{1}{2} (\nu_{i+1} - \mu_{i+1})
\]

\[
\leq r_i + \frac{1}{2} a_i + \frac{1}{4} (l_i - \lambda_i) + \frac{1}{2}.
\]

Hence \( 4t \leq 4r_i + 2a_i + l_i - \lambda_i + 2 \). Similar argument shows \( 4t \leq 4l_i + 2a_i + r_i - \nu_i + 2 \). Adding these two inequalities gives the desired result.

Proposition 4.11. Let \( t = 3s + 2 \) and consider a vertex \( v_i \) with at least \( s+3 \) pebbles for some \( i \in \{2, \ldots, n-1\} \). If the pebble distribution is \( t \)-solvable, then \( v_i \) is also \( t \)-reachable after a smoothing move at \( v_i \).

Proof. Consider a \( t \)-solvable pebble distribution with \( a_i \) pebbles on vertex \( v_i \) of \( P_n \) for all \( i \). Let \( \bar{a}_i \) be the number of pebbles at vertex \( v_i \) after the smoothing move. Also let \( \bar{l}_i, \bar{m}_i, \bar{r}_i, \bar{v}_i, \bar{\lambda}_i, \bar{\mu}_i \), and \( \bar{\nu}_i \) be the usual values after the smoothing move. Proposition 4.10 implies

\[
2s + \frac{1}{10} (\lambda_i + \nu_i) = \frac{4}{5} (3s+2) - \frac{2}{5} (s+3) - \frac{2}{5} + \frac{1}{10} (\lambda_i + \nu_i) \leq \frac{1}{2} (l_i + r_i).
\]

Corollary 4.1(1) now gives

\[
\bar{m}_i = \bar{a}_i + \frac{1}{2} (\bar{l}_i + \bar{r}_i - \bar{\mu}_i)
\]

\[
= a_i - 2 + \frac{1}{2} (l_i + r_i + 1 - \mu_i)
\]

\[
= a_i + \frac{1}{2} (l_i + r_i) - \frac{1}{2} \mu_i - 1
\]

\[
\geq s + 3 + 2s + \frac{1}{10} (\lambda_i + \nu_i) - \frac{1}{2} \mu_i - 1
\]

\[
= 3s + 2 + \frac{1}{10} (\lambda_i + \nu_i) - \frac{1}{2} \mu_i.
\]

Since \( \bar{m}_i \) is an integer, we must have \( \bar{m}_i \geq 3s + 2 = t \).

Proposition 4.12. Assume \( n \geq 2 \) and \( t = 3s + 2 \). There is a solvable pebble distribution on \( P_n \) with \( \rho^*_t(P_n) \) many pebbles such that \( a_1 \geq 2s + 1 \) or \( a_n \geq 2s + 1 \).
Proof. Consider a \( t \)-optimal pebble distribution. Applying all available smoothing moves at vertices \( v_2, \ldots, v_{n-1} \) must end in finitely many steps. This results in a \( t \)-solvable pebble distribution with \( a_i \) pebbles on vertex \( v_i \) for all \( i \). Since no smoothing move is available, we must have \( a_i \leq s + 2 \) for all \( i \in \{2, \ldots, n-1\} \). Using Corollary 4.1(2) repeatedly, we have

\[
\begin{align*}
l_2 &\leq a_1 + \frac{1}{2}l_1 = a_1, \\
l_3 &\leq a_2 + \frac{1}{2}l_2 \leq a_2 + \frac{1}{2}a_1, \\
l_4 &\leq a_3 + \frac{1}{2}l_3 \leq a_3 + \frac{1}{2}a_2 + \frac{1}{2^n}a_1, \\
\vdots \\
l_{n+1} &\leq a_n + \frac{1}{2}l_n \leq a_n + \frac{1}{2}a_{n-1} + \frac{1}{2^n}a_{n-2} + \cdots + \frac{1}{2^{n-2}}a_2 + \frac{1}{2^{n-1}}a_1.
\end{align*}
\]

The assumption \( a_1, a_n \leq 2s \) gives the contradiction

\[
3s + 2 = m_n = l_{n+1} \leq a_n + \left( \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\right) (s+2) + \frac{1}{2^{n-1}} a_1 \\
\leq 2s + (1 - \frac{1}{2^{n-2}})(s+2) + \frac{1}{2^{n-2}} 2s \\
= 2s + s + 2 - \frac{1}{2^{n-2}}s - \frac{2}{2^{n-2}} + \frac{1}{2^{n-2}} 2s \\
= 3s + 2 - \frac{1}{2^{n-3}}.
\]

\[\qed\]

Proposition 4.13. If \( n \geq 1 \) then \( \rho^*_{3s+2}(P_n) \leq \rho^*_{3s+2}(P_{n+1}) - s - 1 \).

Proof. Consider a \((3s+2)\)-optimal pebble distribution on \( P_{n+1} \) with \( a_i \) pebbles on \( v_i \) for all \( i \). By the previous proposition, we can assume without loss of generality that \( a_{n+1} \geq 2s+1 \). We create a new pebble distribution on \( P_n \) with

\[
\tilde{a}_i := \begin{cases} 
  a_i, & i \in \{1, \ldots, n-1\} \\
  a_n + a_{n+1} - s - 1, & i = n
\end{cases}
\]

pebbles on \( v_i \). We show that this new pebble distribution containing \( \rho^*_{3s+2}(P_{n+1}) - s - 1 \) pebbles is \((3s+2)\)-solvable on \( P_n \).

We have

\[
l_{n-1} = a_n + \frac{1}{2}(r_n - \nu_n) = a_n + \frac{1}{2}(a_{n+1} - \nu_n) \\
= a_n + a_{n+1} - \frac{1}{2}(a_{n+1} + \nu_n) \leq a_n + a_{n+1} - \frac{1}{2}(2s + 2) \\
= a_n + a_{n+1} - s - 1 = \tilde{a}_n.
\]

Since the original pebble distribution is \((3s+2)\)-solvable, the No-Cycle-Lemma implies that the new distribution is also \((3s+2)\)-solvable for \( v_1, \ldots, v_{n-1} \).

Now we show that it is also solvable for \( v_n \). We have

\[
\tilde{m}_n = \tilde{a}_n + \frac{1}{2}(l_n + \tilde{r}_n - \tilde{\mu}_n) \\
= a_n + a_{n+1} - s - 1 + \frac{1}{2}(l_n - \lambda_n) \\
= a_n + \frac{1}{2}(l_n + r_n - \mu_n) - \frac{1}{2}(r_n - \mu_n - \lambda_n) + a_{n+1} - s - 1 \\
= m_n - \frac{1}{2}(a_{n+1} - (\mu_n - \lambda_n)) + a_{n+1} - s - 1 \\
\geq 3s + 2 + \frac{1}{2}(a_{n+1} - \mu_n - \lambda_n) - s - 1 \\
\geq 3s + 2 + s + \frac{1}{2} - s = 3s + 2 - \frac{1}{2}.
\]

This implies \( \tilde{m}_n \geq 3s + 2 \) since \( \tilde{m}_n \) is an integer.

\[\qed\]

Proposition 4.14. If \( s \) is a non-negative and \( n \) is a positive integer, then \( \rho^*_{3s+2}(P_n) \geq n + 1 + s(n+2) \).
Proof. We use induction on $n$. We clearly have $\rho_{3s+2}^*(P_1) = 3s + 2$, so the statement holds for $n = 1$. The inductive step follows from the computation

$$
\rho_{3s+2}^*(P_{n+1}) \geq \rho_{3s+2}^*(P_n) + n + 1 \\
\geq n + 1 + s(n + 2) + s + 1 \\
= (n + 1) + 1 + s((n + 1) + 2).
$$

Combining the three cases provides the main result.

Theorem 4.1. If $s$ is non-negative and $n$ is a positive integer, then

1. $\rho_{3s}^*(P_n) = s(n + 2)$;
2. $\rho_{3s+1}^*(P_n) = \left\lceil \frac{n + 1}{2} \right\rceil + s(n + 2)$;
3. $\rho_{3s+2}^*(P_n) = n + 1 + s(n + 2)$.

It is easy to see that the three cases can be combined into the formula

$$\rho_t^*(P_n) = \left\lfloor \frac{t}{3} \right\rfloor (n + 2) + \left\lceil \left( t - 3 \left\lfloor \frac{t}{3} \right\rfloor \right) (n + 1) \right\rceil.$$

References