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Research Article

Variations of central limit theorems and Stirling numbers of the first kind

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Abstract

We construct a new parametrization of double sequences $\{A_{n,k}(s)\}_{n,k}$ between $A_{n,k}(0) = \binom{n-1}{k-1}$ and $A_{n,k}(1) = \frac{1}{n!}\left[{n \atop k}\right]$, where $\left[{n \atop k}\right]$ are the unsigned Stirling numbers of the first kind. For each s, we prove a central limit theorem and a local limit theorem. This extends the de Moivre–Laplace central limit theorem and Goncharov's result that unsigned Stirling numbers of the first kind are asymptotically normal. We also provide several applications.

Keywords: central limit theorem; local limit theorem; probabilistic number theory; singularity analysis.

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1. Introduction

The central limit theorem is one of the most remarkable theorems in science [11–13]. From the de Moivre–Laplace theorem in combinatorics and probability, involving binomial distributions, the central limit theorem culminated in a universal law of nature (see [26], Section 3). In this paper, we construct a parametrization of double sequences $\{A_{n,k}(s)\}_{n,k}$ between $A_{n,k}(0) = \binom{n-1}{k-1}$ and $A_{n,k}(1) = \frac{1}{n!}\left[{n \atop k}\right]$, where $\left[{n \atop k}\right]$ are the unsigned Stirling numbers of the first kind. Finally, we apply Harper's method [19] and prove a central limit theorem and a local limit theorem for each s.

Let $Z_n \in \{0, 1, \ldots, n\}$ denote a random variable with binomial distribution

$$\mathbb{P}\left(Z_n = k\right) = \binom{n}{k} p^k \left(1-p\right)^{n-k},$$

where $0 < p < 1$. The central limit theorem states that the normalized random variables Z_n, converge in distribution to the standard normal distribution $N\left(0,1\right)$:

$$\frac{Z_n - n\,p}{\sqrt{n\,p\,\left(1-p\right)}} \xrightarrow{D} N(0,1).$$

In asymptotic analysis [4, 8], one is interested in the asymptotic normality of sequences. Goncharov [15, 16] proved in 1944 that the unsigned Stirling numbers of the first kind $\left[{n \atop k}\right]$ have this property. More than 20 years later, Harper [19] discovered a more conceptional method, proving also that the Stirling numbers of the second kind are asymptotically normal. When describing Goncharov's proof, Harper writes "Goncharov ... by brute force torturously manipulates the characteristic functions of the distributions until they approach $\exp\left(-x^2/c\right)$, c a positive constant."

Recently, Harper's method has been applied by Gawronski and Neuschel [14] to Euler–Frobenius numbers (see also Kahle and Stump [25]). Note that in several cases, also other non-gaussian distributions need to be considered (e. g. limiting Betti distributions of Hilbert schemes of n points, where the Gumbel distribution was the correct limit distribution, Griffin et al. [17]).

Moreover, related to the topic, since we deal with unimodal sequences, we suggest the analysis of properties of the modes. We utilize a result by Darroch [5, 9] and study the modes of $\{A_{n,k}(s)\}$. This is connected to Erdős' proof [10] of the Hammersley conjecture [18], related to the peaks of $\{\left[{n \atop k}\right]\}$. For other sequences, we refer to Bringmann et al. [7].

We finally want to mention that the method provided in this paper contributes to the following problem in combinatorics and number theory. Let $\{P_n(z)\}_n$ denote the sequence of D'Arcais polynomials [21] defined by

$$\prod_{n=1}^{\infty} \left(1-q^n\right)^{-z} = \sum_{n=0}^{\infty} P_n\left(z\right) q^n \qquad \text{for } |q| < 1. \tag{1}$$

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Lehmer conjectured in 1947 that $P_n(-24) \neq 0$ for all $n \in \mathbb{N}$ [2,28,33]. It is expected that $P_n(-m) \neq 0$ for all even numbers m different from $2, 4, 6, 8, 10, 14, 26$ [24]. More generally, let

$$P_n^s(z) := \frac{z}{n^s} \sum_{k=1}^{n} g(k) P_{n-k}^s(z), \tag{2}$$

with initial value $P_0^s(z) = 1$. Then we obtain the D'Arcais polynomials $P_n(z)$ for $s = 1$ and $g(n) = \sum_{d|n} d$. The family $\{P_n^s(z)\}$ includes Chebyshev and Laguerre polynomials if $g(k) = k$ and $s = 0$ or $s = 1$, respectively (see Lemma 3.3 in [20] and Remark 2.8 in [23]) as $P_n^0(z) = zU_{n-1}\left(\frac{z}{2}+1\right)$ for $s = 0$ and $U_n(z)$ the nth Chebyshev polynomial of second kind and $P_n^1(z) = \frac{z}{n} L_{n-1}^{(1)}(-z)$ for $s = 1$ and $L_n^{(\alpha)}(z)$ the nth α-associated Laguerre polynomial. It is well known from the theory of orthogonal polynomials that Laguerre polynomials are more difficult to study than Chebyshev polynomials. We quote Rahmann–Schmeisser (see [32], Introduction, Page 24): "The Chebyshev polynomials are the only classical orthogonal polynomials whose zeros can be determined in explicit form". One of our goals is to analyse the properties of the solutions of the hereditary difference equation defined by Equation (2).

We have the following property in mind. If the coefficients of the polynomials do not satisfy a central limit theorem and if the associated variance goes to infinity, then the polynomials have roots in the right complex half-plane (see for example [27]).

Therefore, it would be interesting to transfer properties from polynomials related to $s = 0$ to $s = 1$. Unfortunately, the task is apparently complicated. Nevertheless, as a first attempt, already involving Stirling numbers of the first kind the case

$$P_n^s(z) = \frac{z}{n^s} \sum_{k=0}^{n-1} P_k^s(z) = (n!)^{-s} z \prod_{k=1}^{n-1} (z + k^s) \tag{3}$$

for $s \in \mathbb{R}$ and $g(k) = 1$ yields some interesting results. The equivalence of recursion and factorization follows from Example 2.4 of [23].

2. Main results

Let $P_n^s(z)$ be given by (3). We are interested in the coefficients:

$$P_n^s(z) = \sum_{k=0}^{n} A_{n,k}(s)\, z^k.$$

Let $n \geq 1$. Then $A_{n,n}(s) = (n!)^{-s}$ and $A_{n,0}(s) = 0$. We deduce from [22], example 1 and example 3 for the values $s = 0$ and $s = 1$:

$$A_{n,k}(0) = \binom{n-1}{k-1} \text{ and } A_{n,k}(1) = \frac{1}{n!} \begin{bmatrix} n \\ k \end{bmatrix}. \tag{4}$$

The unsigned Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ denote the number of all permutations of a set of n elements with exactly k distinct cycles. We refer to Bóna [6].

We mainly focus on $s \in [0, 1]$ due to (4). Then, we obtain central and local limit theorems for the double sequence $\{A_{n,k}(s)\}_{n,k}$. The case $s \in [-1, 0]$ can be reduced to the case $[0, 1]$:

$$P_n^s(z) = n^s \prod_{k=1}^{n} k^s\, z^{n+1}\, P_n^{-s}\left(z^{-1}\right).$$

Central limit theorem

The classical central limit theorem by de Moivre (1738) and Laplace (1812), was developed from the results in probability theory [13,26] to a general theorem, without direct reference to concepts as random variable, expected value, and variance. We refer to Feller [11] and Canfield [8] for excellent surveys. The modern version of the central limit theorem can also be considered as a theorem on the asymptotic normality of a sequence of non-negative numbers in singularity analysis [4]. In this spirit, we state our first result in the most general form.

Theorem 2.1. *Suppose $s \in [0, 1]$. Then there exist real sequences $\{a_n(s)\}_n$ and $\{b_n(s)\}_n$ with $b_n(s)$ positive for almost all n, such that*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{1}{P_n^s(1)} \sum_{k \leq a_n(s) + x\, b_n(s)} A_{n,k}(s) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{t^2}{2}}\, \mathrm{d}t \right| = 0. \tag{5}$$

Theorem 2.1 is proven with Harper's method [19]. The sequences $\{a_n(s)\}_n$ and $\{b_n(s)\}_n$ are provided by the expected values and variances of a suitable sequence of random variables. It is essential that $P_n^s(z)$ has real roots $-k^s \leq 0$:

$$P_n^s(z) = (n!)^{-s}\, z \prod_{k=1}^{n-1} \left(z + k^s\right). \tag{6}$$

Further, we utilize the Berry–Esseen theorem [8] to control the convergence rate. The corresponding expected values and variances are

$$\mu_n(s) := 1 + \sum_{k=1}^{n-1} \frac{1}{1+k^s}, \tag{7}$$

$$\sigma_n^2(s) := \sum_{k=1}^{n-1} \frac{k^s}{(1+k^s)^2}. \tag{8}$$

Then $\mu_n(0) = \frac{n+1}{2}$ and $\mu_n(1) = H_n$, where H_n is the nth harmonic number. Moreover, let generally $H_n^{(s)} := \sum_{k=1}^n k^{-s}$. We have $\sigma_n^2(0) = \frac{n-1}{4}$ and $\sigma_n^2(1) = H_n - H_n^{(2)}$. Most importantly, let $s \in [0,1]$. Then

$$\lim_{n\to\infty} \sigma_n(s) = \infty.$$

Therefore, we obtain:

Theorem 2.2. *Suppose $s \in [0,1]$. There exists a positive constant C such that*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{1}{P_n^s(1)} \sum_{k \leq \mu_n(s)+x\,\sigma_n(s)} A_{n,k}(s) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{t^2}{2}}\, \mathrm{d}t \right| \leq C\, \frac{1}{\sigma_n(s)}.$$

The standard deviation $\sigma_n(s)$ approaches infinity.

Remark 2.1.
a) The constant can be chosen as $C = 0.7975$ (we refer to [3], and the survey article [30]).
b) Let $s > 1$. Then $\lim_{n\to\infty} \sigma_n^2(s) \leq \lim_{n\to\infty} \sum_{k=1}^{n-1} k^{-s} = \zeta(s) < \infty$. Here we denote by $\zeta(s)$ the Riemann zeta function.

Local limit theorem

We refer to Section 4 for an introduction. We prove:

Theorem 2.3. *Let $s \in \mathbb{R}$. Then there exists a universal constant $K > 0$, such that*

$$\max_k \left| \frac{\sigma_n(s)}{P_n^s(1)} A_{n,k}(s) - \frac{\mathrm{e}^{-\frac{(x_n(s))^2}{2}}}{\sqrt{2\pi}} \right| < \frac{K}{\sigma_n(s)},$$

for $x_n(s) = (k - \mu_n(s))/\sigma_n(s)$. Uniformly for $k - \mu_n(s) = O(\sigma_n(s))$ we have

$$\frac{A_{n,k}(s)}{P_n^s(1)} \sim \frac{\mathrm{e}^{-(x_n(s))^2/2}}{\sigma_n(s)\,\sqrt{2\pi}}.$$

Peaks and plateaux

The polynomials $P_n^s(z)$ are real-rooted. Therefore, a theorem by Newton implies that the sequence $\{A_{n,k}(s)\}_k$ is unimodal and has two modes at most. Either we have one peak, or a plateau.

In the case $s = 0$, we have a peak for n odd at $k = \frac{n+1}{2}$ and a plateau for n even at $k = \frac{n}{2}$ and $\frac{n+2}{2}$. This is obvious, since the $A_{n,k}(0)$ are binomial coefficients. The case $s = 1$ is more delicate. Let $n \geq 3$. Hammersley [18] conjectured in the context of Stirling numbers of the first kind that there is always a peak. This was proved by Erdős [10]. The proof depends on the fact that $\left\{ \left[{n \atop k} \right] \right\}_k$ are natural numbers. This allows Erdős to apply special results related to the prime number theorem and certain divisibility properties of the Stirling numbers of the first kind. Our goal is to contribute to the case $s \in (0,1)$ and obtain information for $s = 1$. But this seems to be very difficult, since in general, the numbers $A_{n,k}(s)$ are not integers. Nevertheless, by utilizing a theorem by Darroch [9] we obtain:

Theorem 2.4. *Let $n \geq 6$. Assume that*

$$k_0 = 1 + \sum_{k=1}^{n-1} \frac{1}{1+k^s}$$

is an integer. Then the sequence $\{A_{n,k}(s)\}$ has a peak at k_0. The number of possible k_0 is given by the number of integers between H_n and $\frac{n+1}{2}$.

3. The probabilistic viewpoint on asymptotic normality

We begin with a useful tool from probability.

The Berry–Esseen theorem (see [8], Theorem 3.2.4)

Let X be a random variable. We denote by $\mathbb{E}(X)$ and $\mathbb{V}(X)$ the expected value and variance of X.

Theorem 3.1. *Let $X_{n,k}$ for $1 \leq k \leq n$ be independent random variables with values in $\{0,1,\ldots,n\}$. Let $\mu_{n,k}$ be the expected values, $\sigma_{n,k}^2$ the variances, and*

$$\rho_{n,k} = \mathbb{E}(|X_{n,k} - \mu_{n,k}|^3) < \infty$$

the absolute third central moments. Let $\mu_n := \sum_{k=1}^n \mu_{n,k}$, $\sigma_n^2 := \sum_{k=1}^n \sigma_{n,k}^2$, and $Z_n := \sum_{k=1}^n X_{n,k}$. Let $Z_n^* = (Z_n - \mu_n)/\sigma_n$. Then

$$\left\| \mathbb{P}\left(Z_n^* < x\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{t^2}{2}} \, \mathrm{d}t \right\|_{\mathbb{R}} \leq C \, \frac{\sum_{k=1}^n \rho_{n,k}}{\sigma_n^3},$$

where $\|f(x)\|_{\mathbb{R}}$ denotes the supremum norm of f on $\mathbb{R}$ and $C > 0$ is a universal constant. This constant can be chosen as $C = 0.7975$ [3].

Let $P_n(z) = \sum_{k=0}^n a(n,k)\, z^k$ be a monic polynomial of degree n with $a(n,k) \geq 0$. Suppose the roots of $P_n(z)$ are real and $P_n(z) = \prod_{k=1}^n (z + r_k)$. Harper [19] introduced a triangular array of Bernoulli random variables $X_{n,j}$ with distribution

$$\mathbb{P}\left(X_{n,j} = 0\right) := \frac{r_j}{1 + r_j} \text{ and } \mathbb{P}\left(X_{n,j} = 1\right) := \frac{1}{1 + r_j}.$$

Let $Z_n := \sum_{j=1}^n X_{n,j}$. Then $\mathbb{P}\left(Z_n = k\right) = \frac{a(n,k)}{P_n(1)}$. Let $X_{n,j}$ be given. Then

$$\mathbb{E}(X_{n,j}) = \frac{1}{1 + r_j}, \quad \mathbb{V}(X_{n,j}) = \frac{r_j}{(1 + r_j)^2}, \quad \mathbb{E}(|X_{n,j} - \mathbb{E}(X_{n,j})|^3) = \frac{r_j(1 + r_j^2)}{(1 + r_j)^4}.$$

This implies that

$$\mathbb{E}(|X_{n,j} - \mathbb{E}(X_{n,j})|^3) < \mathbb{V}(X_{n,j}). \tag{9}$$

We apply Harper's method setting $P_n(z) = (n!)^s \, P_n^s(z)$. The expected value of $Z_n(s)$ is given by $\mu_n(s)$ and the variance by $\sigma_n^2(s)$, as recorded in (7) and (8).

Lemma 3.1. *1. Let $s \in (0,1)$. Then*

$$\mu_n(s) \sim \frac{n^{1-s}}{1-s} \text{ and } \sigma_n^2(s) \sim \frac{n^{1-s}}{1-s}.$$

2. We obtain

$$\mu_n(0) = \frac{n+1}{2} \text{ and } \sigma_n^2(0) = \frac{n-1}{4},$$
$$\mu_n(1) \sim \ln n \text{ and } \sigma_n^2(1) \sim \ln n.$$

Proof. 1. Let $n \geq 2$. We have

$$0 \geq \frac{1}{1 + k^s} - k^{-s} = -\frac{k^{-2s}}{1 + k^{-s}} \geq -k^{-2s}.$$

Therefore,

$$1 + \sum_{k=1}^{n-1} \frac{1}{1 + k^s} = 1 + \sum_{k=1}^{n-1} k^{-s} - \frac{k^{-2s}}{1 + k^{-s}}.$$

Now, for $0 < s < 1$, by Theorem 3.2 (b) of [1], we obtain

$$\sum_{k=1}^{n-1} k^{-s} = -n^{-s} + \sum_{k=1}^{n} k^{-s} = \frac{n^{1-s}}{1-s} + \zeta(s) + O\left(n^{-s}\right)$$

using Landau's O notation. For the remainder term, we obtain similarly the bound

$$0 \leq \sum_{k=1}^{n-1} \frac{k^{-2s}}{1 + k^{-s}} \leq \sum_{k=1}^{n-1} k^{-2s} = \frac{n^{1-2s}}{1-2s} + \zeta(2s) + O\left(n^{-2s}\right).$$

Therefore, for $0 < s < 1$

$$\frac{\mu_n(s)}{\frac{n^{1-s}}{1-s}} = 1 + \frac{1 + \zeta(s) + O(n^{-s}) - O\left(\frac{n^{1-2s}}{1-2s} + \zeta(2s) + O(n^{-2s})\right)}{\frac{n^{1-s}}{1-s}} \to 1.$$

We have

$$\sigma_n^2(s) = \sum_{k=1}^{n-1} \frac{k^{-s}}{(1+k^{-s})^2} = \sum_{k=1}^{n-1} k^{-s} - \frac{2k^{-2s}+k^{-3s}}{(1+k^{-s})^2}, \quad (0 < s < 1) \quad \text{with} \quad 0 \leq \frac{2k^{-2s}+k^{-3s}}{(1+k^{-s})^2} \leq 3k^{-2s}.$$

Similarly as above we obtain

$$\sigma_n^2(s) = \sum_{k=1}^{n-1} \frac{k^{-s}}{(1+k^{-s})^2} = \frac{n^{1-s}}{1-s} + \zeta(s) + O(n^{-s}) - O\left(\frac{n^{1-2s}}{1-2s} + \zeta(2s) + O(n^{-2s})\right).$$

Therefore, $\frac{\sigma_n^2(s)}{\frac{n^{1-s}}{1-s}} \to 1$.

2. The case $s = 0$ follows from the definition. For $s = 1$, by Theorem 3.2 (a) of [1] we obtain

$$\mu_n(1) = \sum_{k=1}^{n} \frac{1}{k} = \ln(n) + \gamma + O(n^{-1})$$

where γ is the Euler–Mascheroni constant. Therefore, $\frac{\mu_n(1)}{\ln(n)} \to 1$. Similarly, we obtain

$$\sigma_n^2(1) = \sum_{k=1}^{n-1} \frac{k^{-1}}{(1+k^{-1})^2} = \sum_{k=1}^{n-1} k^{-1} - \frac{2k^{-2}+k^{-3}}{(1+k^{-1})} = \ln(n) + \gamma + O(n^{-1}) - O(n^{-1} + \zeta(2) + O(n^{-2})).$$

Therefore, $\frac{\sigma_n^2(1)}{\ln(n)} \to 1$.

□

Corollary 3.1. *Let* $s \in [0,1]$. *Then* $\lim_{n\to\infty} \sigma_n(s) = \infty$.

Remark 3.1. *Let* $0 \leq s_1 \leq s_2$. *Then* $\sigma_n(s_1) \geq \sigma_n(s_2)$, *since* $\frac{\mathrm{d}}{\mathrm{d}s}\sigma_n(s) < 0$.

Proof of Theorem 2.2

Let $s \in [0,1]$. Let $X_{n,k}(s) \in \{0,1\}$ for $1 \leq k \leq n$ be a random variable with values 0 and 1. Here

$$\mathbb{P}(X_{n,k}(s) = 1) = \frac{1}{1 + (k-1)^s}.$$

We put $Z_n(s) := \sum_{k=1}^{n} X_{n,k}(s)$. Then

$$\mathbb{P}(Z_n(s) = k) = \frac{A_{n,k}(s)}{P_n^s(1)}.$$

Thus, we obtain

$$\mathbb{E}(Z_n(s)) = \mu_n(s) = 1 + \sum_{k=1}^{n-1} \frac{1}{1+k^s} \text{ and } \mathbb{V}(Z_n(s)) = \sigma_n^2(s) = \sum_{k=1}^{n-1} \frac{k^s}{(1+k^s)^2}.$$

Corollary 3.1 states that the variance approaches infinity. This proves Theorem 2.2.

Proof of Theorem 2.1

This follows from Theorem 2.2. The crucial part of applying Harper's method and the Berry–Esseen theorem is that the variance approaches infinity.

4. Local limit theorem

A double indexed sequence $\{a(n,k)\}_{n,k}$ satisfies a local limit theorem on a set $S \subset \mathbb{R}$ provided that

$$\sup_{x \in S} \left| \frac{\sigma_n\, a(n, \lfloor \mu_n + x\, \sigma_n \rfloor)}{\sum_k a(n,k)} - \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}} \right| \longrightarrow 0$$

(cf. Canfield [8], section 3.7). We recall the following result due to Bender [4].

Theorem 4.1 (Bender). *Suppose that the* $\{a(n,k)\}_k$ *for* $n \in \mathbb{N}$ *are asymptotically normal, and* $\sigma_n^2 \to \infty$. *If for each* n *the sequence* $\{a(n,k)\}_k$ *is log-concave in* k, *then* $\{a(n,k)\}_k$ *satisfies a local limit theorem on* $S = \mathbb{R}$.

Pólya frequency sequences and limit theorems

We follow the excellent survey by Pitman [30] and apply several results to the sequences $\{A_{n,k}(s)\}_k$.

Let $(a_0, a_1, \ldots, a_n)$ be a sequence of non-negative real numbers. Let $P_n(z) := \sum_{k=0}^{n} a_k\, z^k$ be real rooted and $P_n(1) > 0$. Then the sequence is called a (finite) Pólya frequency sequence. Let $Z_n \in \{0, 1, \ldots, n\}$ be a random variable with $\mathbb{P}(Z_n = k) := \frac{a_k}{P_n(1)}$, mean μ, and variance σ^2. Then

$$\max_k \left| \mathbb{P}(0 \leq Z_n \leq k) - \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\frac{k-\mu}{\sigma}} \mathrm{e}^{-\frac{t^2}{2}}\, \mathrm{d}t \right| < \frac{0.7975}{\sigma}$$

(see [30], Formula (24) or [3]). Further, there exists a universal constant K:

$$\max_k \left| \sigma \mathbb{P}(Z_n = k) - \frac{1}{\sqrt{2\,\pi}}\, \mathrm{e}^{-\left(\frac{k-\mu}{\sigma}\right)^2/2} \right| < \frac{K}{\sigma}.$$

The bound is due to Platonov [31] (see also [30], formula (25)).

Proof of Theorem 2.3. It follows from our previous considerations that $\{A_{n,k}(s)\}_k$ is a Pólya frequency sequence for all $s \in \mathbb{R}$. This implies the theorem. $\square$

5. Peaks of $\{A_{n,k}(s)\}_k$

Recall Darroch's theorem [9]. Let $(a_0, a_1, \ldots, a_n)$ be a Pólya frequency sequence. Let $P_n(z) := \sum_{k=0}^{n} a_k z^k$ with $P_n(1) > 0$. Let $\mu_n := \frac{P_n'(1)}{P_n(1)}$. Newton's theorem implies that the sequence $\{a_k\}_k$ is unimodal and has two modes at most. Darroch proved that the modes have distance less than 1 from μ_n. Armed with the results of the previous sections we have the next proof.

Proof of Theorem 2.4. We consider the polynomials $P_n^s(z)$. Then $\mu_n(s) = 1 + \sum_{k=1}^{n-1} \frac{1}{1+k^s}$. Suppose that $\mu_n(s)$ is an integer. Then $\{A_{n,k}(s)\}_k$ has a peak. Let us restrict μ_n to $[0, 1]$. In this case

$$\mu_n : [0,1] \longrightarrow [\mu_n(1), \mu_n(0)], \quad s \mapsto \mu_n(s).$$

Since $s \mapsto \frac{1}{1+k^s}$ is strictly decreasing for $k \geq 2$, μ_n is bijective for $n \geq 3$. Let $n \geq 6$. Then $\mu_n(0) - \mu_n(1) > 1$. This implies that integers $k_0 \in \big(\mu_n(1), \mu_n(0)\big)$ exist and are realizable by suitable $s \in (0, 1)$:

$$k_0 = 1 + \sum_{k=1}^{n-1} \frac{1}{1+k^s}.$$

Let such an s be given. Then $\{A_{n,k}(s)\}_k$ has a peak at k_0. $\square$

Finally, we provide an illustration of Theorem 2.4 by Table 1. Let $M_n(1)$ be the unique mode of $\{A_{n,k}(1)\}$, as proven by Erdős.

n	1	2	3	4	5	6	7	8	9	10	100	1000
H_n	1	1.5	1.83	2.08	2.28	2.45	2.59	2.72	2.83	2.93	5.19	7.49
$M_n(1)$	–	–	2	2	2	2	2	3	3	3	5	7
$\frac{n+1}{2}$	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	50.5	500.5

Table 1: Modes for $s = 1$ and related values.

6. Applications

Approximation of Stirling numbers of the first kind

Wilf [35] contributed to the asymptotic behavior of Stirling numbers of the first kind. Several asymptotic formulas are provided. Wilf also compares his results with Jordan's formula in the case $\left[{100 \atop 5}\right]$ and presents numerical data. Although the focus of this paper is not to obtain best approximations, the numerical value we obtain is already solid. We refer to Theorem 2.3 and the approximation

$$\frac{1}{100!} \begin{bmatrix} 100 \\ 5 \end{bmatrix} \approx \frac{\mathrm{e}^{-\left(\frac{5-\mu_{100}(1)}{\sigma_{100}(1)}\right)^2/2}}{\sigma_{100}(1)\sqrt{2\pi}}. \tag{10}$$

	$\left[{100 \atop 5}\right]/99!$	Error
Exact value	$21.1204415\ldots$	-
Jordan formula	$18.740\ldots$	$\approx 11\%$
3 terms of Equation (9) (Wilf)	$21.24986\ldots$	$\approx 0.613\%$
Theorem 1 (Wilf)	$20.960\ldots$	$\approx 0.76\%$
Approximation (10) (this paper)	$21.062180\ldots$	$\approx 0.28\%$
Equation (7) to order $1/n$ (Wilf)	$21.12070\ldots$	$\approx 0.0012\%$
Equation (7) to order $1/n^2$ (Wilf)	$21.1204409\ldots$	$\approx 0.000003\%$

Table 2: Several approximations of the maximal value for $n = 100$ (see Wilf [35], page 349 for details).

We have $\mu_{100}(1) \approx 5.19$ and $\sigma_{100}(1) \approx 1.88477$. Wilf considers the value of $\left[{100 \atop 5}\right]/99!$ (Table 2).

For uniform approximation in k we refer to [34], Table 2. An asymptotic expansion in the central region is offered in [29].

One mode property of the Stirling numbers of the first kind

Let $n \geq 3$. Erdős [10] proved that $\left\{\left[{n \atop k}\right]\right\}_k$ has one mode. We give a new proof for infinitely many n. A variant of Darroch's theorem [9, 30] states: Let $\{a_k\}_k$ be a Pólya frequency sequence. Let $\mu_n = \frac{P_n'(1)}{P_n(1)}$. Then the sequence has exactly one mode k_0, if

$$k_0 \leq \mu_n < k_0 + \frac{1}{k_0+2} \quad \text{or} \quad k_0 - \frac{1}{n-k_0+2} < \mu_n \leq k_0.$$

Let the sequence $A_{n,k}(1)$ be given. Then $\mu_n(1) = H_n$. Since H_n is unbounded and $H_{n+1} = H_n + \frac{1}{n+1}$, we can find infinitely many pairs (n, k_0), such that $H_n \in [k_0, k_0 + \frac{1}{k_0+2}]$. This implies that $\{A_{n,k}(1)\}_k$ has one mode and thus, $\left\{\left[{n \atop k}\right]\right\}_k$ has one mode.

Example 6.1. *From Table 1 we see that for $n = 10$, we have $k_0 = 3$ and $3 - \frac{1}{9} < 2.9 < H_{10} = \mu_{10}(1) \leq k_0$. For $n = 30$, we have $\mu_{30} = H_{30} \approx 3.995$ and therefore, $k_0 = 4$ is the unique mode, as $4 - \frac{1}{28} < 3.97 < \mu_{30} \leq 4$. For $83 \leq n \leq 95$, we have $5.002 \approx \mu_{83} = H_{83} \leq \mu_n \leq \mu_{95} \approx 5.136$ and therefore, $k_0 = 5$ is the unique mode as $5 \leq \mu_n < 5.14 < 5 + \frac{1}{7}$.*

One can vary the argument and prove the next result.

Theorem 6.1. *Let $n \geq 6$. We consider the sequence $\{A_{n,k}(s)\}_k$ for $s \in (0,1)$. Let $k_0 = \mu_n(s_0) \in \mathbb{N}$ with $s_0 \in (0,1)$. Then there exists an $\varepsilon > 0$, such that for all $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$, the sequences $\{A_{n,k}(s)\}_k$ have exactly one mode.*

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