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Research Article

Rainbow subgraphs in edge-colored planar and outerplanar graphs

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Abstract

Let $\mathcal{G}$ be a class of graphs. The strong rainbow number of the graph H in $\mathcal{G}$ is the minimum number of colors k such that every graph $G \in \mathcal{G}$ admits an edge coloring with at most k colors in which all copies of H are rainbow (i.e., all edges of H have different colors). In this paper, it is shown that the strong rainbow number of any 2-connected graph H in the class of outerplanar graphs is bounded from above by a constant (depending only on H).

Keywords: planar graph; edge coloring; rainbow subgraph.

2020 Mathematics Subject Classification: 05C10, 05C15.

1. Introduction

In this paper, we consider finite simple graphs. Throughout the paper, we use C_n, K_n, P_n, and $K_{m,n}$ to denote the cycle, complete graph, path on n vertices, and the complete bipartite graph with m vertices in one part and n vertices in the other part, respectively. The number of edges of a graph G is denoted by $e(G)$, and the number of vertices of G is denoted by $v(G)$. An edge-colored graph is called rainbow if no two edges have the same color. A rainbow copy of a graph H in an edge-colored graph G is a rainbow subgraph of G isomorphic to H. Edge colorings of graphs with constraints on special subgraphs are very intensively studied. For graphs H and G, the anti-Ramsey number of H in G, denoted by $\mathrm{ar}(G, H)$, is the maximum number of colors in an edge coloring of G containing no rainbow copy of H. This problem was introduced by Erdős, Simonovits, and Sós [10] in 1973 and considered in the classical case when G is K_n. Since then, the study of $\mathrm{ar}(G, H)$ for some special graphs H and G has attracted a lot of attention, see the dynamic survey [12]. During recent years, the anti-Ramsey problem in planar graphs was extensively studied, see e.g. [17]. When, instead of coloring edges to obtain all copies of H non-rainbow using the greatest possible number of colors, one aims to minimize the number of colors to avoid the appearance of a non-rainbow copy of H, we get the dual version of the anti-Ramsey problem. This variant was introduced by Axenovich, Füredi, and Mubayi [2] in a more general setting. Given graphs H and G, and an integer $q \le e(H)$, an (H, q)-coloring of G is an edge coloring of G in which every copy of H is colored with at least q colors. Let $r(G, H, q)$ denote the minimum number of colors in an (H, q)-coloring of G. By the definition $1 \le r(G, H, q) \le e(G)$. The case $r(K_n, K_p, q)$ was first studied systematically by Erdős and Gyárfás [9]. They obtained a general upper bound on $r(K_n, K_p, q)$ for general p and q:

$$r(K_n, K_p, q) \le cn^{\frac{p-2}{\binom{p}{2}-q+1}},$$

where c depends only on p, q (this bound was improved very recently, see [4, 5]). They noted that determining $r(K_n, K_p, q)$ for small values of p, q leads to problems of varying difficulty. For example, any $(K_3, 3)$-coloring of K_n is a proper edge coloring of K_n and vice versa, therefore $r(K_n, K_3, 3)$ is equal to the chromatic index of K_n. On the other hand, determining $r(K_n, K_p, 2)$ is hopeless because it is equivalent to determining the classical Ramsey numbers for multicolorings. They proved that the smallest q for which $r(K_n, K_p, q)$ is linear in n is $\binom{p}{2} - p + 3$ and the smallest q for which $r(K_n, K_p, q)$ is quadratic in n equals to $\binom{p}{2} - \lfloor \frac{p}{2} \rfloor + 2$ for $p \ge 4$. They also showed that the smallest q for which $r(K_n, K_p, q) = \binom{n}{2} - O(1)$ is $q = \binom{p}{2} - \lfloor \frac{p}{4} \rfloor + 1$. Sárközy and Selkow [23] studied the behavior of the function $r(K_n, K_p, q)$ between the linear and quadratic orders of magnitude. They proved that $r(K_n, K_p, q)$ is linear in n for at most $\log_2 p$ values of q. Recently, a family of additional thresholds appeared in [3, 8, 11, 19, 20]. The bipartite case $r(K_{n,n}, K_{p,p}, q)$ was considered by Axenovich, Füredi, and Mubayi [2]. They proved that the smallest q for which $r(K_{n,n}, K_{p,p}, q)$ is linear in n is $p^2 - 2p + 3$, the smallest q for which $r(K_{n,n}, K_{p,p}, q)$ is quadratic in n equals to $p^2 - p + 2$ and the smallest q for which $r(K_{n,n}, K_{p,p}, q) = n^2 - O(1)$ is $q = p^2 - \lfloor \frac{p}{2} \rfloor + 1$. Sárközy and Selkow [21] proved that $r(K_{n,n}, K_{p,p}, q)$ is linear in n for at most $\log_2 p + 1$ values of q. For

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non-balanced bipartite graphs, i.e. for the function $r(K_{n,n}, K_{s,t}, q)$, we refer the interested reader to [13, 18]. Besides the cases, where G and H are complete or complete bipartite graphs, little is known about $r(G, H, q)$. In [16], Krueger studied the asymptotic behavior of $r(K_n, P_m, q)$, Axenovich, Füredi, and Mubayi [2] proved that $r(G, H, q) < cn^{\frac{v-2}{e-q+1}}$ holds for any two graphs G, H with $n = v(G)$, $v = v(H)$, $e = e(H)$, $1 \leq q \leq e$, where $c = c(H, q)$ is a constant. Yet another approach was considered in [6]: Burr, Erdős, Graham, and Sós investigated a function $f(n, e, H)$, defined as the minimum number of colors necessary to color the edges of a graph G on n vertices and e edges such that all copies of H in it are rainbow, see also [7, 22].

In this paper, we consider another approach: Let $\mathcal{G}$ be a class of graphs. We define the strong rainbow number of H in $\mathcal{G}$ as $\mathrm{srb}(\mathcal{G}, H) = \max\{r(G, H, e(H)) : G \in \mathcal{G}\}$. We focus on the cases when $\mathcal{G}$ is the class of outerplanar or planar graphs. Our main surprising result is that $\mathrm{srb}(\mathcal{G}, H)$ can be bounded from above by a constant (depending only on H) for the class of outerplanar graphs $\mathcal{G}$ and every 2-connected graph H. We also show that if $\mathcal{G}$ is the class of planar graphs, then $\mathrm{srb}(\mathcal{G}, C_3) = 3$ and there is no constant upper bound for $\mathrm{srb}(\mathcal{G}, C_k)$ if $k \geq 4$.

2. Results

A drawing of a graph maps each vertex to a point in the plane and each edge to a Jordan arc between its endvertices. A drawing is planar if no two edges intersect each other, except at their endvertices. A planar graph is a graph that has a planar drawing. A planar drawing partitions the plane into connected regions, called faces. There are two types of faces. The bounded faces are internal, while the unbounded face is the outer face. An outerplanar drawing is a planar drawing such that all the vertices are incident to the outer face. An outerplanar graph is a graph that admits an outerplanar drawing. An outerplanar graph is maximal outerplanar if it is not possible to add an edge such that the resulting graph is still outerplanar. For convenience, we often use the abbreviation plane (outerplane) graph for a particular planar (outerplanar) drawing of a planar (outerplanar) graph.

Let G be a 2-connected plane graph. We define the (geometric) dual G^* of G as follows. G^* is a plane multigraph that has precisely one vertex in each face of G. If e is an edge of G, then G^* has an edge e^* crossing e and joining the two vertices of G^* in the two faces of G that contain e on the boundary. Moreover, e^* has no other points in common with G, and all edges of G^* are obtained in this way. Splitting the vertex v^* of G^* corresponding to the outer face of G into the number of copies equal to the size of the outer face of G so that each copy is incident to exactly one edge corresponding to an edge of the outer face results in the semidual graph G^*_s. Figure 1 gives an example of a plane graph G and its dual, semidual graphs.

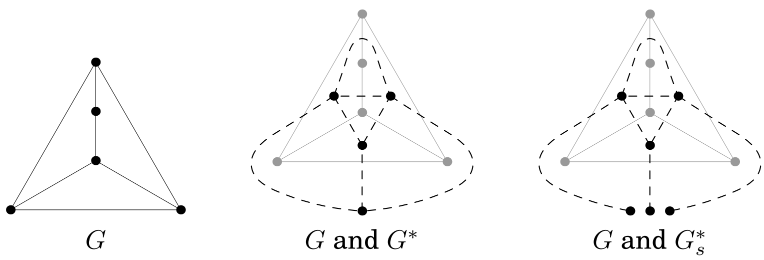

Figure 1: Planar drawings of G, G^* and G^*_s.

Outerplanar graphs

Theorem 2.1. *If $\mathcal{G}$ is the class of outerplanar graphs, then $\mathrm{srb}(\mathcal{G}, C_k) < 2^k$, for any $k \geq 3$.*

Proof. Since any outerplanar graph can be extended (by adding some edges) to a maximal outerplanar graph, it is sufficient to prove that $r(G, C_k, k) < 2^k$ for every maximal outerplanar graph G. Every maximal outerplanar graph is 2-connected ([14], page 107). Every 2-connected outerplanar graph has a unique Hamiltonian cycle [24], hence every 2-connected outerplanar graph has a unique outerplanar drawing. Consequently, every maximal outerplanar graph has a unique embedding. Therefore, in the following we do not distinguish G and its outerplanar drawing.

Let G be a maximal outerplane graph and let G^*_s be its semidual. By [15], G^*_s is a tree with vertices of degree 3 and 1 only. Now we show that G^*_s has an edge coloring with at most $2^k - 3$ colors such that no two edges within distance $k-2$ have the same color. Here, the distance of two edges in G^*_s is defined as the distance of the corresponding vertices in the line graph of G^*_s. (The line graph of a graph F, denoted by $L(F)$, is the graph whose vertices are the edges of F, where two vertices of $L(F)$ are adjacent if and only if the corresponding edges are adjacent in F.) Let $L(G^*_s)$ denote the line graph of G_s^*, and let $L^{k-2}(G_s^*)$ be its $(k-2)$th power, which is the graph obtained from $L(G_s^*)$ by adding the edges between pairs of vertices at distance at most $k-2$. Since G_s^* is a tree with maximum degree 3, the maximum degree of $L^{k-2}(G_s^*)$ is not greater than $2(2+4+8+\cdots+2^{k-2}) = 2(2^{k-1}-2) = 2^k-4$. Therefore, $L^{k-2}(G_s^*)$ admits a proper vertex coloring with at most 2^k-3 colors. This coloring induces an edge coloring of G_s^* such that any two edges within distance $k-2$ have distinct colors.

Let C be a cycle in G of length k and let x, y be two of its edges. Now we show that the edges x_s^*, y_s^* of G_s^* which correspond to x, y are at distance at most $k-2$ in G_s^*. Let $[C]$ be the subgraph of G consisting of C and the edges which are in its interior. The graph G is maximal outerplanar, hence $[C]$ is maximal outerplanar as well. The outerplanar drawing of a k-vertex maximal outerplanar graph has $k-2$ interior faces ([14], page 106). Therefore, a longest path in the semidual of $[C]$ has at most k vertices, consequently, at most $k-1$ edges. From this it follows that the edges x_s^*, y_s^* are at distance at most $k-2$ in $[C]_s^*$, hence they are at distance at most $k-2$ in G_s^*. This means that every edge coloring of G_s^* such that any two edges within distance $k-2$ receive distinct colors induces an edge coloring of G such that any cycle of length k is rainbow. □

The upper bound for $\mathrm{srb}(\mathcal{G}, C_k)$ in Theorem 2.1 is exponential. Next, we show that this cannot be improved in general.

Theorem 2.2. *For every even integer $k \geq 4$, there is an outerplanar graph G such that $r(G, C_k, k) \geq 2^{\frac{k}{2}}$.*

Proof. Let $G_0 = C_4$. For $i \geq 0$, G_{i+1} is the outerplane graph obtained from G_i by replacing each edge uv incident with the outer face with a triangle $uvwu$, see Figure 2 for an illustration.

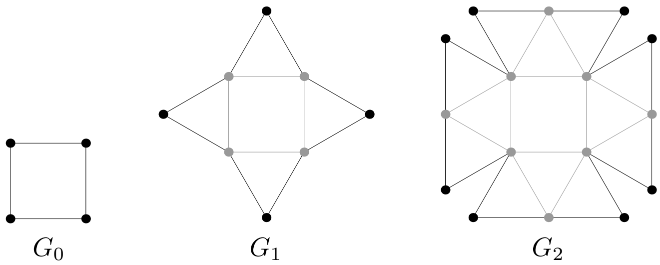

Figure 2: The graphs G_0, G_1, and G_2.

Now, we show that any two edges incident with the outer face of G_i are incident with a cycle of length $2i+4$. We proceed by induction on i. The case $i = 0$ trivially holds. Assume that the claim holds for $i = j$. Now consider the case $i = j+1$. Let e_1 and e_2 be two edges of G_{j+1} incident with the outer face. We distinguish two cases.

Case 1: e_1 and e_2 do not share a vertex of degree 2 in G_{j+1}.
Let $e_1 = u_1u_2$, $e_2 = v_1v_2$ and assume that u_2, v_2 have degree 2 in G_{j+1}. Let u_3 be the other neighbor of u_2 and let v_3 be the other neighbor of v_2. The edges u_1u_3 and v_1v_3 are incident with the outer face of G_j, therefore (by the inductive hypothesis) there is a cycle C in G_j of length $2j+4$ incident with both of them. If we remove the edges u_1u_3 and v_1v_3 from C and add the edges u_1u_2, u_2u_3, v_1v_2, and v_2v_3, then we obtain a cycle in G_{j+1} of length $(2j+4)-2+4 = 2(j+1)+4$ which contains the edges e_1 and e_2.

Case 2: e_1 and e_2 share a vertex of degree 2 in G_{j+1}.
In this case, we choose a third edge e_3 incident with the outer face of G_{j+1}. Since e_1 and e_3 do not share a vertex of degree 2 in G_{j+1} we can find a cycle of length $2(j+1)+4$ which contains both of them. Clearly, this cycle must contain e_2 as well.

Now, consider an edge coloring of G_i in which any cycle of length $2(i+2)$ is rainbow. In this coloring, no two edges incident with the outer face have the same color, since any two such edges are incident with a common cycle of length $2i+4$. It is easy to see that exactly 2^{i+2} edges of G_i are incident with its outer face, so the coloring uses at least 2^{i+2} colors.

This means that $r(G_i, C_{2(i+2)}, 2(i+2)) \geq 2^{i+2}$ or equivalently (with $k = 2(i+2)$) $r(G_{\frac{k}{2}-2}, C_k, k) \geq 2^{\frac{k}{2}}$. □

Theorem 2.3. *If $\mathcal{G}$ is the class of outerplanar graphs and H is a 2-connected graph on k vertices, then $\mathrm{srb}(\mathcal{G}, H) < 2^k$.*

Proof. Clearly, if H is not outerplanar, then $r(G, H, e(H)) = 1$ for any outerplanar graph G. So we can assume that H is outerplanar. The boundary of the outer face of any 2-connected outerplane graph is a cycle, therefore we can use the same arguments as in the proof of Theorem 2.1. □

It is worth mentioning that 2-connectedness of H is necessary in Theorem 2.3. To see this, take an outerplane graph H_1 and let v be an arbitrary vertex of H_1. Now, take n copies of H_1 and identify all v's. The resulting graph H_n is outerplanar, see Figure 3 for an illustration.

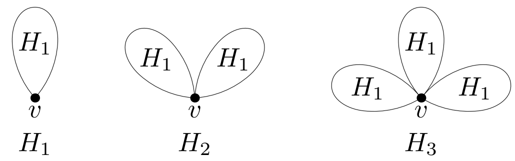

Figure 3: The graphs H_1, H_2, and H_3.

Since any two edges of H_n are incident with a common H_2 we have $r(H_n, H_2, e(H_2)) = e(H_n)$ for every $n \geq 2$.

Conjecture 2.1. *For any positive integer K and any 1-connected (not 2-connected) outerplanar graph H there is an outerplanar graph G such that $r(G, H, e(H)) > K$.*

Planar graphs

Theorem 2.4. *If $\mathcal{G}$ is the class of planar graphs, then $\mathrm{srb}(\mathcal{G}, C_3) = 3$.*

Proof. Clearly, $\mathrm{srb}(\mathcal{G}, C_3) \geq 3$. So it is sufficient to show that $\mathrm{srb}(\mathcal{G}, C_3) \leq 3$. Every planar graph admits a proper vertex coloring with at most four colors [1]. Let G be a planar graph and c be its proper vertex coloring which uses at most four colors. Let the colors be the elements of the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$. Color each edge $e = uv$ of G with color $c(u) + c(v)$. In the obtained edge coloring of G every C_3 is rainbow, see Figure 4. So $r(G, C_3, 3) \leq 3$.

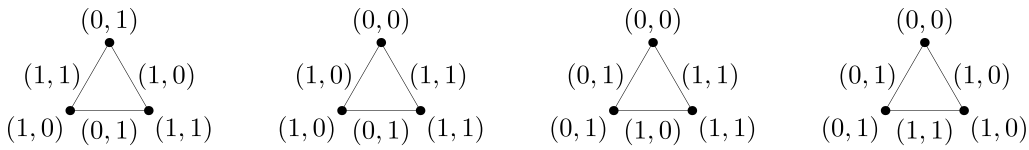

Figure 4: Admissible colorings of C_3.

□

Now we prove that there is no constant upper bound for $\mathrm{srb}(\mathcal{G}, C_k)$ if $k \geq 4$.

Theorem 2.5. *For any two positive integers K and k there is a planar graph G such that $r(G, C_k, k) > K$.*

Proof. Let G_n be a planar graph with vertex set $V(G_n) = \{v, w, u_1, u_2, \ldots, u_n\}$ and edge set $E(G_n) = \{vu_i, wu_i : i = 1, 2 \ldots, n\} \cup \{u_i u_{i+1} : i = 1, 2 \ldots, n-1\}$, see Figure 5 for illustration.

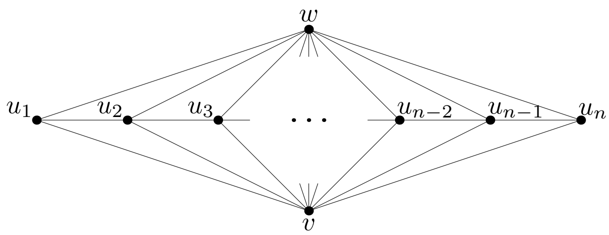

Figure 5: The graph G_n.

It suffices to show that for sufficiently large $n > K$ any two edges incident with v are incident with a common cycle C_k, because this implies $r(G_n, C_k, k) \geq n$. Let $n = \max\{K+1, 3k\}$ and let vu_i, vu_j, $i < j$, be two edges of G_n.

If $i > k$, then $w, u_j, v, u_i, u_{i-1}, u_{i-2}, \ldots, u_{i-(k-4)}, w$ is a cycle of length k containing vu_i, vu_j.

If $j < 2k$, then $w, u_i, v, u_j, u_{j+1}, u_{j+2}, \ldots, u_{j+(k-4)}, w$ is a cycle of length k containing vu_i, vu_j.

If $i \leq k$ and $j \geq 2k$, then $w, u_i, v, u_j, u_{j-1}, u_{j-2}, \ldots, u_{j-(k-4)}, w$ is a cycle of length k containing vu_i, vu_j. □

Theorem 2.6. *If $\mathcal{G}$ is the class of planar graphs, then $\mathrm{srb}(\mathcal{G}, K_4) = 6$.*

Proof. We proceed as in the proof of Theorem 2.4. We color the vertices with colors $0, 1, 2, 4$. □

We finish the paper with the following conjecture.

Conjecture 2.2. *If $\mathcal{G}$ is the class of planar graphs and H is a 3-connected planar graph, then $\mathrm{srb}(\mathcal{G}, H)$ can be bounded from above by a constant (depending only on H).*

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