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Research Article

The largest eigenvalue conditions for Hamiltonian and traceable graphs

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Abstract

In this article, sufficient conditions based on the largest eigenvalue, minimum degree, and connectivity for Hamiltonian and traceable graphs are presented.

Keywords: largest eigenvalue; minimum degree; connectivity; Hamiltonian graph; traceable graph.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. For a graph G, we use n to denote its order $|V(G)|$. The minimum degree and connectivity of a graph G are denoted by $\delta(G)$ and $\kappa(G)$, respectively. A subset V_1 of the vertex set $V(G)$ of G is independent if no two vertices in V_1 are adjacent in G. A maximum independent set in a graph G is an independent set of the largest possible size. The independence number, denoted $\alpha(G)$, of a graph G is the cardinality of a maximum independent set in G. For disjoint vertex subsets X and Y of $V(G)$, we define $E(X,Y)$ as $\{\, e : e = xy \in E, x \in X, y \in Y\,\}$. A graph G is semiregular if G is bipartite and all the vertices in the same part of bipartition have the same degree. The eigenvalues of a graph G, denoted $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, are defined as the eigenvalues of its adjacency matrix $A(G)$. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path.

In 2010, Fiedler and Nikiforov [4] obtained the following spectral results for the Hamiltonicity and traceability of graphs.

Theorem 1.1. [4] *Let G be a graph of order n.*

(i). *If $\lambda_1(G) \geq n-2$, then G contains a Hamiltonian path unless $G = K_{n-1} + v$; if strict inequality holds, then G contains a Hamiltonian cycle unless $G = K_{n-1} + e$.*

(ii). *If $\lambda_1(G^c) \leq \sqrt{n-1}$, then G contains a Hamiltonian path unless $G = K_{n-1} + v$.*

(iii). *If $\lambda_1(G^c) \leq \sqrt{n-2}$, then G contains a Hamiltonian cycle unless $G = K_{n-1} + e$.*

Inspired by Theorem 1.1, several researchers obtained additional spectral results for the Hamiltonicity and traceability of graphs; some of them can be found in [1, 7–9, 11–13]. In this paper, we present new conditions based on the largest eigenvalue, minimum degree, and connectivity for the Hamiltonicity and traceability of graphs. The main results of the present paper are the next two theorems.

Theorem 1.2. *Let G be a graph of order $n \geq 3$ vertices and e edges with connectivity κ ($\kappa \geq 2$). If*

$$\lambda_1 \leq \sqrt{\frac{(\kappa+1)\delta^2}{n} + \frac{e^2}{n(n-\kappa-1)}},$$

then G is Hamiltonian or G is $K_{\kappa,\,\kappa+1}$.

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Theorem 1.3. *Let G be a graph of order $n \geq 12$ with connectivity κ ($\kappa \geq 1$). If*

$$\lambda_1 \leq \sqrt{\frac{(\kappa+2)\delta^2}{n} + \frac{e^2}{n(n-\kappa-2)}},$$

then G is traceable or G is $K_{\kappa,\,\kappa+2}$.

It is well known that a graph G is Hamiltonian if $\kappa(G) \geq 2$ and $\alpha(G) \leq \kappa(G)$ (see [3]). If $\alpha(G) \geq \kappa(G)+1$, then $e \geq \sum_{u \in I} d(u) \geq \alpha(G)\delta(G) \geq (\kappa(G)+1)\delta(G)$, where I is a maximum independent set in G. Now, we have

$$\delta\sqrt{\frac{\kappa+1}{n-\kappa-1}} \leq \sqrt{\frac{(\kappa+1)\delta^2}{n} + \frac{e^2}{n(n-\kappa-1)}}.$$

Thus, Theorem 1.2 is a generalization of the next result.

Theorem 1.4. [8] *Let G be a graph of order $n \geq 3$ with connectivity κ ($\kappa \geq 2$). If*

$$\lambda_1 \leq \delta\sqrt{\frac{\kappa+1}{n-\kappa-1}},$$

then G is Hamiltonian or G is $K_{\kappa,\,\kappa+1}$.

It is well known that a graph G is traceable if $\kappa(G) \geq 1$ and $\alpha(G) \leq \kappa(G)+1$ (see [3]). If $\alpha(G) \geq \kappa(G)+2$, then $e \geq \sum_{u \in I} d(u) \geq \alpha(G)\delta(G) \geq (\kappa(G)+2)\delta(G)$, where I is a maximum independent set in a graph G. Now, we have

$$\delta\sqrt{\frac{\kappa+2}{n-\kappa-2}} \leq \sqrt{\frac{(\kappa+2)\delta^2}{n} + \frac{e^2}{n(n-\kappa-2)}}.$$

Thus, Theorem 1.3 is a generalization of the following result.

Theorem 1.5. [8] *Let G be a graph of order $n \geq 12$ with connectivity κ ($\kappa \geq 1$). If*

$$\lambda_1 \leq \delta\sqrt{\frac{\kappa+2}{n-\kappa-2}},$$

then G is traceable or G is $K_{\kappa,\,\kappa+2}$.

2. Lemmas

We need the following results as lemmas when we prove Theorems 1.2 and 1.3.

Lemma 2.1. [10] *Let G be a balanced bipartite graph of order $2n$ with bipartition (A, B). If $d(x) + d(y) \geq n+1$ for any $x \in A$ and any $y \in B$ with $xy \notin E$, then G is Hamiltonian.*

Lemma 2.2. [5] *Let G be a graph of order n with degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\lambda_1 \geq \sqrt{\frac{d_1^2 + d_2^2 + \cdots + d_n^2}{n}}.$$

Lemma 2.3. [6] *Let G be a 2-connected bipartite graph with bipartition (A, B), where $|A| \geq |B|$. If each vertex in A has degree at least k and each vertex in B has degree at least l, then G contains a cycle of length at least $2\min(|B|, k+l-1, 2k-2)$.*

3. Proofs

Proof of Theorem 1.2

Let G be a graph satisfying the conditions in Theorem 1.2. Suppose, to the contrary, that G is not Hamiltonian. Then $n \geq 2\kappa+1$ (otherwise $\delta \geq \kappa \geq \frac{n}{2}$ and G is Hamiltonian). Since $\kappa \geq 2$, G has a cycle. Choose a longest cycle C in G and give an orientation on C. Since G is not Hamiltonian, there exists a vertex $u_0 \in V(G) - V(C)$. By Menger's theorem, we can find s ($s \geq \kappa$) pairwise disjoint (except for u_0) paths $P_1, P_2, \ldots, P_s$ between u_0 and $V(C)$. Let v_i be the end vertex of P_i on C, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of $v_1, v_2, \ldots, v_s$ agrees with the orientation of C. We use v_i^+ to denote the successor of v_i along the orientation of C, where $1 \leq i \leq s$. Since C is a longest cycle in G, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s$ and the index $s+1$ is regarded as 1. Moreover, $S := \{u_0, v_1^+, v_2^+, \ldots, v_\kappa^+\}$

is independent (otherwise G would have cycles which are longer than C). Let $u_i = v_i^+$ for each i with $1 \leq i \leq \kappa$. Set $T := V - S = \{ w_1, w_2, \ldots, w_{n-\kappa-1} \}$. Notice that

$$\sum_{u \in S} d(u) = |E(S, V-S)| \leq \sum_{w \in V-S} d(w)$$

and

$$\sum_{u \in S} d(u) + \sum_{w \in V-S} d(w) = 2e.$$

We have that

$$\sum_{u \in S} d(u) \leq e \leq \sum_{w \in V-S} d(w).$$

By the conditions of Theorem 1.2, Lemma 2.2, and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sqrt{\frac{(\kappa+1)\delta^2}{n} + \frac{e^2}{n(n-\kappa-1)}} &\geq \lambda_1 \\ &\geq \sqrt{\frac{\sum_{u \in V} d^2(u)}{n}} = \sqrt{\frac{\sum_{u \in S} d^2(u)}{n} + \frac{\sum_{w \in V-S} d^2(w)}{n}} \\ &\geq \sqrt{\frac{(\kappa+1)\delta^2}{n} + \frac{\left(\sum_{w \in V-S} d(w)\right)^2}{n(n-\kappa-1)}} \\ &\geq \sqrt{\frac{(\kappa+1)\delta^2}{n} + \frac{e^2}{n(n-\kappa-1)}}.\end{aligned}$$

Thus, all the inequalities above become equalities. Therefore,

$$d(u_0) = d(u_1) = \cdots = d(u_\kappa) = \delta, \ d(w_1) = d(w_2) = \cdots = d(w_{n-\kappa-1}) := \delta_1,$$

and

$$e = \sum_{u \in S} d(u) = \sum_{w \in V-S} d(w).$$

Since $e = \sum_{u \in S} d(u)$, there is no edge between any pair of vertices in $V - S$. Thus $V - S$ is independent. Again, notice that

$$e = \sum_{u \in S} d(u) = (\kappa+1)\delta = \sum_{w \in V-S} d(w) = (n-\kappa-1)\delta_1 \geq (n-\kappa-1)\delta,$$

we have that $n \leq 2\kappa + 2$. Since $n \geq 2\kappa + 1$, then $n = 2\kappa + 1$ or $n = 2\kappa + 2$.

When $n = 2\kappa + 1$, then $n - \kappa - 1 = \kappa$. Since $d(u_i) = \delta \geq \kappa$ for i with $0 \leq i \leq \kappa$, $u_i w_j \in E$ for each i with $0 \leq i \leq \kappa$ and for each j with $1 \leq j \leq n - \kappa - 1$. Hence G is $K_{\kappa, \kappa+1}$.

When $n = 2\kappa + 2$, then $n - \kappa - 1 = \kappa + 1$ and G is a balanced bipartite graph. By Lemma 2.1, G is Hamiltonian, a contradiction. □

Proof of Theorem 1.3

Let G be a graph satisfying the conditions in Theorem 1.3. Suppose, to the contrary, that G is not traceable. Then $n \geq 2\kappa+2$ (otherwise $\delta \geq \kappa \geq \frac{n-1}{2}$ and G is traceable). Choose a longest path P in G and give an orientation on P. Let x and y be the two end vertices of P. Since G is not traceable, there exists a vertex $u_0 \in V(G) - V(P)$. By Menger's theorem, we can find s ($s \geq \kappa$) pairwise disjoint (except for u_0) paths $P_1, P_2, \ldots, P_s$ between u_0 and $V(P)$. Let v_i be the end vertex of P_i on P, where $1 \leq i \leq s$. Without loss of generality, we assume that the appearance of $v_1, v_2, \ldots, v_s$ agrees with the orientation of P. Since P is a longest path in G, $x \neq v_i$ and $y \neq v_i$, for each i with $1 \leq i \leq s$; otherwise, G would have paths that are longer than P. We use v_i^+ to denote the successor of v_i along the orientation of P, where $1 \leq i \leq s$. Since P is a longest path in G, we have that $v_i^+ \neq v_{i+1}$, where $1 \leq i \leq s-1$. Moreover, $S := \{u_0, v_1^+, v_2^+, \ldots, v_\kappa^+, x\}$ is independent (otherwise G would have paths which are longer than P). Let $u_i = v_i^+$ for each i with $1 \leq i \leq \kappa$ and $u_{\kappa+1} = x$. Set $T := V - S = \{ w_1, w_2, \ldots, w_{n-\kappa-2} \}$. Notice again that

$$\sum_{u \in S} d(u) = |E(S, V-S)| \leq \sum_{w \in V-S} d(w)$$

and

$$\sum_{u \in S} d(u) + \sum_{w \in V-S} d(w) = 2e.$$

We have that

$$\sum_{u\in S} d(u) \le e \le \sum_{w\in V-S} d(w).$$

By the conditions of Theorem 1.3, Lemma 2.2, and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\sqrt{\frac{(\kappa+2)\delta^2}{n}+\frac{e^2}{n(n-\kappa-2)}} &\ge \lambda_1 \\
&\ge \sqrt{\frac{\sum_{u\in V} d^2(u)}{n}} = \sqrt{\frac{\sum_{u\in S} d^2(u)}{n}+\frac{\sum_{w\in V-S} d^2(w)}{n}} \\
&\ge \sqrt{\frac{(\kappa+2)\delta^2}{n}+\frac{\left(\sum_{w\in V-S} d(w)\right)^2}{n(n-\kappa-2)}} \\
&\ge \sqrt{\frac{(\kappa+2)\delta^2}{n}+\frac{e^2}{n(n-\kappa-2)}}.
\end{aligned}$$

Thus, all the inequalities above become equalities. Therefore,

$$d(u_0)=d(u_1)=\cdots=d(u_{\kappa+1})=\delta,\ d(w_1)=d(w_2)=\cdots=d(w_{n-\kappa-2}):=\delta_1,$$

and

$$e=\sum_{u\in S} d(u)=\sum_{w\in V-S} d(w).$$

Since $e=\sum_{u\in S} d(u)$, there is no edge between any pair of vertices in $V-S$. Thus, $V-S$ is independent. Again, notice that

$$e=\sum_{u\in S} d(u)=(\kappa+2)\delta=\sum_{w\in V-S} d(w)=(n-\kappa-2)\delta_1\ge(n-\kappa-2)\delta,$$

we have that $n\le 2\kappa+4$. Since $n\ge 2\kappa+2$, then $n=2\kappa+2$, $n=2\kappa+3$, or $n=2\kappa+4$.

When $n=2\kappa+2$, then $n-\kappa-2=\kappa$. Since $d(u_i)=\delta\ge\kappa$ for i with $0\le i\le\kappa+1$, $u_iw_j\in E$ for each i with $0\le i\le\kappa+1$ and for each j with $1\le j\le n-\kappa-2$. Hence G is $K_{\kappa,\,\kappa+2}$.

When $n=2\kappa+3$, then $n-\kappa-2=\kappa+1$. Notice that $\kappa\ge 5$ since $n=2\kappa+3\ge 12$. Notice further that each vertex in S or T has degree at least $\delta\ge\kappa$. By Lemma 2.3, G has a cycle of length $2\kappa+2$. Since $n=2\kappa+3$ and $\kappa\ge 5$, G has a path containing all the vertices of G. Namely, G is traceable, a contradiction.

When $n=2\kappa+4$, then $n-\kappa-2=\kappa+2$. Notice that $\kappa\ge 4$ since $n=2\kappa+4\ge 12$. Notice further that each vertex in S or T has degree at least $\delta\ge\kappa$. By Lemma 2.3, G has a cycle of length $2\kappa+4$, which implies that G is traceable, a contradiction. □

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