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Research Article

A new 4-chromatic edge critical Koester graph

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Abstract

Let S be a decomposition of a simple 4-regular plane graph into edge-disjoint cycles such that every two adjacent edges on a face belong to different cycles of S. Such graphs, called Grötzsch–Sachs graphs, may be considered as a result of a superposition of simple closed curves in the plane with tangencies disallowed. Koester studied the coloring of Grötzsch–Sachs graphs when all curves are circles. In 1984, he presented the first example of a 4-chromatic edge critical plane graph of order 40 formed by 7 circles. In the present paper, a new 4-chromatic edge critical graph generated by circles in the plane is presented.

Keywords: plane graph; 4-critical graph; Grötzsch–Sachs graph; Koester graph.

2020 Mathematics Subject Classification: 05C10, 05C15.

1. Introduction

A simple graph is called k-*chromatic* if its chromatic number is equal to k. A graph is *edge (vertex) 4-critical* if it is 4-chromatic and the removal of any edge (vertex) decreases its chromatic number. Numerous results and problems related to critical graphs are collected in [1, 11]. Consider a graph $G = G(S)$ formed by the superposition of a set S of simple closed curves in the plane, no two of which are tangent and no three of which meet at a point. Crossing points and arcs of S correspond to vertices and edges of G, respectively. Since every two closed curves in the plane have an even number of crossing points, G is always a 4-regular plane graph of even order. Such 4-regular graphs are called *Grötzsch–Sachs graphs*. If all curves are circles, then graphs of this class will be referred to as *Koester graphs* (see, for example, Figure 1). To describe mutual positions of curves of S, it is convenient to use the *characteristic graph* $H = H(S)$ which is the intersection graph of the curves: vertices of $H(S)$ correspond to curves of S and two vertices are adjacent if and only if the corresponding curves intersect. We also write $H = H(G)$ if G is defined by a set of curves S.

The first studies concerning the coloring of graphs generated by a set of curves in the plane are due to H. Grötzsch. H. Sachs discussed problems concerning such graphs at a number of conferences. Some results of the coloring of Grötzsch–Sachs graphs and related problems can be found in [8–10, 12–18].

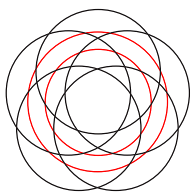

Figure 1: The 4-chromatic edge critical Koester graph K^1.

F. Jaeger proved that if $\chi(H(G)) \leq 3$, then $\chi(G) \leq 3$ [8, 9]. Various examples and infinite families of 4-chromatic Grötzsch–Sachs graphs were described in [3–6]. In particular, Grötzsch–Sachs–Koester's conjecture stating that if $\chi(H(G)) = 4$ then $\chi(G) \leq 3$, was disproved in [3]. Except for two counterexamples of order 18 to the conjecture, all the other Grötzsch–Sachs graphs with up to 18 vertices are 3-chromatic. An infinite family of vertex 4-critical graphs was described in [2, 3] and two infinite sets of edge 4-critical Grötzsch–Sachs graphs were constructed in [4, 6]. The minimal 4-chromatic Koester graph G has 20 vertices and $H(G) \cong K_5$ [12–14]. Fourteen 4-chromatic Koester graphs of order 28 and derived infinite families were presented in [7]. In 1984, G. Koester constructed the first example of an edge 4-critical Grötzsch–Sachs graph K^1 generated by a set of 7 circles in the plane (see Figure 1) [12–14]. It has 40 vertices and

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$H(K^1) \cong K_7 - e$. So far, it has been the only known example of this kind. Attempts to find similar graphs lead to the following problem [7].

Problem 1.1. *Construct new 4-chromatic edge critical Koester graphs.*

In this paper, a new edge 4-critical Koester graph K^2 of order 40 is presented.

2. Construction of graph K^2

Let K^2 be a graph generated by seven circles in the plane as shown in Figure 2a. It is a 4-regular plane graph with 40 vertices and 80 edges. Vertex numbering of K^2 is given in Figure 2b. The curve set of K^2 has the unique pair of non-crossing (red) circles, i. e. $\chi(K^2) = K_7 - e$. Graphs K^1 and K^2 are not isomorphic. Indeed, K^2 has two faces of size 7 while the maximal size of faces of K^1 is equal to 6. Because of the structure of K^2, there is an automorphism that fixes six edges in the central part of K^2 (edges (1,4), (2,3), (16,18), (26,36), (24,31), and (9,11)).

Proposition 2.1. *Koester graph K^2 is edge 4-critical.*

Proof. First, we show that K^2 is a 4-chromatic graph. Since K^2 contains triangles, $\chi(K^2) \geq 3$. By Brooks' theorem, $\chi(K^2) \leq 4$. Suppose that K^2 is a 3-chromatic graph and try to color vertices of K^2. The initial step of the coloring process is to assign colors to vertices of some pentagonal face (5-face) in all possible ways. Then we demonstrate that any extension of the initial coloring leads to the improper coloring of K^2 by three colors.

Choose face $f_5 = (33, 23, 24, 31, 30)$ for the initial coloring (see Figure 2b). We depict graph vertices by red, white, and green circles. To color vertices of any 5-face in a 3-chromatic graph, one needs exactly 3 colors. One vertex of a 5-face always has a color that is distinct from the colors of the other four vertices. This unique color will be red. The total number of 3-colorings of the initial 5-face is five. Because of the symmetry of graph K^2, it is sufficient to examine only three different colorings of 5-face.

During the 3-coloring procedure, the following useful property will be used. Let a graph G be obtained from the simple triangle by joining a new pendant vertex to two vertices of the triangle. If two pendant vertices of G have the same color in some proper 3-coloring, then the unique vertex of degree two of G must have this color.

All possible extensions of the initial colorings are presented in Figures 2cdef. The number near every vertex indicates the number of the step at which this vertex gets a forced color during the coloring procedure. The question mark is located near an edge whose vertices should receive the same color, i. e. this edge cannot be properly colored. If the color of a vertex is determined, say, in step 8 by two vertex colors obtained in steps 3 and 4, we will write $s(3, 4) \to s(8)$.

Case 1. Let vertex 33 of face f_5 be red (see Figure 2c). The following sequence of coloring steps uniquely defines vertex colors: $s(2, 3) \to s(6)$; $s(4, 5) \to s(7)$; $s(3, 4) \to s(8)$; $s(6, 8) \to s(9)$; $s(7, 8) \to s(10)$; $s(6, 9) \to s(11)$; $s(9, 10) \to s(12)$; $s(7, 10) \to s(13)$; $s(11, 12) \to s(14)$; $s(12, 13) \to s(15)$. Then end-vertices of the edge (1,4) have the same color.

Case 2. Let vertex 23 of face f_5 be red (see Figure 2d). Then $s(2, 3) \to s(6)$, $s(3, 4) \to s(7)$, $s(4, 5) \to s(8)$, $s(6, 7) \to s(9)$, $s(7, 9) \to s(10)$, $s(8, 10) \to s(11)$, $s(10, 11) \to s(12)$, $s(9, 12) \to s(13)$, $s(6, 13) \to s(14)$, $s(13, 14) \to s(15)$, $s(12, 15) \to s(16)$, $s(15, 16) \to s(17)$, $s(8, 11) \to s(18)$. As a result, edge (11,32) has end-vertices with the same color.

Case 3. Let vertex 24 of face f_5 be red (see Figure 2ef). We have $s(1, 2) \to s(6)$, $s(2, 3) \to s(7)$, $s(3, 4) \to s(8)$, $s(4, 5) \to s(9)$, $s(8, 9) \to s(10)$, $s(8, 10) \to s(11)$, $s(9, 10) \to s(12)$, $s(7, 11) \to s(13)$, $s(7, 13) \to s(14)$, $s(6, 14) \to s(15)$. Then vertex 38 can be colored in two ways in step 16.

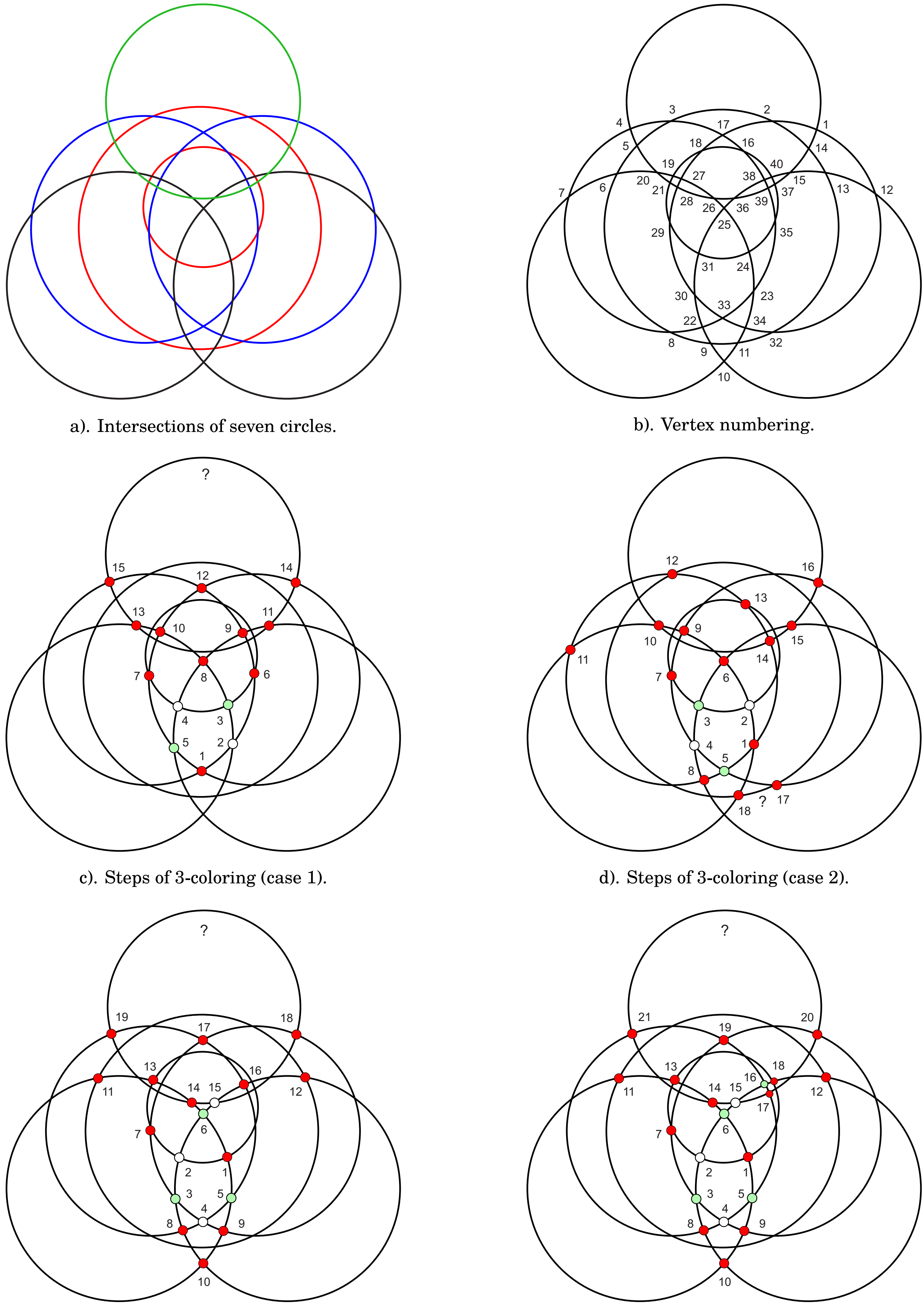
Case 3a. Assume that vertex 38 is red (see Figure 2e). We have $s(13, 16) \to s(17)$, $s(12, 17) \to s(18)$, $s(11, 17) \to s(19)$, and end-vertices of the edge (1,4) have the same color.

Case 3b. Assume that vertex 38 is green (see Figure 2f). Then $s(15, 16) \to s(17)$, $s(12, 17) \to s(18)$, $s(13, 18) \to s(19)$ $s(12, 19) \to s(20)$, $s(11, 19) \to s(21)$, and end-vertices of the edge (1,4) cannot be colored properly.

The considered cases lead to the impossibility of 3-coloring of graph K^2 and, therefore, $\chi(K^2) = 4$.

To prove that every edge of K^2 is critical, we present 3-coloring of graph $K^2 - e$ for all edges e of K^2. Because of the symmetry of K^2, it is sufficient to check 43 edges. Table 1 collects 3-colorings of $K^2 - e$ for 37 edges of the right part of K^2 and 6 edges of the central part of K^2. The vertex numbering of the graph is given in the header of Table 1. Columns of this table contain the colors of the corresponding vertices. As an illustration, 3-coloring of $K^2 - (1, 4)$ is depicted in Figure 3. □

It would be interesting to find a 4-chromatic edge critical Koester graph of order $n < 40$.



a). Intersections of seven circles.

b). Vertex numbering.

c). Steps of 3-coloring (case 1).

d). Steps of 3-coloring (case 2).

e). Steps of 3-coloring (case 3a).

f). Steps of 3-coloring (case 3b).

Figure 2: The edge 4-critical Koester graph K^2 of order 40.

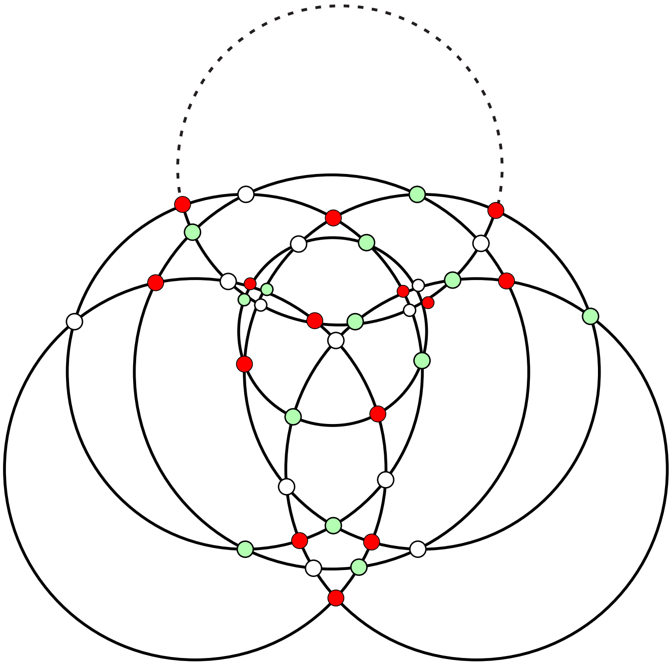


Figure 3: A 3-coloring of the graph $K^2 - (1, 4)$.

Table 1: The 3-colorings of the graph $K^2 - e$.

N	e	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	(1, 2)	1	1	3	2	1	2	1	3	1	2	3	3	2	3	1	1	2	3	2	3	1	2	1	2	1	2	1	3	2	1	3	1	3	2	3	3	2	2	1	3
2	(1, 4)	1	2	3	1	2	1	3	2	3	1	2	2	1	3	2	2	1	3	1	3	2	1	3	1	3	1	2	3	1	3	2	3	2	1	2	2	1	1	3	3
3	(1,12)	1	2	3	2	1	2	1	3	1	3	2	1	2	3	1	2	1	3	1	3	2	2	2	3	1	3	2	1	3	1	2	3	3	1	1	2	2	1	3	3
4	(1,14)	1	2	1	2	3	1	3	2	3	1	2	3	2	1	3	2	3	1	3	2	1	1	1	2	3	1	2	3	2	3	1	1	2	3	3	2	2	3	1	1
5	(2, 3)	1	2	2	3	1	3	2	1	3	1	2	2	1	3	2	2	3	1	3	2	1	2	2	1	2	1	2	3	2	1	3	3	3	1	3	3	1	1	2	3
6	(2,14)	1	2	1	2	3	1	3	2	3	2	1	3	1	2	3	2	3	1	3	2	1	1	1	2	3	1	2	3	2	3	1	2	2	3	3	2	2	3	1	1
7	(2,17)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	1	2	3	2	3	1	2	1	3	1	3	1	2	3	1	2	1	3	2	2	2	3	3	1	2
8	(9,11)	1	2	3	2	1	2	1	3	1	3	1	2	1	3	2	2	1	3	1	3	2	2	3	1	3	1	2	3	1	3	2	3	1	2	2	2	1	1	3	3
9	(10,11)	1	2	3	2	1	2	3	1	2	1	1	3	1	3	2	2	1	3	1	3	2	3	1	3	1	3	2	1	3	1	2	2	2	3	2	2	1	1	3	3
10	(10,12)	1	2	3	2	1	2	3	1	3	2	1	2	1	3	2	2	1	3	2	3	1	2	1	2	1	2	1	3	2	1	3	3	3	2	3	3	1	1	2	3
11	(11,32)	1	2	1	3	2	3	1	2	3	2	1	3	2	3	1	1	3	2	3	1	2	1	1	2	1	2	1	3	1	2	3	1	3	2	3	3	2	2	1	3
12	(11,34)	1	2	3	2	1	2	1	3	1	3	2	2	1	3	2	2	1	3	1	3	2	2	3	1	3	1	2	3	1	3	2	3	1	2	2	2	1	1	3	3
13	(12,13)	1	2	3	2	1	2	1	3	1	3	2	2	2	3	1	2	1	3	1	3	2	2	2	3	1	3	2	1	3	1	2	3	3	1	1	2	2	1	3	3
14	(12,32)	1	2	3	2	1	2	3	1	3	1	2	3	2	3	1	3	1	2	1	3	2	2	2	3	1	2	3	1	3	1	2	3	3	1	1	3	3	1	2	2
15	(13,14)	1	2	1	3	2	3	1	2	1	3	2	2	3	3	2	2	3	1	2	1	3	3	1	2	1	2	3	1	2	1	3	1	2	3	3	3	1	1	2	3
16	(13,15)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	2	2	1	3	1	3	2	2	3	1	3	1	2	3	1	3	2	1	1	2	2	2	1	1	3	3
17	(13,32)	1	2	3	2	1	2	1	3	1	2	3	3	1	3	2	2	1	3	1	3	2	2	3	1	3	1	2	3	1	3	2	1	1	2	2	2	1	1	3	3
18	(14,15)	1	2	3	2	1	2	3	1	2	1	3	3	2	3	3	3	1	2	1	3	2	3	3	2	3	2	3	1	3	2	1	1	1	2	1	1	2	2	3	1
19	(15,37)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	2	1	3	1	3	2	2	3	1	3	1	2	3	1	3	2	1	1	2	2	2	1	1	3	3
20	(15,40)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	2	1	3	1	3	2	2	3	1	3	1	2	3	1	3	2	1	1	2	2	2	3	3	1	1
21	(16,17)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	1	1	3	1	3	2	2	3	1	3	1	2	3	1	3	2	1	1	2	2	2	3	3	1	2
22	(16,18)	1	2	1	2	3	1	3	2	3	1	2	2	1	3	2	2	3	2	3	2	1	1	2	3	2	3	1	2	3	2	1	3	3	1	1	1	3	3	2	1
23	(16,38)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	2	1	3	2	3	1	2	3	2	3	2	1	3	2	3	1	1	1	2	1	1	2	2	3	3
24	(16,40)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	2	1	3	1	3	2	2	3	1	3	1	2	3	1	3	2	1	1	2	2	2	3	3	1	2
25	(23,24)	1	2	3	2	1	2	1	3	1	3	2	2	1	3	2	2	1	3	2	3	1	2	2	2	1	2	1	3	2	1	3	3	3	1	3	3	1	1	2	3
26	(23,33)	1	2	3	2	1	3	1	2	3	2	1	3	1	3	2	2	1	3	1	2	3	1	2	3	2	3	2	1	2	3	1	2	2	3	1	1	3	3	2	1
27	(23,34)	1	2	3	2	1	2	1	3	1	2	3	3	1	3	2	2	1	3	2	3	1	2	1	2	1	2	1	3	2	1	3	2	3	1	3	3	1	1	2	3
28	(23,35)	1	2	3	2	1	2	1	3	1	3	2	2	1	3	2	2	1	3	1	3	2	2	2	3	1	3	2	1	3	1	2	3	3	1	2	2	1	1	3	3
29	(24,25)	1	2	3	2	1	2	1	3	1	3	2	2	1	3	2	2	1	3	2	3	1	2	2	1	1	2	1	3	2	1	3	3	3	1	3	3	1	1	2	3
30	(24,31)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	2	1	3	1	3	2	2	3	2	3	1	2	3	1	3	2	1	1	2	1	2	2	1	3	3
31	(24,35)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	2	1	3	1	3	2	2	3	1	3	1	2	3	1	3	2	1	1	2	1	2	2	1	3	3
32	(25,36)	1	2	3	2	1	2	1	3	1	2	3	3	1	3	2	2	1	3	1	3	2	2	2	3	1	3	2	1	3	1	2	2	3	1	1	1	3	3	2	1
33	(26,36)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	2	1	3	2	3	1	2	3	2	3	2	1	3	2	3	1	1	1	2	1	2	2	1	3	3
34	(32,34)	1	2	3	2	1	2	1	3	1	2	3	3	1	3	2	2	1	3	1	3	2	2	3	1	3	1	2	3	1	3	2	2	1	2	2	2	1	1	3	3
35	(33,34)	1	2	3	2	1	2	1	3	1	2	3	3	1	3	2	2	1	3	1	3	2	2	3	1	3	1	2	3	1	3	2	2	1	1	2	2	1	1	3	3
36	(35,37)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	2	1	3	1	3	2	2	3	1	3	1	2	3	1	3	2	1	1	2	2	2	2	1	3	3
37	(35,39)	1	2	3	2	1	2	1	3	1	2	3	3	1	3	2	2	1	3	1	3	2	2	2	3	1	3	2	1	3	1	2	2	3	1	1	2	3	3	1	1
38	(36,38)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	2	1	3	2	3	1	2	3	2	3	2	1	3	2	3	1	1	1	2	1	1	2	1	3	3
39	(36,39)	1	2	3	2	1	2	1	3	1	2	3	3	1	3	2	2	1	3	1	3	2	2	2	3	1	3	2	1	3	1	2	2	3	1	1	2	3	3	2	1
40	(37,39)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	2	1	3	2	3	1	2	1	2	1	2	1	3	2	1	3	1	3	2	3	3	2	1	2	3
41	(37,40)	1	2	3	2	1	2	3	1	2	1	3	3	1	3	2	2	1	3	2	3	1	3	3	2	1	2	1	3	2	1	3	2	2	1	1	3	3	1	2	3
42	(38,39)	1	2	3	2	1	2	1	3	1	2	3	3	2	3	1	2	1	3	2	3	1	2	1	2	1	2	1	3	2	1	3	1	3	2	3	3	2	1	1	3
43	(38,40)	1	2	3	2	1	2	1	3	1	2	3	3	1	3	2	3	1	2	1	3	2	2	2	3	1	2	3	1	3	1	2	2	3	1	1	3	3	1	2	1

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