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Research Article

A common approach to three open problems in number theory

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Abstract

The following system of equations $\{x_1 \cdot x_1 = x_2,\ x_2 \cdot x_2 = x_3,\ 2^{2^{x_1}} = x_3,\ x_4 \cdot x_5 = x_2,\ x_6 \cdot x_7 = x_2\}$ has exactly one solution in $(\mathbb{N} \setminus \{0,1\})^7$, namely $(2,4,16,2,2,2,2)$. Conjecture 2.1 states that if a system $\mathcal{S}$ of equations has at most five equations and at most finitely many solutions in $(\mathbb{N} \setminus \{0,1\})^7$, then each such solution $(x_1,\dots,x_7)$ satisfies $x_1,\dots,x_7 \leqslant 16$, where $\mathcal{S} \subseteq \{x_i \cdot x_j = x_k :\ i,j,k \in \{1,\dots,7\}\} \cup \{2^{2^{x_j}} = x_k :\ j,k \in \{1,\dots,7\}\}$. Conjecture 2.1 implies that there are infinitely many composite numbers of the form $2^{2^n}+1$. Conjectures 3.1 and 4.1 are of similar kind. Conjecture 3.1 implies that if the equation $x!+1=y^2$ has at most finitely many solutions in positive integers x and y, then each such solution (x,y) belongs to the set $\{(4,5),(5,11),(7,71)\}$. Conjecture 4.1 implies that if the equation $x(x+1)=y!$ has at most finitely many solutions in positive integers x and y, then each such solution (x,y) belongs to the set $\{(1,2),(2,3)\}$. Semi-algorithms $\texttt{sem}_j$ $(j=2,3,4)$ that never terminate are described. For every $j \in \{2,3,4\}$, if Conjecture j.1 is true, then $\texttt{sem}_j$ endlessly prints consecutive positive integers starting from 1. For every $j \in \{2,3,4\}$, if Conjecture j.1 is false, then $\texttt{sem}_j$ prints a finite number (including zero) of consecutive positive integers starting from 1.

Keywords: Brocard's problem; Brocard-Ramanujan equation; Erdős' equation $x(x+1)=y!$; composite Fermat numbers.

2020 Mathematics Subject Classification: 11D61, 11D85.

1. Epistemic notions increase the scope of mathematics

Nicolas Goodman observed that epistemic notions increase the scope of mathematics, see [2]. For many finite sets $\mathcal{X} \subseteq \mathbb{N}^m$, we know an algorithm that decides $\mathcal{X}$, but no known algorithm computes a positive integer n satisfying $\mathcal{X} \subseteq [0,n]^m$. This holds because, for many Diophantine equations, the number of rational solutions is finite by Faltings' theorem. Faltings' theorem tells us that certain curves have at most finitely many rational points, but no known proof gives any bound on the sizes of the numerators and denominators of the coordinates of those points.

In Sections 2–4, our knowledge (including conjectures) about the set $\mathcal{X}$ is different. The considerations in Section 2 imply the existence of the set $\mathcal{X}_2 \subseteq (\mathbb{N} \setminus \{0,1\})^7$ whose finiteness/infiniteness is unknown, although we conjecture that $\mathrm{card}(\mathcal{X}_2) < \omega \Rightarrow \mathcal{X}_2 \subseteq [2,16]^7$. The considerations in Section 3 imply the existence of the set $\mathcal{X}_3 \subseteq (\mathbb{N} \setminus \{0\})^6$ whose finiteness/infiniteness is unknown, although we conjecture that $\mathrm{card}(\mathcal{X}_3) < \omega \Rightarrow \mathcal{X}_3 \subseteq [1,(24!)!]^6$. The considerations in Section 4 imply the existence of the set $\mathcal{X}_4 \subseteq (\mathbb{N} \setminus \{0\})^6$ whose finiteness/infiniteness is unknown, although we conjecture that $\mathrm{card}(\mathcal{X}_4) < \omega \Rightarrow \mathcal{X}_4 \subseteq [1,720!]^6$. For every $j \in \{2,3,4\}$, we know an algorithm that decides the set $\mathcal{X}_j$.

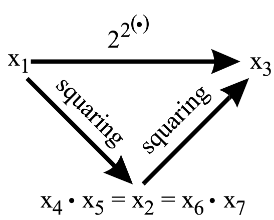
2. Composite numbers of the form $2^{2^n}+1$

Let $\mathcal{A}$ denote the following system of equations:

$$\Big\{x_i \cdot x_j = x_k :\ i,j,k \in \{1,\dots,7\}\Big\} \cup \Big\{2^{2^{x_j}} = x_k :\ j,k \in \{1,\dots,7\}\Big\}$$

The following subsystem of $\mathcal{A}$

$$\begin{cases} x_1 \cdot x_1 = x_2 \\ x_2 \cdot x_2 = x_3 \\ 2^{2^{x_1}} = x_3 \\ x_4 \cdot x_5 = x_2 \\ x_6 \cdot x_7 = x_2 \end{cases}$$



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has exactly one solution in $(\mathbb{N} \setminus \{0,1\})^7$, namely $(2,4,16,2,2,2,2)$.

Conjecture 2.1. *If a system of equations $\mathcal{S} \subseteq \mathcal{A}$ has at most five equations and at most finitely many solutions in $(\mathbb{N} \setminus \{0,1\})^7$, then each such solution $(x_1, \ldots, x_7)$ satisfies $x_1, \ldots, x_7 \leqslant 16$.*

Lemma 2.1 (see p. 109 in [8]). *For every pair of non-negative integers x and y, the equation $x+1=y$ holds if and only if*

$$2^{2^x} \cdot 2^{2^x} = 2^{2^y}.$$

Theorem 2.1. *Conjecture 2.1 implies that $2^{2^{x_1}}+1$ is composite for infinitely many integers x_1 greater than 1.*

Proof. Assume, on the contrary, that Conjecture 2.1 holds and $2^{2^{x_1}}+1$ is composite for at most finitely many integers x_1 greater than 1. Then, the equation

$$x_2 \cdot x_3 = 2^{2^{x_1}} + 1$$

has at most finitely many solutions in $(\mathbb{N} \setminus \{0,1\})^3$. By Lemma 2.1, in positive integers greater than 1, the following subsystem of $\mathcal{A}$

$$\begin{cases} 2^{2^{x_1}} = x_5 \\ 2^{2^{x_5}} = x_6 \\ 2^{2^{x_4}} = x_7 \\ x_6 \cdot x_6 = x_7 \\ x_2 \cdot x_3 = x_4 \end{cases}$$

$$\begin{tikzcd} x_1 \arrow[r, "2^{2^{(\cdot)}}"] & x_5 \arrow[r, dashed, "+1"] \arrow[d, "2^{2^{(\cdot)}}"] & x_4 = x_2 \cdot x_3 \arrow[d, "2^{2^{(\cdot)}}"] \\ & x_6 \arrow[r, "\text{squaring}"] & x_7 \end{tikzcd}$$

has at most finitely many solutions in $(\mathbb{N} \setminus \{0,1\})^7$ and expresses that

$$\begin{cases} x_2 \cdot x_3 &=& 2^{2^{x_1}} + 1 \\ x_4 &=& 2^{2^{x_1}} + 1 \\ x_5 &=& 2^{2^{x_1}} \\ x_6 &=& 2^{2^{2^{2^{x_1}}}} \\ x_7 &=& 2^{2^{2^{2^{x_1}}+1}} \end{cases}.$$

Since $641 \cdot 6700417 = 2^{2^5}+1 > 16$, we get a contradiction. $\square$

Most mathematicians believe that $2^{2^n}+1$ is composite for every integer $n \geqslant 5$, see [3, p. 23].

Problem 2.1 (see p. 159 in [4]). *Are there infinitely many composite numbers of the form $2^{2^n}+1$?*

Primes of the form $2^{2^n}+1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n}+1$ is prime, see [4, p. 1]. Fermat remarked that $2^{2^0}+1=3$, $2^{2^1}+1=5$, $2^{2^2}+1=17$, $2^{2^3}+1=257$, and $2^{2^4}+1=65537$ are all prime, see [4, p. 1].

Problem 2.2 (see p. 158 in [4]). *Are there infinitely many prime numbers of the form $2^{2^n}+1$?*

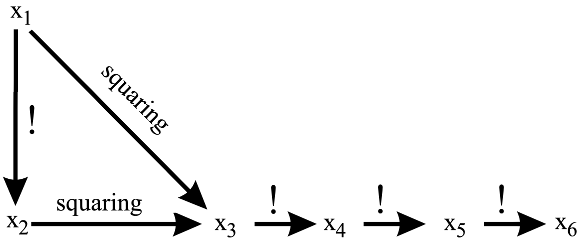
3. The Brocard-Ramanujan equation $x!+1=y^2$

Let $\mathcal{B}$ denote the following system of equations:

$$\{x_i \cdot x_j = x_k : i,j,k \in \{1,\ldots,6\}\} \cup \{x_j! = x_k : (j,k \in \{1,\ldots,6\}) \wedge (j \neq k)\}.$$

The following subsystem of $\mathcal{B}$

$$\begin{cases} x_1! = x_2 \\ x_1 \cdot x_1 = x_3 \\ x_2 \cdot x_2 = x_3 \\ x_3! = x_4 \\ x_4! = x_5 \\ x_5! = x_6 \end{cases}$$



has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(2, 2, 4, 24, 24!, (24!)!)$.

Conjecture 3.1. *If a system of equations $\mathcal{S} \subseteq \mathcal{B}$ has at most finitely many solutions in positive integers $x_1, \ldots, x_6$, then each such solution $(x_1, \ldots, x_6)$ satisfies $x_1, \ldots, x_6 \leqslant (24!)!$.*

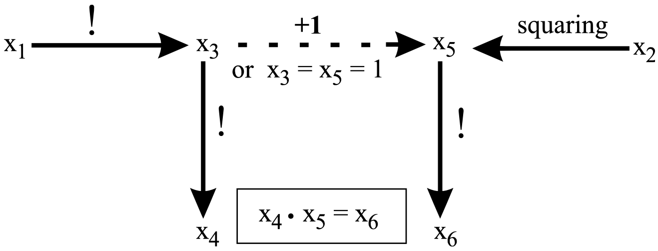
Lemma 3.1. *For every pair of positive integers x and y, the equation $x! \cdot y = y!$ holds if and only if*

$$(x + 1 = y) \vee (x = y = 1).$$

Theorem 3.1. *Conjecture 3.1 implies that if the equation $x_1! + 1 = x_2^2$ has at most finitely many solutions in positive integers x_1 and x_2, then each such solution (x_1, x_2) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.*

Proof. The following system of equations $\mathcal{B}_1$

$$\begin{cases} x_1! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_4 \cdot x_5 = x_6 \\ x_2 \cdot x_2 = x_5 \end{cases}$$



is a subsystem of $\mathcal{B}$. By Lemma 3.1, in positive integers, the system $\mathcal{B}_1$ expresses that $x_1 = \ldots = x_6 = 1$ or

$$\begin{cases} x_1! + 1 &= x_2^2 \\ x_3 &= x_1! \\ x_4 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)!\,. \end{cases}$$

If the equation $x_1! + 1 = x_2^2$ has at most finitely many solutions in positive integers x_1 and x_2, then $\mathcal{B}_1$ has at most finitely many solutions in positive integers $x_1, \ldots, x_6$ and Conjecture 3.1 implies that every tuple $(x_1, \ldots, x_6)$ of positive integers that solves $\mathcal{B}_1$ satisfies $(x_1! + 1)! = x_6 \leqslant (24!)!$. Hence, $x_1 \in \{1, \ldots, 23\}$. If $x_1 \in \{1, \ldots, 23\}$, then $x_1! + 1$ is a square only for $x_1 \in \{4, 5, 7\}$. $\square$

It is conjectured that $x! + 1$ is a square only for $x \in \{4, 5, 7\}$, see [10, p. 297]. A weak form of Szpiro's conjecture implies that the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers, see [7].

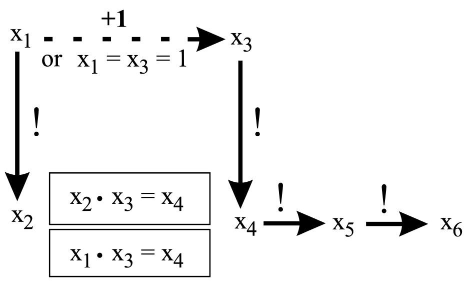
4. The Erdős' equation $x(x + 1) = y!$

Let $\mathcal{C}$ denote the following system of equations:

$$\{x_i \cdot x_j = x_k : (i, j, k \in \{1, \ldots, 6\}) \wedge (i \neq j)\} \cup \{x_j! = x_k : (j, k \in \{1, \ldots, 6\}) \wedge (j \neq k)\}.$$

The following subsystem of $\mathcal{C}$

$$\begin{cases} x_1! = x_2 \\ x_3! = x_4 \\ x_2 \cdot x_3 = x_4 \\ x_1 \cdot x_3 = x_4 \\ x_4! = x_5 \\ x_5! = x_6 \end{cases}$$



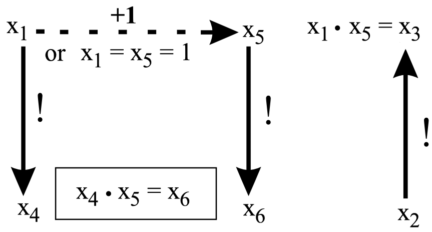
has exactly three solutions in positive integers, namely $(1,\ldots,1)$, $(1,1,2,2,2,2)$, and $(2,2,3,6,720,720!)$.

Conjecture 4.1. *If a system of equations $\mathcal{S} \subseteq \mathcal{C}$ has at most finitely many solutions in positive integers $x_1,\ldots,x_6$, then each such solution $(x_1,\ldots,x_6)$ satisfies $x_1,\ldots,x_6 \leqslant 720!$.*

Theorem 4.1. *Conjecture 4.1 implies that if the equation $x_1(x_1+1) = x_2!$ has at most finitely many solutions in positive integers x_1 and x_2, then each such solution (x_1,x_2) belongs to the set $\{(1,2),(2,3)\}$.*

Proof. The following system of equations $\mathcal{C}_1$

$$\begin{cases} x_1! = x_4 \\ x_5! = x_6 \\ x_4 \cdot x_5 = x_6 \\ x_2! = x_3 \\ x_1 \cdot x_5 = x_3 \end{cases}$$



is a subsystem of $\mathcal{C}$. By Lemma 3.1, in positive integers, the system $\mathcal{C}_1$ expresses that $x_1 = \ldots = x_6 = 1$ or

$$\begin{cases} x_1 \cdot (x_1+1) &= x_2! \\ x_3 &= x_1 \cdot (x_1+1) \\ x_4 &= x_1! \\ x_5 &= x_1+1 \\ x_6 &= (x_1+1)!\,. \end{cases}$$

If the equation $x_1(x_1+1) = x_2!$ has at most finitely many solutions in positive integers x_1 and x_2, then $\mathcal{C}_1$ has at most finitely many solutions in positive integers $x_1,\ldots,x_6$ and Conjecture 4.1 implies that every tuple $(x_1,\ldots,x_6)$ of positive integers that solves $\mathcal{C}_1$ satisfies $x_2! = x_3 \leqslant 720!$. Hence, $x_2 \in \{1,\ldots,720\}$. If $x_2 \in \{1,\ldots,720\}$, then $x_2!$ is a product of two consecutive positive integers only for $x_2 \in \{2,3\}$ because the following *MuPAD* program

```
for x2 from 1 to 720 do
x1:=round(sqrt(x2!+(1/4))-(1/2)):
if x1*(x1+1)=x2! then print(x2) end_if:
end_for:
```

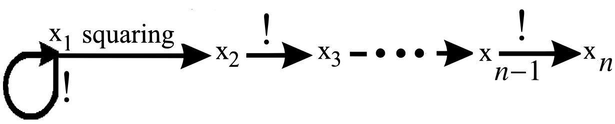
returns 2 and 3. □

The question of solving the equation $x(x+1) = y!$ was posed by Erdős, see [1]. Luca proved that the abc conjecture implies that the equation $x(x+1) = y!$ has only finitely many solutions in positive integers, see [5].

5. Conjectures 3.1 and 4.1 cannot be generalized to an arbitrary number of variables

Let $f(1) = 2$, $f(2) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geqslant 2$. Let $\mathcal{W}_1$ denote the system of equations $\{x_1! = x_1\}$. For an integer $n \geqslant 2$, let $\mathcal{W}_n$ denote the following system of equations:

$$\begin{cases} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2,\ldots,n-1\}\ x_i! = x_{i+1} \end{cases}$$



For every positive integer n, the system $\mathcal{W}_n$ has exactly two solutions in positive integers $x_1, \ldots, x_n$, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$. For a positive integer n, let Ψ_n denote the following statement: *if a system of equations*

$$\mathcal{S} \subseteq \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\} \cup \{x_j! = x_k : j, k \in \{1, \ldots, n\}\}$$

has at most finitely many solutions in positive integers $x_1, \ldots, x_n$*, then each such solution* $(x_1, \ldots, x_n)$ *satisfies* $x_1, \ldots, x_n \leqslant f(n)$. The statements Ψ_n are discussed in [9].

Theorem 5.1. *Every factorial Diophantine equation can be algorithmically transformed into an equivalent system of equations of the forms* $x_i \cdot x_j = x_k$ *and* $x_j! = x_k$*. (It means that this system of equations satisfies a modified version of Lemma 4 in [8].)*

Proof. It follows from Lemmas 2–4 of [8] and Lemma 3.1. □

For every $n \in \mathbb{N} \setminus \{0\}$, the statement Ψ_n is dubious. By Theorem 5.1, this statement implies that there is an algorithm which takes as input a factorial Diophantine equation and returns an integer which is greater than the solutions in positive integers, if these solutions form a finite set. This conclusion is strange because properties of factorial Diophantine equations are similar to properties of exponential Diophantine equations and a computable upper bound on non-negative integer solutions does not exist for exponential Diophantine equations with a finite number of solutions, see [6].

6. Equivalent forms of Conjectures 2.1–4.1

If $k \in [10^{19}, 10^{20} - 1] \cap \mathbb{N}$, then there are uniquely determined non-negative integers $a(0), \ldots, a(19) \in \{0, \ldots, 9\}$ such that

$$\Big(a(19) \geqslant 1\Big) \wedge \Big(k = a(19) \cdot 10^{19} + a(18) \cdot 10^{18} + \ldots + a(1) \cdot 10^1 + a(0) \cdot 10^0\Big).$$

Definition 6.1. *For an integer* $k \in [10^{19}, 10^{20} - 1]$*,* $\mathcal{S}_k$ *stands for the smallest system of equations* $\mathcal{S}$ *satisfying conditions* (1) *and* (2).

(1) *If* $i \in \{0, 4, 8, 16\}$ *and* $a(i)$ *is even, then the equation* $x_{a(i+1)} \cdot x_{a(i+2)} = x_{a(i+3)}$ *belongs to* $\mathcal{S}$ *when it belongs to* $\mathcal{A}$.

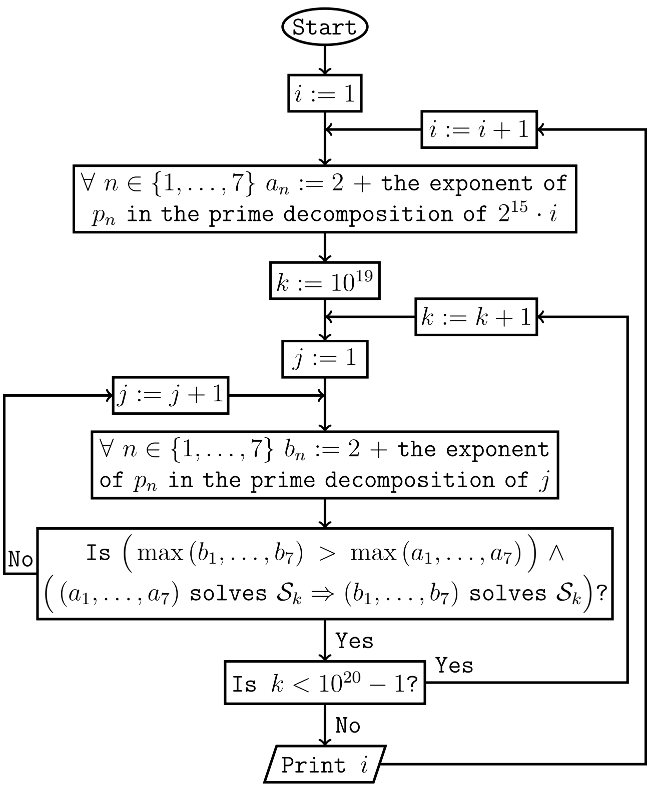
(2) *If* $i \in \{0, 4, 8, 16\}$ *and* $a(i)$ *is odd, then the equation* $2^{2^{x_{a(i+1)}}} = x_{a(i+2)}$ *belongs to* $\mathcal{S}$ *when it belongs to* $\mathcal{A}$.

Lemma 6.1. $\{\mathcal{S}_k : k \in [10^{19}, 10^{20} - 1] \cap \mathbb{N}\} = \{\mathcal{S} : (\mathcal{S} \subseteq \mathcal{A}) \wedge (\mathrm{card}(\mathcal{S}) \leqslant 5)\}$.

Proof. It follows from the equality $5 \cdot 4 = 20$. □

For a positive integer n, let p_n denote the n-th prime number.

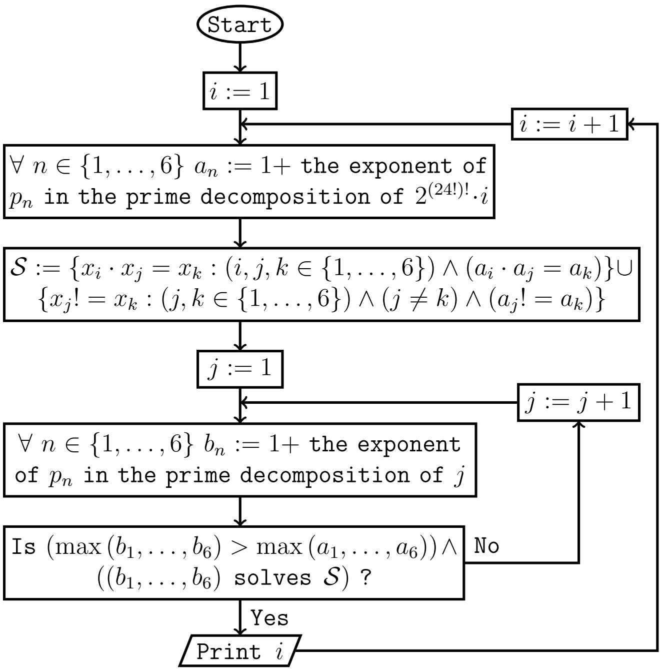
Theorem 6.1. *The following semi-algorithm* $\texttt{sem}_2$ *never terminates.*



If Conjecture 2.1 is true, then $\texttt{sem}_2$ *endlessly prints consecutive positive integers starting from* 1*. If Conjecture 2.1 is false, then* $\texttt{sem}_2$ *prints a finite number (including zero) of consecutive positive integers starting from* 1*.*

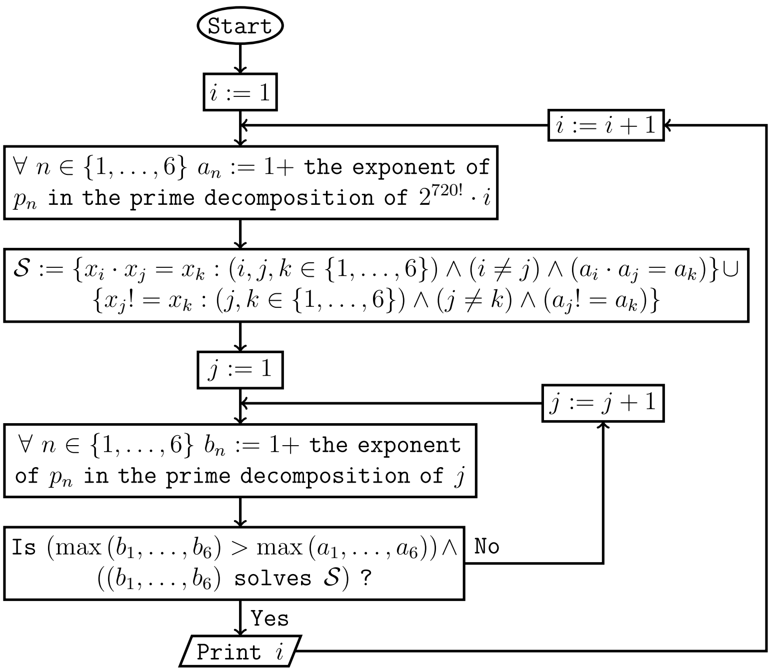
Proof. It follows from Lemma 6.1. □

Theorem 6.2. *The following semi-algorithm* $\texttt{sem}_3$ *never terminates.*



If Conjecture 3.1 is true, then $\texttt{sem}_3$ *endlessly prints consecutive positive integers starting from* 1*. If Conjecture 3.1 is false, then* $\texttt{sem}_3$ *prints a finite number (including zero) of consecutive positive integers starting from* 1*.*

Theorem 6.3. *The following semi-algorithm* $\texttt{sem}_4$ *never terminates.*



If Conjecture 4.1 is true, then $\texttt{sem}_4$ *endlessly prints consecutive positive integers starting from* 1*. If Conjecture 4.1 is false, then* $\texttt{sem}_4$ *prints a finite number (including zero) of consecutive positive integers starting from* 1*.*

References

[1] D. Berend, J. E. Harmse, On polynomial-factorial Diophantine equations, *Trans. Amer. Math. Soc.* **358** (2006) 1741–1779.

[2] N. D. Goodman, The knowing mathematician, In: H. Leblanc, E. Mendelson, A. Orenstein (Eds.), *Foundations: Logic, Language, and Mathematics,* Springer, Dordrecht, 1984, 21–38,

[3] J.-M. De Koninck, F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers,* American Mathematical Society, Providence, 2012.

[4] M. Křížek, F. Luca, L. Somer, *17 Lectures on Fermat Numbers: From Number Theory to Geometry,* Springer, New York, 2001.

[5] F. Luca, The Diophantine equation $P(x) = n!$ and a result of M. Overholt, *Glas. Mat. Ser. III* **37** (2002) 269–273.
[6] Y. V. Matiyasevich, Existence of noneffectivizable estimates in the theory of exponential Diophantine equations, *J. Sov. Math.* **8** (1977) 299–311.
[7] M. Overholt, The Diophantine equation $n! + 1 = m^2$, *Bull. London Math. Soc.* **25** (1993) 104.
[8] A. Tyszka, A hypothetical upper bound on the heights of the solutions of a Diophantine equation with a finite number of solutions, *Open Comput. Sci.* **8** (2018) 109–114.
[9] A. Tyszka, Statements and open problems on decidable sets $\mathcal{X} \subseteq \mathbb{N}$, *Pi Mu Epsilon J.* **15** (2023) 493–504.
[10] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics,* 2nd Edition, CRC, Boca Raton, 2002.