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Research Article

Optimal t-rubbling on complete graphs and paths

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Abstract

Given a distribution of pebbles on the vertices of a graph, a rubbling move places one pebble at a vertex and removes a pebble each at two not necessarily distinct adjacent vertices. One pebble is the cost of transportation. A vertex is t-reachable if at least t pebbles can be moved to the vertex using rubbling moves. The optimal t-rubbling number of a graph is the minimum number of pebbles in a pebble distribution that makes every vertex t-reachable. The optimal t-rubbling numbers of complete graphs and paths are determined.

Keywords: optimal t-rubbling; pebbling.

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1. Introduction

Graph pebbling is a simple model for the transportation of perishable resources. Let G be a connected simple graph with vertex set V. A *pebble distribution* $p : V \to \{0, 1, 2, \ldots\}$ on G is a placement of some pebbles at the vertices of G. A *pebbling move* $(v \to u)$ removes two pebbles from v and places one pebble at the adjacent vertex u. We think of the lost pebble as the cost of transportation along the edge vu. A vertex r is *t-reachable* from a pebble distribution if at least t pebbles can be moved to r by a sequence of moves. A pebble distribution is *t-solvable* if every vertex is t-reachable. A recent guide to the extensive literature of graph pebbling can be found in [13]. Another useful reference is [12].

The *t-pebbling number* of G is the smallest number $\pi_t(G)$ of pebbles in a pebble distribution that forces the pebble distribution to be t-solvable. The *optimal t-pebbling number* of G is the least number $\pi_t^*(G)$ of pebbles we need to create a t-solvable pebble distribution. Deciding whether $\pi_1^*(G) \leq k$ is an NP-complete problem [18]. The t-pebbling number of some graph families has been found [6, 8, 11, 16, 17]. Some optimal t-pebbling numbers were determined in [10, 19–21].

Graph rubbling allows for an extra move. A *strict rubbling move* $(v, w \to u)$ removes one pebble each from the distinct vertices v and w and places one pebble at the common neighbor vertex u. This time the pebbles are moved along the edges vu and wu and this transportation costs one pebble. A *rubbling move* is a pebbling or a strict rubbling move. Graph rubbling was introduced in [4] and further developed in [1–3, 7, 9, 14, 15].

The *t-rubbling number* of G is the smallest number $\rho_t(G)$ of pebbles in a pebble distribution that forces the pebble distribution to be t-solvable. The *optimal t-rubbling number* is the least number $\rho_t^*(G)$ of pebbles we need to create a t-solvable pebble distribution. In this paper we determine the optimal t-rubbling numbers of complete graphs and paths.

2. Preliminaries

We start with some basic results about graph rubbling. If the total number of pebbles on the vertices that are adjacent to a vertex v is a, then the maximum number of pebbles we can transfer to v using only these pebbles is $\lfloor \frac{a}{2} \rfloor$. This is because the pebbles can be paired up and used in rubbling moves until we run out of pebbles. Transferring pebbles between the vertices adjacent to v, instead of directly moving them to v, has no benefit.

Since the expression $\lfloor \frac{a}{2} \rfloor$ plays an important role in our calculations, we collect some tools that help handling it. Let $\operatorname{pty}(k)$ be the parity of the integer k. That is, $\operatorname{pty}(k) := 0$ if k is even and $\operatorname{pty}(a) := 1$ if a is odd. Then $\lfloor \frac{1}{2} a \rfloor = \frac{1}{2}(a - \operatorname{pty}(a))$. For $x \in \mathbb{R}$ and $a \in \mathbb{Z}$ we often use the identities

$$\lceil -x \rceil = -\lfloor x \rfloor, \quad \lfloor a + x \rfloor = a + \lfloor x \rfloor, \quad \lceil a + x \rceil = a + \lceil x \rceil.$$

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We will denote the vertex set of the graph G by $V = \{v_1, \dots, v_n\}$. Let m_i be the maximum number of pebbles that we can move to vertex v_i using rubbling moves. A pebble distribution is t-solvable if $m_i \geq t$ for all i. A pebble distribution is called t-*optimal* if it is t-solvable and contains $\rho_t^*(G)$ pebbles.

The *transition digraph* of a sequence of rubbling moves on G is a directed multigraph with vertex set V. Every rubbling move in the rubbling sequence contributes two arrows to the transition digraph. The move $(v, w \to u)$ adds the arrows (v, u) and (w, u). The *No-Cycle Lemma* of [4] essentially states that if a vertex is t-reachable, then it is also t-reachable with a rubbling sequence whose transition digraph has no directed cycles. In particular, we can avoid moving pebbles back and forth along an edge.

If a pebble distribution is s-solvable and another pebble distribution is t-solvable, then the sum of these pebble distributions is clearly $(s+t)$-solvable. Hence $\rho_{s+t}^*(G) \leq \rho_s^*(G) + \rho_t^*(G)$.

3. Optimal t-rubbling on the complete graph

In this section we find the optimal t-rubbling number $\rho_t^*(K_n)$ of the complete graph K_n with n vertices. It was shown in [4] that $\rho^*(K_n) = 2$ for $n \geq 2$.

Proposition 3.1. *If n and t are positive integers, then $\rho_t^*(K_n) = \left\lceil \frac{2nt}{n+1} \right\rceil$.*

Proof. Consider a t-optimal pebble distribution with a_i pebbles on vertex v_i for all i. Let $a := \sum_{i=1}^n a_i$. Then

$$t \leq m_i = a_i + \left\lfloor \frac{a - a_i}{2} \right\rfloor \leq \frac{1}{2}a_i + \frac{1}{2}a.$$

Adding these inequalities for all i gives

$$nt \leq \frac{1}{2}a + \frac{1}{2}na = \frac{1}{2}(n+1)a.$$

This implies $\frac{2nt}{n+1} \leq a$. Since a is an integer, we must have $\left\lceil \frac{2nt}{n+1} \right\rceil \leq a = \rho_t^*(K_n)$.

Now we show that $\left\lceil \frac{2nt}{n+1} \right\rceil$ pebbles are sufficient. Let $s := \left\lfloor \frac{2t}{n+1} \right\rfloor$ and $2t = s(n+1) + r$ with $0 \leq r \leq n$. Then

$$\left\lceil \frac{2nt}{n+1} \right\rceil = \left\lceil \frac{2(n+1)t - 2t}{n+1} \right\rceil = 2t - \left\lfloor \frac{2t}{n+1} \right\rfloor = s(n+1) + r - s = sn + r.$$

We verify that the pebble distribution

$$a_i := \begin{cases} s, & i \in \{1, \dots, n-1\} \\ s + r, & i = n \end{cases}$$

containing $\left\lceil \frac{2nt}{n+1} \right\rceil$ pebbles is t-optimal. If $i \in \{1, \dots, n-1\}$, then

$$m_i = a_i + \left\lfloor \frac{(n-1)s + r}{2} \right\rfloor = s + \left\lfloor \frac{2t - 2s}{2} \right\rfloor = s + t - s = t.$$

We also have

$$m_n = a_n + \left\lfloor \frac{(n-1)s}{2} \right\rfloor = s + r + \left\lfloor \frac{2t - 2s - r}{2} \right\rfloor$$
$$= s + r + t - s + \left\lfloor \frac{-r}{2} \right\rfloor = t + r - \left\lceil \frac{r}{2} \right\rceil \geq t.$$

□

4. Optimal t-rubbling on the path

In this section we find the optimal t-rubbling number $\rho_t^*(P_n)$ of the path P_n with n vertices. It was shown in [4] that $\rho^*(P_n) = \lceil \frac{n+1}{2} \rceil$ for $n \geq 1$.

Preliminary results

We start with developing some tools. Consider a pebble distribution with a_i pebbles on vertex v_i of P_n for $i \in \{1, \dots, n\}$. For $i \in \{2, \dots, n+1\}$ let l_i be the maximum number of pebbles we can move to vertex v_{i-1} using only the pebbles on $v_1, \dots, v_{i-1}$. For $i \in \{0, \dots, n-1\}$ let r_i be the maximum number of pebbles we can move to vertex v_{i+1} using only the pebbles on $v_{i+1}, \dots, v_n$. We define $l_1 := 0$ and $r_n := 0$ to simplify some formulas. Note that $m_1 = r_0$ and $m_n = l_{n+1}$. We define

$$\lambda_i := \operatorname{pty}(l_i), \quad \mu_i := \operatorname{pty}(l_i + r_i), \quad \nu_i := \operatorname{pty}(r_i).$$

The No-Cycle Lemma implies the following result.

Corollary 4.1. *For $i \in \{1, \dots, n\}$ we have*

(1) $m_i = a_i + \lfloor \frac{1}{2}(l_i + r_i) \rfloor = a_i + \frac{1}{2}(l_i + r_i - \mu_i)$*;*

(2) $l_{i+1} = a_i + \lfloor \frac{1}{2} l_i \rfloor = a_i + \frac{1}{2}(l_i - \lambda_i)$*;*

(3) $r_{i-1} = a_i + \lfloor \frac{1}{2} r_i \rfloor = a_i + \frac{1}{2}(r_i - \nu_i)$*.*

We can now express m_i without a_i.

Proposition 4.1. *For $i \in \{1, \dots, n\}$ we have*

(1) $m_i = l_{i+1} + \frac{1}{2}(r_i + \lambda_i - \mu_i)$*;*

(2) $m_i = r_{i-1} + \frac{1}{2}(l_i + \nu_i - \mu_i)$*.*

Proof. Formulas (2) and (3) of Corollary 4.1 imply $a_i = l_{i+1} - \frac{1}{2}(l_i - \lambda_i)$ and $a_i = r_{i-1} - \frac{1}{2}(r_i - \nu_i)$. Substituting these into Corollary 4.1(1) give the desired results. □

We prove an identity.

Proposition 4.2. *If $k \in \{1, \dots, n\}$ then*

$$\sum_{i=1}^{k} l_i = 2\sum_{i=1}^{k} a_i - 2l_{k+1} - \sum_{i=1}^{k} \lambda_i.$$

Proof. We use induction on k. The statement is true for $k = 1$ since

$$l_1 = 0 = 2a_1 - 2a_1 - 0 = 2a_1 - 2l_2 - \lambda_1.$$

The inductive step uses Corollary 4.1(2):

$$\begin{aligned}\sum_{i=1}^{k+1} l_i &= 2\sum_{i=1}^{k} a_i - 2l_{k+1} - \sum_{i=1}^{k} \lambda_i + l_{k+1} = 2\sum_{i=1}^{k} a_i - l_{k+1} - \sum_{i=1}^{k} \lambda_i \\ &= 2\sum_{i=1}^{k+1} a_i - 2a_{k+1} - l_{k+1} + \lambda_{k+1} - \sum_{i=1}^{k+1} \lambda_i = 2\sum_{i=1}^{k+1} a_i - 2l_{k+2} - \sum_{i=1}^{k+1} \lambda_i.\end{aligned}$$

□

The following result is an important special case.

Corollary 4.2. *If n is a positive integer, then*

$$\sum_{i=1}^{n} l_i = 2\sum_{i=1}^{n} a_i - 2l_{n+1} - \sum_{i=1}^{n} \lambda_i.$$

Reversing the path gives the following result.

Corollary 4.3. *If n is a positive integer, then*

$$\sum_{i=1}^{n} r_i = 2\sum_{i=1}^{n} a_i - 2r_0 - \sum_{i=1}^{n} \nu_i.$$

Now we prove a formula connecting the sum of the m_i with the sum of the a_i.

Proposition 4.3. *If n is a positive integer, then*

$$\sum_{i=1}^{n} a_i = \frac{1}{3}(m_1 + m_n + \sum_{i=1}^{n} m_i + \frac{1}{2}\sum_{i=1}^{n}(\lambda_i + \nu_i + \mu_i)).$$

Proof. Applying the previous two corollaries to the sum of the formulas in Corollary 4.1(1) gives

$$\begin{aligned}\sum_{i=1}^{n} m_i &= \sum_{i=1}^{n} a_i + \frac{1}{2}\sum_{i=1}^{n} l_i + \frac{1}{2}\sum_{i=1}^{n} r_i - \frac{1}{2}\sum_{i=1}^{n} \mu_i \\ &= \sum_{i=1}^{n} a_i + (\sum_{i=1}^{n} a_i - l_{n+1} - \frac{1}{2}\sum_{i=1}^{n} \lambda_i) + (\sum_{i=1}^{n} a_i - r_0 - \frac{1}{2}\sum_{i=1}^{n} \nu_i) - \frac{1}{2}\sum_{i=1}^{n} \mu_i \\ &= 3\sum_{i=1}^{n} a_i - m_1 - m_n - \frac{1}{2}\sum_{i=1}^{n}(\lambda_i + \nu_i + \mu_i).\end{aligned}$$

□

The case $t = 3s$

Proposition 4.4. *If s and n are positive integers, then $\rho^*_{3s}(P_n) \le s(n+2)$.*

Proof. It is clear that the pebble distribution with 2 pebbles each on vertices v_1 and v_n and 1 pebble each on vertices $v_2, \dots, v_{n-1}$ is 3-solvable and contains $n+2$ pebbles. Multiplying the number of pebbles on every vertex by s creates a $3s$-solvable distribution with $s(n+2)$ pebbles. □

Proposition 4.5. *If s and n are positive integers, then $\rho^*_{3s}(P_n) \ge s(n+2)$.*

Proof. Consider a $3s$-solvable pebble distribution with a_i pebbles on vertex v_i of P_n for all i. Since $m_i \ge 3s$ for all $i \in \{1, \dots, n\}$, Proposition 4.3 implies

$$\sum_{i=1}^{n} a_i \ge \frac{1}{3}(3s + 3s + n3s) = s(n+2).$$ □

Corollary 4.4. *If s and n are positive integers, then $\rho^*_{3s}(P_n) = s(n+2)$.*

The case $t = 3s + 1$

Proposition 4.6. *If s is a non-negative and n is a positive integer, then $\rho^*_{3s+1}(P_n) \le \lceil \frac{n+1}{2} \rceil + s(n+2)$.*

Proof. Since $\rho^*(P_n) = \lceil \frac{n+1}{2} \rceil$ and $\rho^*_{3s}(P_n) = s(n+2)$, the result follows from the inequality $\rho^*_{s+t}(G) \le \rho^*_s(G) + \rho^*_t(G)$. □

To find a lower bound for $\rho^*_{3s+1}(P_n)$, we need a preliminary result.

Lemma 4.1. *For $i \in \{1, \dots, n-1\}$ we have*

$$m_i + m_{i+1} = \frac{3}{2}(r_i + l_{i+1}) + \frac{1}{2}(\lambda_i + \nu_{i+1} - \mu_i - \mu_{i+1}).$$

Proof. Proposition 4.1 implies

$$m_i + m_{i+1} = l_{i+1} + \frac{1}{2}(r_i + \lambda_i - \mu_i) + r_i + \frac{1}{2}(l_{i+1} + \nu_{i+1} - \mu_{i+1}),$$

which simplifies to the desired formula. □

Proposition 4.7. *If s is a non-negative and n is a positive integer, then $\rho^*_{3s+1}(P_n) \ge \lceil \frac{n+1}{2} \rceil + s(n+2)$.*

Proof. Consider a $(3s+1)$-solvable pebble distribution with a_i pebbles on vertex v_i of P_n for all i. First we show that the inequality

$$m_i + m_{i+1} + \lambda_{i+1} + \nu_i \ge 6s + 3$$

holds for all $i \in \{1, \dots, n-1\}$. Note that $m_i + m_{i+1} \ge 6s + 2$ by $(3s+1)$-solvability. If l_{i+1} or r_i is odd, then $\lambda_{i+1} + \nu_i \ge 1$ and the inequality holds.

Next assume that l_{i+1} and r_i are both even. Then $\mu_i = \lambda_i$ and $\mu_{i+1} = \nu_{i+1}$, which implies $\lambda_i + \nu_{i+1} - \mu_i - \mu_{i+1} = 0$. Hence $m_i + m_{i+1} = \frac{3}{2}(r_i + l_{i+1})$ by Lemma 4.1. This implies that $m_i + m_{i+1}$ is divisible by 3. Since we also have $m_i + m_{i+1} \ge 6s + 2$, we must have $m_i + m_{i+1} \ge 6s + 3$ as desired.

Using Proposition 4.3 and our inequality, now we have

$$\begin{aligned}
\sum_{i=1}^{n} a_i &= \frac{1}{3}(m_1 + m_n + \sum_{i=1}^{n} m_i + \frac{1}{2}\sum_{i=1}^{n}(\lambda_i + \nu_i + \mu_i)) \\
&= \frac{1}{3}(m_1 + m_n) + \frac{1}{6}\sum_{i=1}^{n}(2m_i + \lambda_i + \nu_i + \mu_i) \\
&= \frac{1}{3}(m_1 + m_n) + \frac{1}{6}(m_1 + m_n + \lambda_1 + \nu_n + \mu_n) + \frac{1}{6}\sum_{i=1}^{n-1}(m_i + m_{i+1} + \lambda_{i+1} + \nu_i + \mu_i) \\
&\ge \frac{1}{3}(m_1 + m_n) + \frac{1}{6}(m_1 + m_n) + \frac{1}{6}\sum_{i=1}^{n-1}(6s+3) \\
&\ge \frac{2}{3}(3s+1) + \frac{2}{6}(3s+1) + \frac{1}{6}(n-1)(6s+3) \\
&= 3s + 1 + (n-1)s + \frac{1}{2}(n-1) \\
&= (n+2)s + \frac{1}{2}(n+1).
\end{aligned}$$

□

Corollary 4.5. *If s is a non-negative and n is a positive integer, then $\rho^*_{3s+1}(P_n) = \lceil \frac{n+1}{2} \rceil + s(n+2)$.*

The case $t = 3s + 2$

Proposition 4.8. *If s is a non-negative and n is a positive integer, then $\rho^*_{3s+2}(P_n) \leq n + 1 + s(n+2)$.*

Proof. It is clear that the pebble distribution with 2 pebbles on vertex v_1 and 1 pebble each on vertices $v_2, \ldots, v_n$ is 2-solvable and contains $n+1$ pebbles. Since $\rho^*_{3s}(P_n) = s(n+2)$, the result follows from the inequality $\rho^*_{s+t}(G) \leq \rho^*_s(G) + \rho^*_t(G)$. □

Finding a lower bound is a bit harder than it was in the previous two cases. We need a tool often used in optimal pebbling. A *smoothing move* removes two pebbles at a vertex of degree two and places one pebble each on the two neighboring vertices. The proof of the following is essentially the same as that of [5, Lemma 6].

Proposition 4.9. *Let v be a vertex of degree two with at least two pebbles and u be a vertex different from v. If u is t-reachable from a pebble distribution, then u is also t-reachable from the pebble distribution created by a smoothing move at v.*

Proposition 4.10. *If the pebble distribution on P_n with a_i pebbles on vertex v_i is t-solvable, then*

$$\frac{4}{5}t - \frac{2}{5}a_i - \frac{2}{5} + \frac{1}{10}(\lambda_i + \nu_i) \leq \frac{1}{2}(l_i + r_i)$$

for all $i \in \{2, \ldots, n-1\}$.

Proof. Proposition 4.1(2) and Corollary 4.1(2) imply

$$\begin{aligned} t \leq m_{i+1} &= r_i + \frac{1}{2}(l_{i+1} + \nu_{i+1} - \mu_{i+1}) \\ &= r_i + \frac{1}{2}(a_i + \frac{1}{2}(l_i - \lambda_i) + \nu_{i+1} - \mu_{i+1}) \\ &= r_i + \frac{1}{2}a_i + \frac{1}{4}(l_i - \lambda_i) + \frac{1}{2}(\nu_{i+1} - \mu_{i+1}) \\ &\leq r_i + \frac{1}{2}a_i + \frac{1}{4}(l_i - \lambda_i) + \frac{1}{2}. \end{aligned}$$

Hence $4t \leq 4r_i + 2a_i + l_i - \lambda_i + 2$. Similar argument shows $4t \leq 4l_i + 2a_i + r_i - \nu_i + 2$. Adding these two inequalities gives the desired result. □

Proposition 4.11. *Let $t = 3s+2$ and consider a vertex v_i with at least $s+3$ pebbles for some $i \in \{2, \ldots, n-1\}$. If the pebble distribution is t-solvable, then v_i is also t-reachable after a smoothing move at v_i.*

Proof. Consider a t-solvable pebble distribution with a_i pebbles on vertex v_i of P_n for all i. Let $\tilde{a}_i$ be the number of pebbles at vertex v_i after the smoothing move. Also let $\tilde{l}_i$, $\tilde{m}_i$, $\tilde{r}_i$, $\tilde{\lambda}_i$, $\tilde{\mu}_i$, and $\tilde{\nu}_i$ be the usual values after the smoothing move. Proposition 4.10 implies

$$2s + \frac{1}{10}(\lambda_i + \nu_i) = \frac{4}{5}(3s+2) - \frac{2}{5}(s+3) - \frac{2}{5} + \frac{1}{10}(\lambda_i + \nu_i) \leq \frac{1}{2}(l_i + r_i).$$

Corollary 4.1(1) now gives

$$\begin{aligned} \tilde{m}_i &= \tilde{a}_i + \frac{1}{2}(\tilde{l}_i + \tilde{r}_i - \tilde{\mu}_i) \\ &= a_i - 2 + \frac{1}{2}(l_i + 1 + r_i + 1 - \mu_i) \\ &= a_i + \frac{1}{2}(l_i + r_i) - \frac{1}{2}\mu_i - 1 \\ &\geq s + 3 + 2s + \frac{1}{10}(\lambda_i + \nu_i) - \frac{1}{2}\mu_i - 1 \\ &= 3s + 2 + \frac{1}{10}(\lambda_i + \nu_i) - \frac{1}{2}\mu_i. \end{aligned}$$

Since $\tilde{m}_i$ is an integer, we must have $\tilde{m}_i \geq 3s + 2 = t$. □

Proposition 4.12. *Assume $n \geq 2$ and $t = 3s + 2$. There is a solvable pebble distribution on P_n with $\rho^*_t(P_n)$ many pebbles* such that $a_1 \geq 2s+1$ or $a_n \geq 2s+1$.

Proof. Consider a t-optimal pebble distribution. Applying all available smoothing moves at vertices $v_2, \dots, v_{n-1}$ must end in finitely many steps. This results in a t-solvable pebble distribution with a_i pebbles on vertex v_i for all i. Since no smoothing move is available, we must have $a_i \le s+2$ for all $i \in \{2, \dots, n-1\}$. Using Corollary 4.1(2) repeatedly, we have

$$\begin{aligned}
l_2 &\le a_1 + \frac{1}{2} l_1 = a_1, \\
l_3 &\le a_2 + \frac{1}{2} l_2 \le a_2 + \frac{1}{2} a_1, \\
l_4 &\le a_3 + \frac{1}{2} l_3 \le a_3 + \frac{1}{2} a_2 + \frac{1}{2^2} a_1, \\
&\vdots \\
l_{n+1} &\le a_n + \frac{1}{2} l_n \le a_n + \frac{1}{2} a_{n-1} + \frac{1}{2^2} a_{n-2} + \cdots + \frac{1}{2^{n-2}} a_2 + \frac{1}{2^{n-1}} a_1.
\end{aligned}$$

The assumption $a_1, a_n \le 2s$ gives the contradiction

$$\begin{aligned}
3s+2 = m_n = l_{n+1} &\le a_n + (\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-2}})(s+2) + \frac{1}{2^{n-1}} a_1 \\
&\le 2s + (1 - \frac{1}{2^{n-2}})(s+2) + \frac{1}{2^{n-1}} 2s \\
&= 2s + s + 2 - \frac{1}{2^{n-2}} s - \frac{2}{2^{n-2}} + \frac{1}{2^{n-1}} 2s \\
&= 3s + 2 - \frac{1}{2^{n-3}}.
\end{aligned}$$

□

Proposition 4.13. *If* $n \ge 1$ *then* $\rho^*_{3s+2}(P_n) \le \rho^*_{3s+2}(P_{n+1}) - s - 1$.

Proof. Consider a $(3s+2)$-optimal pebble distribution on P_{n+1} with a_i pebbles on v_i for all i. By the previous proposition, we can assume without loss of generality that $a_{n+1} \ge 2s+1$. We create a new pebble distribution on P_n with

$$\tilde{a}_i := \begin{cases} a_i, & i \in \{1, \dots, n-1\} \\ a_n + a_{n+1} - s - 1, & i = n \end{cases}$$

pebbles on v_i. We show that this new pebble distribution containing $\rho^*_{3s+2}(P_{n+1}) - s - 1$ pebbles is $(3s+2)$-solvable on P_n.

We have

$$\begin{aligned}
l_{n-1} = a_n + \frac{1}{2}(r_n - \nu_n) &= a_n + \frac{1}{2}(a_{n+1} - \nu_n) \\
&= a_n + a_{n+1} - \frac{1}{2}(a_{n+1} + \nu_n) \le a_n + a_{n+1} - \frac{1}{2}(2s+2) \\
&= a_n + a_{n+1} - s - 1 = \tilde{a}_n.
\end{aligned}$$

Since the original pebble distribution is $(3s+2)$-solvable, the No-Cycle-Lemma implies that the new distribution is also $(3s+2)$-solvable for $v_1, \dots, v_{n-1}$.

Now we show that it is also solvable for v_n. We have

$$\begin{aligned}
\tilde{m}_n &= \tilde{a}_n + \frac{1}{2}(\tilde{l}_n + \tilde{r}_n - \tilde{\mu}_n) \\
&= a_n + a_{n+1} - s - 1 + \frac{1}{2}(l_n - \lambda_n) \\
&= a_n + \frac{1}{2}(l_n + r_n - \mu_n) - \frac{1}{2}(r_n - \mu_n - \lambda_n) + a_{n+1} - s - 1 \\
&= m_n - \frac{1}{2}(a_{n+1} - \mu_n - \lambda_n) + a_{n+1} - s - 1 \\
&\ge 3s + 2 + \frac{1}{2} a_{n+1} - s - 1 \\
&\ge 3s + 2 + s + \frac{1}{2} - s - 1 = 3s + 2 - \frac{1}{2}.
\end{aligned}$$

This implies $\tilde{m}_n \ge 3s+2$ since $\tilde{m}_n$ is an integer. □

Proposition 4.14. *If* s *is a non-negative and* n *is a positive integer, then* $\rho^*_{3s+2}(P_n) \ge n + 1 + s(n+2)$.

Proof. We use induction on n. We clearly have $\rho^*_{3s+2}(P_1) = 3s+2$, so the statement holds for $n=1$. The inductive step follows from the computation

$$\begin{aligned}\rho^*_{3s+2}(P_{n+1}) &\geq \rho^*_{3s+2}(P_n) + n + 1\\ &\geq n + 1 + s(n+2) + s + 1\\ &= (n+1) + 1 + s((n+1)+2).\end{aligned}$$

□

Combining the three cases provides the main result.

Theorem 4.1. *If s is non-negative and n is a positive integer, then*

(1) $\rho^*_{3s}(P_n) = s(n+2)$;

(2) $\rho^*_{3s+1}(P_n) = \lceil \frac{n+1}{2} \rceil + s(n+2)$;

(3) $\rho^*_{3s+2}(P_n) = n + 1 + s(n+2)$.

It is easy to see that the three cases can be combined into the formula

$$\rho^*_t(P_n) = \left\lfloor \frac{t}{3} \right\rfloor (n+2) + \left\lceil \frac{(t - 3\lfloor \frac{t}{3} \rfloor)(n+1)}{2} \right\rceil .$$

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