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Research Article

S-Motzkin paths with catastrophes and air pockets

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Abstract

The so-called S-Motzkin paths are combined with the concepts 'catastrophes' and 'air pockets'. The enumeration is done by properly setting up bivariate generating functions which can be expanded using the kernel method.

Keywords: catastrophe; S-Motzkin path; kernel method; air pocket.

2020 Mathematics Subject Classification: 05A15.

1. Introduction

Dyck paths consist of up-steps $(1,1)$ and down-steps $(1,-1)$, start at the origin and do not go below the x-axis; they appear in many texts, we just give two major references, [1,12]. Typically, these paths return to the x-axis at the end, but we also consider the scenario of open paths, where the paths end at level j, say. A popular variation of Dyck paths are Motzkin paths; the difference is just that now a horizontal step $(1,0)$ is also allowed.

In this paper, we concentrate on S-Motzkin paths (see Figure 1 for an example), which form a subfamily of the family of all Motzkin paths: all three steps (up, level, down) must appear n times, and, ignoring the down-steps, the sequence is $(1,0)(1,1)(1,0)(1,1)(1,0)(1,1)\dots(1,0)(1,1)$. Figure 2 shows how these paths are recognized: the two layers enforce that the flat and up steps are interlaced. Only paths that end in the origin are S-Motzkin but we consider all paths wherever they end. Figure 1 is an example of such an S-Motzkin path with 15 steps.

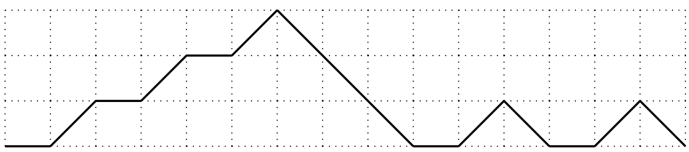

Figure 1: Example of an S-Motzkin path.

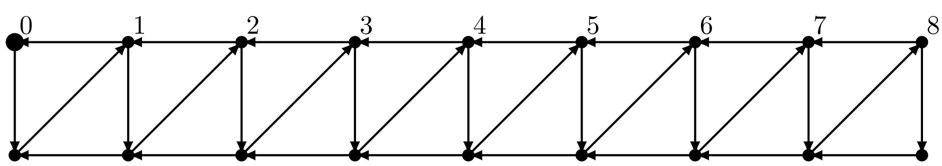

Figure 2: Graph to recognize S-Motzkin paths; they start and end at the special state (origin).

Now, we present a graph (automaton) to recognize exactly the S-Motzkin paths (see Figure 2). The meaning of this automaton is that each path from the origin back to the origin represents exactly such an S-Motzkin path. Arrows to the right in Figure 2 correspond to up-steps, to the left to down-steps, and to vertical ones to level-steps. For automata that appear later the interpretation is similar; we tried to use colors and different line types to help the reader.

This subfamily of Motzkin paths originated from a question in a student competition; see [7] and [10] for history and analysis. In the following, we will combine this family with *catastrophes* and *air pockets*, both originating in papers by Jean-Luc Baril et al. [4,5]; the older paper by Banderier and Wallner [2] might be called the standard reference for lattice

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paths with catastrophes. The very recent papers [3, 5] contain some bijective aspects. The paper [8] investigates the situation in the context of skew Dyck paths.

Dyck (and other lattice) paths with catastrophes are characterized by additional steps ('catastrophes') that bring the path back to the x-axis in just one step from any level $j \geq 2$. For S-Motzkin paths the definition is similar, and the graphical description in Figure 3 is easiest to understand; the catastrophes are drawn in special colors.

In the last section, S-Motzkin paths with air pockets will be discussed. Briefly, down steps of any length are now allowed, but no two down steps may follow each other.

2. S-Motzkin paths with catastrophes

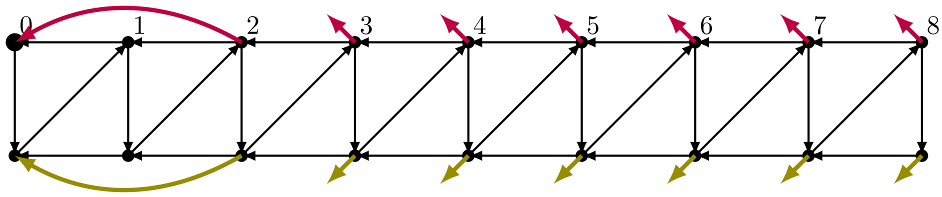

Figure 3: Graph to recognize S-Motzkin paths with catastrophes. Purple arrows lead to the initial state. Olive arrows lead to the level 0 state in the second layer.

In the sequel, we analyze the paths as in Figure 3, which are generated by the automaton. We introduce generating functions $f_i = f_i(z)$, where the coefficient of z^n counts the number of paths starting at the origin (=the big circle) and ending after n steps at state i (=level i) in the upper layer. The generating functions $g_i = g_i(z)$, where the coefficient of z^n counts the number of paths starting at the origin (=the big circle) and ending after n steps at state i (=level i) in the lower layer are also needed. For a similar but simpler instance, we refer to [9].

The following recursions are easy to see:

$$\begin{aligned} f_0 &= 1 + z(f_1 + f_2 + f_3 + f_4 + \cdots), \\ f_i &= zg_{i-1} + zf_{i+1}, \ i \geq 1, \\ g_0 &= zf_0 + z(g_1 + g_2 + g_3 + g_4 + \cdots), \\ g_i &= zf_i + zg_{i+1}, \ i \geq 1. \end{aligned}$$

Since f_0 and g_0 are somewhat special, we leave them out for the moment and compute the other ones, $f_i, g_i, i \geq 1$. Eventually we will solve the equations for f_0 and g_0, which will turn out to be just linear. Therefore we introduce the bivariate generating functions

$$F(u) = F(u,z) = \sum_{i\geq 1} u^{i-1} f_i, \quad G(u) = G(u,z) = \sum_{i\geq 1} u^{i-1} g_i$$

and we treat f_0 and g_0 as parameters. Summing the recursions over all possible values of i, we get

$$F(u) = zg_0 + zuG(u) + \frac{z}{u}[F(u) - f_1], \quad G(u) = zF(u) + \frac{z}{u}[G(u) - g_1].$$

Note that $f_1 = F(0)$ and $g_1 = G(0)$. We compute from this (of course with a computer)

$$F(u) = \frac{z(-u^2 g_0 + zug_0 + uf_1 - zf_1 + zu^2 g_1)}{z^2u^3 - u^2 + 2zu - z^2},$$

$$G(u) = \frac{z(-zu^2 g_0 + ug_1 + zuf_1 - zg_1)}{z^2u^3 - u^2 + 2zu - z^2}.$$

To factor the denominator, we set $u = zv$, and also $z^3 = x = t(1-t)^2$ to get

$$z^2(vt-1)(v^2t^2 - 2tv^2 + vt + v^2 - 2v + 1).$$

These and similar substitutions have appeared already in [10]. Therefore the three roots (expressed again in the variable u) are given by

$$u_1 = \frac{z}{t}, \qquad u_2 = -z\frac{t - 2 + \sqrt{4t - 3t^2}}{2(1-t)^2}, \qquad u_3 = -z\frac{t - 2 - \sqrt{4t - 3t^2}}{2(1-t)^2}$$

and so

$$z^2u^3 - u^2 + 2zu - z^2 = z^2(u-u_1)(u-u_2)(u-u_3).$$

These three roots appear already in [7], where more details are provided. Therefore

$$F(u) = \frac{-u^2g_0 + zug_0 + uf_1 - zf_1 + zu^2g_1}{z(u-u_1)(u-u_2)(u-u_3)} \quad \text{and} \quad G(u) = \frac{-zu^2g_0 + ug_1 + zuf_1 - zg_1}{z(u-u_1)(u-u_2)(u-u_3)}.$$

Cancelling the bad factors $(u-u_2)(u-u_3)$ out, we get

$$F(u) = \frac{-g_0 + zg_1}{z(u-u_1)} \quad \text{and} \quad G(u) = \frac{-g_0}{(u-u_1)}.$$

As a general remark, factors are bad if $\frac{1}{u-\overline{u}}$ has no power series expansion around $z=0$, $u=0$. This is part of the kernel method, see [6] for a user-friendly collection of examples. Plugging in $u=0$, we get

$$f_1 = \frac{g_0 - zg_1}{zu_1} \quad \text{and} \quad g_1 = \frac{g_0}{u_1} = \frac{g_0t}{z} \quad \text{and thus} \quad f_1 = g_0\frac{1-\frac{z}{u_1}}{zu_1} = g_0\frac{t(1-t)}{z^2}.$$

Now we can solve for f_0 and g_0:

$$f_0 = 1 + z(f_1+f_2+f_3+f_4+\cdots) = 1 + zF(1) = 1 + \frac{-g_0+zg_1}{1-u_1},$$
$$g_0 = zf_0 + z(g_1+g_2+g_3+g_4+\cdots) = zf_0 + \frac{-zg_0}{1-u_1}.$$

Therefore

$$f_0 = \frac{-t+z-zt}{-t+z-2zt+zt^2} \quad \text{and} \quad g_0 = \frac{z(z-t)}{-t+z-2zt+zt^2}.$$

Using the Lagrange inversion formula (or contour integration), we get the expansion (as in [7, 10])

$$t = \sum_{n\ge1} \frac{1}{n}\binom{3n-2}{n-1}x^n = \sum_{n\ge1} \frac{1}{n}\binom{3n-2}{n-1}z^{3n}.$$

Note that the equation $x = t(1-t)^2$ is exactly of the type needed for the Lagrange inversion, and so one can express t as a power series in x.

Using this, we can compute the power series expansion of f_0 and g_0; note that f_0 enumerates the S-Motzkin paths with catastrophes:

$$f_0 = 1 + z^3 + z^5 + 3z^6 + z^7 + 7z^8 + 13z^9 + 11z^{10} + 43z^{11} + 70z^{12} + 89z^{13} + 264z^{14} + 424z^{15} + 650z^{16} + 1657z^{17} + \cdots,$$
$$g_0 = z + 2z^4 + 2z^6 + 7z^7 + 2z^8 + 15z^9 + 32z^{10} + 23z^{11} + 96z^{12} + 174z^{13} + 192z^{14} + 604z^{15} + 1048z^{16} + 1434z^{17} + \cdots.$$

The coefficients are not 'nice', in the sense that there are no simple expressions available for them. Consequently, f_k and g_k also do not have nice coefficients, although the factor $\frac{1}{u-u_1}$ leads to nice coefficients, as can be seen from [7]. Many similar series expansions can be found in [11].

Now, we move to **asymptotics**.

As can be seen from the discussion in [10], the asymptotic enumeration of S-Motzkin paths is driven by a square-root type singularity, as it often happens in the enumeration of trees and lattice paths:

$$t \sim \frac{1}{3} - \frac{2}{3\sqrt{3}}\Big(1 - \frac{27x}{4}\Big)^{1/2}, \quad x \sim \frac{4}{27}$$

and the closest singularity (in x) is at $\frac{4}{27}$. Switching to the z-notation, as we have to in the context of catastrophes, we must look at the three roots closest to origin of modulus $\left(\frac{4}{27}\right)^{1/3} = 0.5291336839$. Consequently, the exponential growth of S-Motzkin paths is given by the reciprocal: $1.889881574 85^n$. The exponent n refers to the length n of the S-Motzkin path. There are only paths when n is divisible by 3, but that is of no concern.

For the case of catastrophes, we get a closer (to the origin) singularity. We need the dominant zero of the denominator $-t+z-2zt+zt^2$. A computer provides the value $\overline{z} = 0.5248885986\ldots$ and the corresponding value $\overline{t} = 0.2755080409\ldots$. As we can see, the value is slightly smaller: $0.5248885986 < 0.5291336839$. Consequently, this number leads to a *simple* pole, and the exponential growth is larger, as is not too surprising, considering the additional steps that are possible. The calculations are given at the start of the next page.

We must expand f_0 and g_0 at the simple pole $z=\overline{z}$. First note that

$$\frac{t}{dz}=\frac{dx}{dz}\frac{dt}{dx}\frac{t}{dt}=3z^2\frac{1}{(1-t)(1-3t)}\quad\text{and}\quad\frac{t}{dz}\Big|_{z=\overline{z},t=\overline{t}}=\frac{3\overline{z}^2}{(1-\overline{t})(1-3\overline{t})}=$$

$$\begin{aligned}-t+z-2zt+zt^2&\sim\frac{d}{dz}(-t+z-2zt+zt^2)\Big|_{z=\overline{z}}(z-\overline{z})=\Big(-\frac{t}{dz}+1-2t-2z\frac{t}{dz}+t^2+2zt\frac{t}{dz}\Big)\Big|_{z=\overline{z}}(z-\overline{z})\\&\sim-11.0530836206(z-\overline{z})\sim 21.0579609634(1-1.905166167z),\end{aligned}$$

and further

$$f_0=\frac{-t+z-zt}{-t+z-2zt+zt^2}\sim 0.0049752931\frac{1}{1-1.905166167z}.$$

Since $f_0(z)$ is the generating function of S-Motzkin paths with catastrophes, we got an asymptotic equivalent of these numbers of length n via $[z^n]f_0(z)\sim 0.0049752931(1.905166167)^n$.

A similar computation leads to $[z^n]g_0(z)\sim 0.0062160344(1.905166167)^n$. We continue with

$$F(u)=\frac{-g_0+zg_1}{z(u-u_1)}=\frac{g_0(1-t)t}{z^2(1-u\frac{t}{z})}$$

and therefore

$$[u^k]F(u)=\frac{(z-t)(1-t)}{-t+z-2zt+zt^2}\frac{t^{k+1}}{z^{k+1}}.$$

Similarly,

$$[u^k]G(u)=[u^k]\frac{t(z-t)}{-t+z-2zt+zt^2}\frac{1}{(1-\frac{ut}{z})}=\frac{t(z-t)}{-t+z-2zt+zt^2}\frac{t^k}{z^k}.$$

We note that

$$G(1)=\frac{zt}{-t+z-2zt+zt^2}.$$

Now, we move to partial S-Motzkin paths with *arbitrary* endpoint. In terms of generating functions, it just means $u:=1$, and we found the generating function

$$\begin{aligned}f_0(z)+F(1,z)+g_0(z)+G(1,z)&=\frac{1}{-t+z-2zt+zt^2}\Big[(-t+z-zt)+(1-t)t+z(z-t)+zt\Big]\\&=\frac{z+z^2-zt-t^2}{-t+z-2zt+zt^2}\\&=1+z+z^2+2z^3+3z^4+5z^5+10z^6+16z^7+30z^8+58z^9+98z^{10}+189z^{11}+\cdots.\end{aligned}$$

The asymptotic behaviour of the coefficients is also of the form const.$(1.905166167)^n$ since the denominator is always the same.

3. Right-to-left S-Motzkin paths with catastrophes

Here, we enumerate the S-Motzkin paths with catastrophes from right to left. To make the concept more clear, we might take an S-Motzkin with catastrophes and read it from right to left. Basically, it means that the orientation of the arrows changes. It is, however, easier to think about them in a mirrored fashion, and the automaton reads the symbols from left to right, as usual. If one lands again in the origin, we obtain the results as before. However, when we decide to stop at any step of our choice, the enumerations translate no longer from the left-right model. We use similar generating functions as before, namely $a_i(z)$ for the top layer, and $b_i(z)$ for the bottom layer. Then

$$\begin{aligned}&a_0=1+zb_0,\quad a_1=zb_1+za_0,\quad a_i=zb_i+za_{i-1}+za_0,\ i\ge 2,\\&b_0=za_1,\quad b_1=zb_0+za_2,\quad b_i=za_{i+1}+zb_{i-1}+zb_0,\ i\ge 2.\end{aligned}$$

These recursions can be derived from Figure 4. As before, we introduce

$$A(u)=\sum_{i\ge1}u^{i-1}a_i,\quad B(u)=\sum_{i\ge1}u^{i-1}b_i.$$

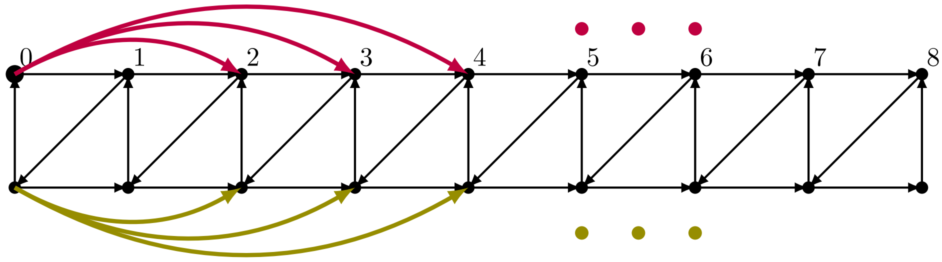

Figure 4: Graph to recognize S-Motzkin paths with catastrophes from right-to-left.

We compute in a similar fashion as before

$$\begin{aligned}A(u)&=a_1+\sum_{i\ge2}u^{i-1}a_i=a_1+z\sum_{i\ge2}u^{i-1}[b_i+a_{i-1}+a_0]\\&=a_1+z\sum_{i\ge2}u^{i-1}b_i+z\sum_{i\ge2}u^{i-1}a_{i-1}+z\sum_{i\ge2}u^{i-1}a_0\\&=a_1+zB(u)-zb_1+zuA(u)+\frac{zu}{1-u}a_0\end{aligned}$$

and

$$\begin{aligned}B(u)&=b_1+\sum_{i\ge2}u^{i-1}b_i=b_1+z\sum_{i\ge2}u^{i-1}[a_{i+1}+b_{i-1}+b_0]\\&=b_1+z\sum_{i\ge2}u^{i-1}a_{i+1}+z\sum_{i\ge2}u^{i-1}b_{i-1}+z\sum_{i\ge2}u^{i-1}b_0\\&=b_1+\frac{z}{u}\sum_{i\ge0}u^{i}a_{i+1}-\frac{z}{u}a_1-za_2+zuB(u)+\frac{zu}{1-u}b_0\\&=\frac{z}{u}A(u)-\frac{z}{u}a_1+zuB(u)+\frac{z}{1-u}b_0.\end{aligned}$$

We rewrite this system in the form

$$\begin{aligned}A(u)&=zuA(u)+zB(u)+\Phi(u),\quad \Phi(u)=a_1-zb_1+\frac{zu}{1-u}a_0,\\B(u)&=\frac{z}{u}A(u)+zuB(u)+\Psi(u),\quad \Psi(u)=-\frac{z}{u}a_1+\frac{z}{1-u}b_0.\end{aligned}$$

The solution is

$$A(u)=\frac{z(zub_0-zu^2a_0-za_1+zua_1+ua_0)}{(z^2u^3-2zu^2+u-z^2)(1-u)},\quad B(u)=\frac{z(-zu^2b_0-zu^2a_1+a_1u+b_0u+zua_1+za_0-a_1)}{(z^2u^3-2zu^2+u-z^2)(1-u)}.$$

Recall that if

$$u_1=\frac{z}{t},\qquad u_2=-z\frac{t-2+\sqrt{4t-3t^2}}{2(1-t)^2},\qquad u_3=-z\frac{t-2-\sqrt{4t-3t^2}}{2(1-t)^2};$$

then

$$z^2u^3-2zu^2+u-z^2=z^2\Big(u-\frac{1}{u_1}\Big)\Big(u-\frac{1}{u_2}\Big)\Big(u-\frac{1}{u_3}\Big);$$

this can be checked directly, compare also [7]. Since $\frac{1}{u_1}=\frac{t}{z}\sim z^2$, the factor $\big(u-\frac{1}{u_1}\big)$ is 'bad' and must cancel out. Plugging in $u=\frac{t}{z}$ into the numerators, we must get 0. Dividing out the factor $u-\frac{t}{z}$, the solutions now look like

$$A(u)=\frac{z(zb_0+za_1-uza_0-a_0t+a_0)}{z^2\big(u-\frac{1}{u_2}\big)\big(u-\frac{1}{u_3}\big)(1-u)},\quad B(u)=\frac{z(-zub_0+za_1-zua_1+a_1+b_0-tb_0-ta_1)}{z^2\big(u-\frac{1}{u_2}\big)\big(u-\frac{1}{u_3}\big)(1-u)}.$$

Note that

$$\Big(u-\frac{1}{u_2}\Big)\Big(u-\frac{1}{u_3}\Big)=u^2+\frac{t-2}{z}u+\frac{z}{t}.$$

Now we can plug in $u=0$ to get

$$A(0)=a_1=\frac{z(zb_0+za_1-a_0t+a_0)}{(1-t)^2},\quad\text{and}\quad a_1=\frac{z(zb_0+a_0-a_0t)}{-z^2+1-2t+t^2}.$$

Since $a_0 = 1 + zb_0$, we get $a_1 = \dfrac{z(2zb_0 + 1 - t - ztb_0)}{-z^2 + 1 - 2t + t^2}$. We also get $B(0) = b_1 = \dfrac{z(za_1 + a_1 + b_0 - tb_0 - ta_1)}{(1-t)^2}$.
From $b_0 = za_1$ we find

$$b_0 = \frac{z^2(zb_0 + a_0 - a_0 t)}{-z^2 + 1 - 2t + t^2} = \frac{t(1-t)}{-t + z - 2zt + zt^2}, \quad \text{and} \quad a_0 = 1 + zb_0 = f_0 = \frac{-t + z - zt}{-t + z - 2zt + zt^2}.$$

In principle, one could also write formulæ for general a_k and b_k, by using partial fraction decomposition. Since the results look very complicated and do not provide extra insight, we refrain from giving such explicit results.

A brief comment about asymptotics: Since the denominators are the same as in the left-right instance, we get again an exponential behaviour, with the same rate as before. The concept of open end does not make sense here since there are infinitely many such paths of a given length $n \geq 1$.

While one might be tempted to attack the current question using some bijective tricks, it is worthwhile to note that our approach is very flexible, and, e. g., subsets of the catastrophes may be considered, with little extra efforts.

4. S-Motzkin paths and air pockets

Now, we move to another model popularized by Baril et al., namely introducing *air pockets*, see [4]. These are maximal chains of downsteps, but this time only counted as *one* step. Since the main issue of S-Motzkin paths to keep the pattern flat, up, flat, up, flat, up, . . . alive, the downsteps live their own live, and we managed to construct a graph with 4 layers of states, describing all possible scenarios. Note that the wavy edges represent transitions without reading a symbol. The

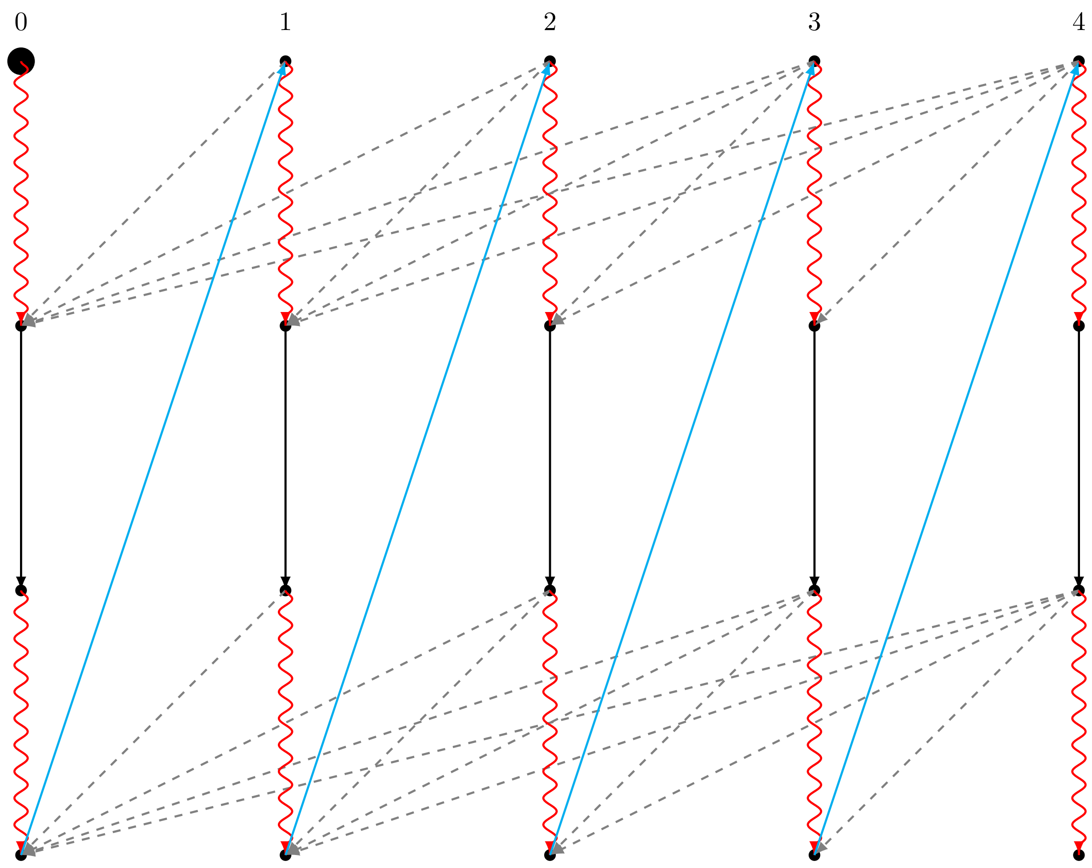

Figure 5: Four layers of states. Only blue edges bring us to the next level, red wavy lines and black lines stay at the same level, only dashed grey arrows represent down-steps, this time by and amount, not necessarily down to the x-axis.

generating functions for the four layers, as seen in Figure 5, reaching level i, can be read off from the diagram; note that the wavy edge is labelled by 1, not by z, since there is no step done. As in the theory of finite automata, such edges where no symbol is read, could be avoided in principle, but we found it convenient to use such transitions. There is no ambiguity here, since for each Motzkin-like path in our context there is exactly one path in the automaton, and the transitions without reading a symbol (represented by the wavy lines) will be ignored when considering the relevant path.

$$a_0 = 1, \quad a_i = zd_{i-1},\ i \geq 1, \quad b_i = a_i + z\sum_{j>i} a_j,$$
$$c_i = zb_i, \quad d_i = c_i + z\sum_{j>i} c_j.$$

The bivariate generating functions are $A(u) = \sum_{i\geq 0} u^i a_i$, $B(u) = \sum_{i\geq 0} u^i b_i$, $C(u) = \sum_{i\geq 0} u^i c_i$. Summing the recursions over all possible values of i, we find the system

$$A(u) = 1 + zuD(u), \quad B(u) = A(u) + \frac{z}{1-u}[A(1) - A(u)],$$
$$C(u) = zB(u), \quad D(u) = C(u) + \frac{z}{1-u}[C(1) - C(u)].$$

The system can be reduced to two equations

$$C(u) = zA(u) + \frac{z^2}{1-u}[A(1) - A(u)],$$
$$\frac{A(u)-1}{u} = zC(u) + \frac{z^2}{1-u}[C(1) - C(u)];$$

and so, by solving,

$$A(u) = \frac{-z^2uC(1) + z^3u^2A(1) - 1 + 2u + z^4uA(1) - u^2 + z^2u^2C(1) - z^3uA(1)}{-1 + 2u - u^2 + z^2u - 2z^2u^2 - 2z^3u + z^2u^3 + 2z^3u^2 + z^4u},$$

$$C(u) = \frac{\big(z^2u^2C(1) - u^2 - zu + 2u + zA(1)u + z^3uC(1) - z^2uC(1) - 1 + z - zA(1)\big)\, z}{-1 + 2u - u^2 + z^2u - 2z^2u^2 - 2z^3u + z^2u^3 + 2z^3u^2 + z^4u}.$$

Plugging in $u = 1$ gives the void equations $A(1) = A(1)$ and $C(1) = C(1)$. Therefore the denominator has to be investigated. We find that

$$-1 + 2u - u^2 + z^2u - 2z^2u^2 - 2z^3u + z^2u^3 + 2z^3u^2 + z^4u = z^2(u-\rho)(u-\sigma)(u-\tau);$$

the explicit forms provided by Maple are useless, but fortunately gfun (in Maple) allows manipulations with the relevant series:

$$\begin{aligned}\rho =& z^{-2} - 2z - 2z^3 - z^4 - 2z^5 - 6z^6 - 4z^7 - 15z^8 - 22z^9 - 33z^{10} - 86z^{11} - 115z^{12} - 256z^{13} - 486z^{14} - 804z^{15} - 1783z^{16}\\ &- 3074z^{17} - 6049z^{18} - 12104z^{19} - 21902z^{20} - 44918z^{21} - 85235z^{22} - 165124z^{23} - 331137z^{24} - 631740z^{25} - 1261785z^{26}\\ &- 2477694z^{27} + \cdots\end{aligned}$$

The other roots are ugly but we compute the simpler $(u-\sigma)(u-\tau) = u^2 + Ku + L$ with

$$K = -2z^2 - 2z^5 - z^6 - 2z^7 - 6z^8 - 4z^9 - 15z^{10} - 22z^{11} - 33z^{12} - 86z^{13} - 115z^{14} - 256z^{15} - 486z^{16} + \cdots,$$
$$L = z^2 + 2z^5 + 2z^7 + 5z^8 + 2z^9 + 14z^{10} + 16z^{11} + 27z^{12} + 74z^{13} + 86z^{14} + 222z^{15} + 395z^{16} + \cdots.$$

So, we must divide this term out from the numerator and denominator. Therefore

$$A(u) = \frac{z^3A(1) + z^2C(1) - 1}{z^2(u-\rho)}, \quad \text{and} \quad C(u) = \frac{z^2C(1) - 1}{z(u-\rho)},$$

and by $u = 1$ and solving,

$$A(1) = \frac{-1+\rho}{z^2(-1+\rho+z)^2}, \quad \text{and} \quad C(1) = \frac{1}{z(-1+\rho+z)}.$$

Since these values are known, we find

$$A(u) = \frac{\rho}{\rho - u} \quad \text{and} \quad C(u) = \frac{1-\rho}{z(1-\rho-z)(\rho-u)},$$

where the identity $1 - 2\rho + \rho^2 - z^2\rho + 2z^2\rho^2 + 2z^3\rho - z^2\rho^3 - 2z^3\rho^2 - z^4\rho = 0$ was used for simplification. One sees $A(0) = 1$, which is clear from combinatorial reasons. Further

$$a_k = [u^k]A(u) = \rho^{-k} \quad \text{and} \quad c_k = [u^k]C(u) = \frac{1-\rho}{z(1-\rho-z)}\rho^{-k-1}.$$

The other quantities are then $b_k = \frac{1}{z}c_k$ and $d_k = \frac{1}{z}a_{k+1}$, for any $k \geq 0$.

We leave the analysis of this air pocket model from right to left, as well as other parameters, to the interested reader. The factorization $(u-\rho^{-1})(u-\sigma^{-1})(u-\tau^{-1})$ will play a role here, and only one factor is bad, namely $(u-\rho^{-1})$.

It is possible to consider catastrophes and air pockets at the same time; we leave such considerations to enthusiastic young researchers.

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