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Research Article

On Boolean functions defined on bracket sequences

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Abstract

In the paper [B. Bakos, N. Hegyvári, M. Pálfy, X. H. Yan, *Discrete Math. Lett.* **4** (2020) 31–36], the authors introduced the so-called pseudo-recursive sequences which generalize bracket sequences. In the present article, Boolean functions are defined on hypergraphs with edges having big intersections induced by bracket sequences and hypergraphs that are thinly intersecting. These Boolean functions related to combinatorial number theory are new in this area.

Keywords: bracket sequences; additive combinatorics; Boolean cube; basic Fourier analysis.

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1. Introduction

Many problems in combinatorial number theory are concerned with the representation of positive integers as the sum of elements from a given set. Many such problems require that the number of members of the representation is bounded; for example, Lagrange's four-square theorem; however, there are many that do not. The motivation for this article is the case when there is no such a requirement.

Let $X \subseteq \mathbb{N}$. The set of subset sums of X is defined in the following way:

$$P(X) := \Big\{ \sum_{x\in Y} x : Y \subseteq X; |Y| < \infty \Big\}. \tag{1}$$

For the empty set, let $\sum_{x\in\emptyset} x = 0$. The set X is said to be complete if all sufficiently large integers belong to $P(X)$. Birch, Erdős, Roth, and Szekeres are some of the prominent names in connection with this subject (see for example [3,4,6]).

A challenging problem is when X is a bracket sequence. Answering a question of Erdős, Graham proved that: $S(t,\alpha) = \{t\lfloor\alpha^n\rfloor\}_{n=1}^{\infty}$ is complete if $0 < t < 1$ and $1 < \alpha \le \sqrt[3]{5}$. Furthermore, Erdős and Graham conjectured that $S(t,\alpha)$ is complete for $1 < \alpha < \varrho$, the golden number.

Another challenging problem is when $X = A_{\alpha\beta} := \{\lfloor 2^n\alpha\rfloor\}_{n=1}^{\infty} \cup \{\lfloor 2^m\beta\rfloor\}_{m=1}^{\infty}$. Erdős and Graham conjectured in [3] that $A_{\alpha\beta}$ is complete provided α/β is irrational. I made some progress towards this conjecture (see e.g. in [5]) and I think it is sufficient to assume that $\alpha/\beta \neq 2^k;\ k \in \mathbb{Z}$. The bracket sequences have a pseudo-recursive definition as well. These sequences were used for certain cryptographic and combinatorial problems (see the details in [1,2]).

When X is a finite set one can write (1) in the form $P(X) = \{\sum_{i=1}^{n} \varepsilon_i x_i :\ \varepsilon_i \in \{0,1\}\}$, where $X = \{x_i\}_{i=1}^{n}$. In this case, $P(X)$ can be considered as an image set of a function $f_X : (\varepsilon_1, \varepsilon_2, \dots \varepsilon_n)$ which maps from the cube $\{0,1\}^n$ to $\mathbb{R}$.

A Boolean function B can have several definitions; sometime $B : \{0,1\}^n \to \{0,1\}$ and sometime $B : \{0,1\}^n \to \mathbb{R}$. Throughout the paper, both of these definitions are used. In either case, it will be clarified which definition is being used. The function $f_X : (\varepsilon_1, \varepsilon_2, \dots \varepsilon_n) \mapsto \varepsilon_1 x_1 + \varepsilon_2 x_2 + \dots \varepsilon_n x_n \pmod 2$ is a Boolean function in the first meaning.

The goal of this article is to combine bracket sequences ($A_\alpha := \{\lfloor 2^n\alpha\rfloor\}$ and $C_\alpha := \{\lfloor 3^n\alpha\rfloor\}$) with special Boolean functions. The sequence A_α (and C_α too) looks like a "pseudo-random" sequence (indeed α is a random number, 2^n and 3^n are regular). These Boolean functions related to combinatorial number theory are new in this area.

We now explain our functions using *hypergraphs* which will define on the vertex set $[n] := \{1, 2, \dots, n\}$ and the elements of any edge correspond to a subset of variables. Formally $H = (V, E)$ is a hypergraph, where V is a set of vertices and E is a set of non-empty subsets of V called hyperedges (or shortly edges). A hypergraph H is said to be k-uniform if all edges contain exactly k vertices. We concentrate on some hypergraphs in which there are many pairs of edges with "large" intersections (Section 2), and also an opposite situation when the "total intersection" is small (Section 3).

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2. Highly intersecting hypergarphs

In this section, we consider a hypergraph in which there are many pairs of edges with "large" intersections. More precisely our graph is a k-uniform cycle hypergraph, with $k-1$ many common elements in the connected edges, i.e. $H=(V,E)$, $V=\{x_1,x_2,\dots,x_n\}$ and $E=\{\{x_1,x_2,\dots,x_k\},\{x_2,x_3,\dots,x_{k+1}\},\dots,\{x_n,x_1,\dots,x_{k+1}\}\}$. So, let

$$F_\alpha(x_1,x_2,\dots,x_n)=a_1(x_1+x_2\cdots+x_k)+a_2(x_2+x_3\cdots+x_{k+1})+\dots \tag{2}$$

$$\dots+a_{n-k+1}(x_{n-k+1}+x_{n-k+2}\cdots+x_n)+\dots+a_n(x_n+x_1\cdots+x_{k-1}) \pmod 2.$$

where $x=(x_1,x_2,\dots,x_n)\in\{0,1\}^n$, $a_i\in A_\alpha$. Here k is fixed, n does not depend on k, but is also fixed, and α varies. For a chosen α denote by $\mathcal{F}(\alpha)$ the set of these functions.

It is important to remark that F_α is a polynomial form of the function. F_α has a Fourier representation as well, where $B_\alpha(x_1,x_2,\dots,x_n)=\sum_{S\subseteq[n]}\widehat{F_\alpha}(S)\chi_S(x)$, and we mean that for every $(x_1,x_2,\dots,x_n)\in\{0,1\}^n$, $B_\alpha(x_1,x_2,\dots,x_n)=F_\alpha(x_1,x_2,\dots,x_n)$ (see the details in Section 5).

Take now the Fourier expansion of F_α. An interesting question is to bound the number of terms in the representation. More precisely we define the following

Definition 2.1. *Let* $\mathcal{M}=\{S\subseteq[n]:\widehat{F}_\alpha(S)\neq 0\}$*, i.e.* $|\mathcal{M}|$ *is the number of terms of the Fourier representation of* F_α*.*

We will prove

Theorem 2.1. *Drawn* α *uniformly at random from* $[0,1]$*. Then* $|\mathcal{M}|>r$ *holds with probability at least*

$$1-\frac{e-1}{2^{n-(k-1)r-r\log n}}.$$

For example when $k\sim\log n$ and we ask the chance that $|\mathcal{M}|>\varepsilon\frac{n}{\log n}$ then the probability of it is at least $1-\frac{c}{2^{(1-2\varepsilon)n}}$.

A related multiplicative function would be $T_\alpha(x):=sign\{G_\alpha\}$, where $G_\alpha:\{-1,1\}^n\to\mathbb{R}$ defined by

$$G_\alpha(x_1,x_2,\dots,x_n)=a_1x_1x_2\cdots x_k+a_2x_2x_3\cdots x_{k+1}+\dots+a_{n-k+1}x_{n-k+1}x_{n-k+2}\cdots x_n+$$

$$+a_{n-k+2}x_{n-k+2}x_{n-k+2}\cdots x_{n+1}+\dots+a_nx_nx_1\dots x_{k-1},$$

and $sign\{x\}=1$ if $x>0$ and $sign\{x\}=-1$ otherwise. The function T_α called *threshold* function, namely we examine when T_α takes a positive value. We will show that T_α depends only on k variable (which in computer science is sometimes said to be *junta*).

Proposition 2.1. *The sign function* T_α *depends only on the variables* $x_1,\dots,x_{k-1},x_n$*.*

3. Thinly intersecting hypergarphs

Our next function will be the opposite of the previous functions; we will assume that the total amount of intersections is "small".

Definition 3.1 (ε–thin system of sets). *The system of sets* $\mathcal{S}=(S_1,S_2,\dots,S_r);S_i\subseteq[n]$*, is said to be* ε*–thin system if* $\sum_{1\le i<j\le r}|S_i\cap S_j|<\varepsilon r$*.*

In Section 4, we are going to investigate the cardinality of the output domain of the function $H:\{0,1\}^n\to\mathbb{R}$: $H(x_1,x_2,\dots,x_n):=\sum_{i=1}^r c_i\prod_{j\in S_i}(-1)^{x_j}$, where $\{c_i\}_{i=1}^r$; $(r\le n)$ is a bracket sequence, $c_k:=\lfloor 3^k\alpha\rfloor:\alpha\in\mathbb{R}^+$ and $\mathcal{S}=(S_1,S_2,\dots,S_r)$ forms an ε–thin system. Let us denote by $Im(H)$ the image set of H, i.e.

$$Im(H):=\Big\{y\in R:\exists\,(x_1,x_2,\dots,x_n)\in\{0,1\}^n;\ y=H(x_1,x_2,\dots,x_n)\Big\}.$$

Clearly $|Im(H)|\le 2^r$. Note that the equality does not necessarily hold. Let e.g. $n=2$ and let

$$H(x_1,x_2):=c_1(-1)^{x_1}+c_2(-1)^{x_1}(-1)^{x_2}+c_3(-1)^{x_2}.$$

It is easy to check that $c_1+c_2-c_3$ is not in $Im(H)$. Nevertheless, we prove the next result.

Theorem 3.1. *Let* $H(x_1,x_2,\dots,x_n):=\sum_{i=1}^r c_i\prod_{j\in S_i}(-1)^{x_j}$ *where* $\{S_i\}_{i=1}^r$ *is an* ε*-thin system. Then* $|Im(H)|\ge 2^{(1-\varepsilon)r}$*.*

4. Preliminaries and notations

The set $\{1, 2, \dots n\}$ will be denoted by $[n]$. For $S \subseteq [n]$ the corresponding input is $x = (x_1, x_2, \dots, x_n) \in \{0, 1\}^n$; where $x_i = 1$ if $i \in S$ and $x_i = 0$ otherwise. A basis function or character is defined by $\chi_x(y) := (-1)^{\langle x,y\rangle_2}$, where $\langle x, y\rangle_2 := \sum_{i=1}^n x_i y_i$. Sometimes we write $\langle S, y\rangle_2$ if x is the corresponding input of S. Let f, g be two Boolean functions. The expected value of f is $\mathbb{E}(f) := \frac{1}{2^n}\sum_{x\in\{0,1\}^n} f(x)$. and the inner product of f and g is $\langle f, g\rangle := \mathbb{E}(fg)$. For a set $S \subseteq [n]$ the Fourier transform of f is $\widehat{f}(S) = \langle f, \chi_S\rangle$. Every Boolean function has a unique Fourier expansion in the form $f = \sum_{S\subseteq[n]} \widehat{f}(S)\chi_S$.

Let us denote by$Pr_{x\in\{0,1\}^n}(\cdot)$ the uniform probability distribution on the discrete $n-$cube. The influence $Inf_i(f)$ of the i^{th} variable on a Boolean function f is the probability that when we flip the value of the i^{th} variable the value of f is flipped as well. More formally $Inf_i(g) = Pr_{x\in\{0,1\}^n}[g(x) \neq g(x + e_i)]$, where $e_i = (0, \dots, 1, \dots, 0)$; the i^{th} coordinate is 1 the other coordinates are 0. In the next lemma we show that the bracket sequence fulfills the *pseudo-recursive* condition (see also [1]).

Lemma 4.1. *Let $\alpha \in \mathbb{R}$, $\alpha > 1$ and write $a_n = \lfloor 2^n\alpha\rfloor$. Then the recursion $a_{n+1} = 2a_n + \alpha_n$ holds, where the binary representation of α is $\alpha = 1.\alpha_1\alpha_2 \dots \alpha_n \dots$. We assume that are infinitely many digits equal to 1.*

Proof. Since infinitely many digits equal to 1, the representation is unique. Then $a_n = 1\alpha_1\alpha_2 \dots \alpha_n = \lfloor 2^n\alpha\rfloor$ and $a_{n+1} = 1\alpha_1\alpha_2 \dots \alpha_{n+1} = \lfloor 2^{n+1}\alpha\rfloor$ in base 2. Hence clearly $a_{n+1} = 2a_n + \alpha_n$, ($\alpha_n \in \{0, 1\}$) holds. □

It is easy to see that one can rearrange F_α in the linear form and hence

$$F_\alpha = \sum_{i=1}^n (a_{i-k+1} + \dots a_i)x_i \equiv \sum_{i=1}^n (\alpha_{i-k+1} + \dots \alpha_i)x_i \pmod 2.$$

This form of the function can be considered as a dual form of the previous. In this version, α digits are considered as a k-uniform cyclic hypergraph.

5. Proofs

First, we prove Proposition 2.1.

Proof of Proposition 2.1. Our task is to characterize those variables $\{x_1, x_2, \dots, x_n\} \in \{-1, 1\}^n$, for which

$$G_\alpha(x) = \sum_{i=1}^n a_i(x_i x_{i+1} \cdots x_{i+k-1}) > 0.$$

This inequality is equivalent to

$$\sum_{i=1}^n a_i\Big(\frac{1 + x_i x_{i+1} \cdots x_{i+k-1}}{2}\Big) > \frac{1}{2}\sum_{i=1}^n a_i.$$

For every i ($1 \le i \le n$), let us introduce the variable

$$\varepsilon_i = \frac{1 + x_i x_{i+1} \cdots x_{i+k-1}}{2}.$$

Observe that $\varepsilon_i \in \{0, 1\}$ for every $1 \le i \le n$. Hence we are looking for the n-tuples $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\} \in \{0, 1\}^n$ for which $\sum_{i=1}^n \varepsilon_i a_i > \frac{1}{2}\sum_{i=1}^n a_i$ and we have to check that there is a realization of an n-tuples in variables $\{x_1, x_2, \dots, x_n\} \in \{-1, 1\}^n$. From Lemma 4.1 one can easily prove the following lemma.

Lemma 5.1. *For every $n \ge 2$ $P(\{a_1, a_2, \dots, a_n\}) = P(\{a_1, a_2, \dots, a_{n-1}\}) + \{0, a_n\}$ and $a_n > \sum_{j=1}^{n-1} a_j$.*

It implies that $a_n > \frac{1}{2}\sum_{j=1}^n a_j$. If $\varepsilon_n = 1$ then we have $\sum_{i=1}^n \varepsilon_i a_i \ge a_n > \frac{1}{2}\sum_{i=1}^n a_i$. Furthermore if $\varepsilon_n = 0$, then

$$\sum_{i=1}^n \varepsilon_i a_i \le \sum_{i=1}^{n-1} a_i < \frac{1}{2}\sum_{i=1}^n a_i.$$

So, we get G_α is positive if and only if $\varepsilon_n = 1$ on other words T_α depends only on $x_1, \dots, x_{k-1}, x_n$. □

To prove Theorem 2.1 we turn the polynomial form into the Fourier expansion form of F_α. Then we have the next result.

Proposition 5.1. *Let $1 \le i \le n$. Then for every S, $i \in S$, $\widehat{F_\alpha}(S) = 0$ holds, if and only if $\sum_{j=i-k+1}^{i} \alpha_j \equiv 0 \pmod 2$. We mean that $\alpha_s = \alpha_t$ when $s \equiv t \pmod n$.*

We derive Proposition 5.1 from the next lemma.

Lemma 5.2. *Let* $\eta_i \in \{0,1\}$. *Then* $Inf_i(F_\alpha) = \eta_i$ *if and only if* $\sum_{j=i-k+1}^{i} \alpha_j \equiv \eta_i \pmod 2$, *where*

$$Inf_i(g) = Pr_{x\in\{0,1\}^n}[g(x) \neq g(x+e_i)] \quad (e_i \textit{ is the } i^{th} \textit{ basis vector}).$$

Now, the implication follows from the following result.

Lemma 5.3. *For every Boolean function* $f : \{0,1\}^n \to \{0,1\}$ *we have* $Inf_i(f) = 4\sum_{S\subseteq[n]:i\in S} \widehat{f}^2(S)$.

Lemma 5.3 implies that if the i^{th} influence is 0 then every Fourier coefficient $\widehat{f}(S)$ is zero containing the element i. The proof of Lemma 5.3 can be found e.g. in [7]. For the sake of completeness, we include a short proof.

Proof. The value of $|f(x) - f((x+e_i)|$ is zero or one, hence $Inf_i(f) = \mathbb{E}_x[(f(x) - f((x+e_i))^2]$. Furthermore by the Fourier representation of $f(x)$ and $f(x+e_i)$ we have $|f(x) - f(x+e_i)| = 2\sum_{i\in S} \widehat{f}(S)(-1)^{\langle S,x\rangle}$. (If $i \notin S$, a term in $f(x)$ cancels the term in $f(x+e_i)$, otherwise it will be doubled). Thus we have

$$Inf_i(f) = \mathbb{E}_x[f(x) - f((x+e_i))^2] = \langle f(x) - f(x+e_i), f(x) - f(x+e_i)\rangle = 4\left\langle \sum_{i\in S} \widehat{f}(S)\chi_S, \sum_{i\in T} \widehat{f}(T)\chi_T \right\rangle = 4\sum_{i\in S} |\widehat{f}(S)|^2.$$

$\square$

Proof of Lemma 5.2. Recall that our task is to show that $Inf_i(F_\alpha) = \eta_i$ if and only if $\sum_{j=i-k+1}^{i} \alpha_j \equiv \eta_i \pmod 2$, where $\eta_i \in \{0,1\}$. Now let $x \in \{0,1\}^n$ and flip its variable x_i to the opposite. Take F_α in the form $F_\alpha = \sum_{i=1}^{n}(a_{i-k+1} + \dots a_i)x_i$. So, F_α changes its value if and only if $\sum_{j=i-k+1}^{i} \alpha_j \equiv 1 \pmod 2$. Hence the statement holds. $\square$

Proof of Theorem 2.1. Since F_α depends only on finite many digits of α, we estimate a discrete probability (i.e. the desired event is a union of finite many subintervals of $[0,1]$). In our model write the n many digits in a circle. We call the consecutive digits in the circle to block, i.e. the sequence of digits $\{\alpha_{i-k+1}, \dots \alpha_i\}$ is a block. Let us denote by X the event, that the number of the blocks where $\sum_{j=i-k+1}^{i} \alpha_j \equiv 1 \pmod 2$ is at most r. Furthermore write $X = \cup_{t=1}^{r} X_t$, where X_t denotes the event that the number of such blocks is exactly t. We estimate $Pr(X_t)$. So we have t many blocks which can be identified at their first digits α_{i-k+1}. Hence we can select at most $\binom{n}{t}$ many blocks. There are 2^{k-1} cases when $\sum_{j=i-k+1}^{i} \alpha_j \equiv 1 \pmod 2$.

Call a block *1-block* if $\sum_{j=i-k+1}^{i} \alpha_j \equiv 1 \pmod 2$, and *0-block* if $\sum_{j=i-k+1}^{i} \alpha_j \equiv 0 \pmod 2$. Now if $\{\alpha_{i-k+1}, \dots \alpha_i\}$ is a 1-block, and the consecutive block $\{\alpha_{i-k+2}, \dots \alpha_{i+1}\}$ is 0-block (or a 1-block), then α_{i+1} is the opposite of α_{i-k+1} (or the same) and carry on like so. If $\{\alpha_{i-k+3}, \dots \alpha_{i+2}\}$ is an ε-block ($\varepsilon \in \{0,1\}$) then α_{i+2} is the same or the opposite as α_{i-k+2} depending on ε; i.e. the digits outside of the blocks are determined uniquely. Hence

$$Pr(X) \le \sum_{t=1}^{r} Pr(X_t) \le \sum_{t=1}^{r} \frac{\binom{n}{t}(2^{k-1})^t}{2^n} \le \sum_{t=1}^{r} \frac{(n2^{k-1})^t}{t!2^n} \le \frac{(n2^{k-1})^r}{2^n}\sum_{t=1}^{r}\frac{1}{t!} < \frac{e-1}{2^{n-(k-1)r-r\log n}}.$$

$\square$

Functions associated to ε–thin sets

The aim of this subsection is to give an estimation to the cardinality of the image set of H. Recall that we defined H in the form

$$H(x_1, x_2, \dots, x_n) := \sum_{i=1}^{r} c_i \prod_{j\in S_i} (-1)^{x_j},$$

where $\{c_i\}_{i=1}^{r} = \{\lfloor 3^i\alpha\rfloor\}_{i=1}^{r}$, $\alpha \ge 1$. First, note that all sums in the form $\sum_{i=1}^{r} \varepsilon_i c_i$; $\varepsilon_i \in \{-1,1\}$ are pairwise distinct. Indeed if $\sum_{i=1}^{r} \varepsilon_i c_i = \sum_{i=1}^{n} \varepsilon'_i c_i$ then rearranging it we obtain that $\sum_{i=1}^{r} \eta_i c_i = \sum_{i=1}^{n} \eta'_i c_i$; $\eta_i, \eta'_i \in \{0,1,2\}$. It remains to show the following lemma.

Lemma 5.4. *Let* $k \in \mathbb{N}$. *For every* $y = \sum_{i=1}^{k} \eta_i c_i$; $\eta_i \in \{0,1,2\}$ *has a unique representation.*

Proof. First observe that for every k, $2\sum_{i=1}^{k} c_i < c_{k+1}$. Indeed, since $\alpha > 1$

$$2\sum_{i=1}^{k} c_i < 2\sum_{i=1}^{k} \alpha 3^i = \alpha(3^{k+1} - 1) < \alpha 3^{k+1} - 1 < \lfloor 3^{k+1}\alpha\rfloor = c_{k+1}.$$

We prove the lemma by induction on k. For $k = 1$ it is obvious. Assume now that for $k \geq 1$ the statement is true. So let $y = \sum_{i=1}^{k+1} \eta_i c_i = \sum_{i=1}^{k+1} \eta'_i c_i;\ , \eta_i, , \eta'_i \in \{0, 1, 2\}$. If $\eta_{k+1} = \eta'_{k+1}$ then by our hypothesis for every $i = 1, 2, \dots k$ $\eta_i = \eta'_i$. Let $\eta_{k+1} > \eta'_{k+1}$.

$$(\eta_{k+1} - \eta'_{k+1})c_{k+1} + \sum_{i=1}^{k} \eta_i c_i > c_{k+1} > 2\sum_{i=1}^{k} c_i \geq \sum_{i=1}^{k} \eta'_i c_i$$

which implies $\sum_{i=1}^{k+1} \eta'_i c_i < \sum_{i=1}^{k+1} \eta_i c_i$. ☐

Since the sums $\sum_{i=1}^{r} \varepsilon_i c_i;\ \varepsilon_i \in \{-1, 1\}$ are pairwise distinct, there is a one-to one map between $Im(H)$ and a subset V of $\{-1, 1\}^r$ and we note that $|V| = |Im(H)| \leq 2^r$. We say that $x = (x_1, x_2, \dots, x_n)$ corresponds to $v_x = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$, if $H(x_1, x_2, \dots, x_n) = \sum_{i=1}^{r} \varepsilon_i c_i$. Let

$$R(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r) = \left| \{(x_1, x_2, \dots, x_n) \in \{0, 1\}^n :\ H(x_1, x_2, \dots, x_n) = \sum_{i=1}^{r} \varepsilon_i c_i\} \right|,$$

i.e. this representation function counts the number of points x in the Boolean cube which correspond to the vector $v = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$. So we have

$$2^n = \sum_{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r) \in V} R(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r) \leq |V| \max_{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r) \in V} R(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r). \tag{3}$$

Lemma 5.5. *For all* $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r) \in V$, $R(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r) \leq 2^{n-r+\varepsilon r}$.

Now Lemma 5.5 and (3) implies Theorem 3.1.

Proof of Lemma 5.5. Write $\Delta := \cup_{i \neq j} S_i \cap S_j = \{i_1, i_2, \dots, i_{|\Delta|}\}$, and $S'_i = S_i \setminus \Delta$, $i = 1, 2, \dots, r$. Let us fix the element $(x_{i_1}, x_{i_2}, \dots, x_{i_{|\Delta|}}) \in \{0, 1\}^{|\Delta|}$. The number of $x = (x_1, x_2, \dots, x_n)$ in which $x_{i_j},\ 1 \leq j \leq |\Delta|$, are fixed and $\prod_{j \in S_i} (-1)^{x_j} = \varepsilon_i$, is $2^{|S"_i|-1}$. There are $2^{|\Delta|}$ 0–1 sequences $(x_{i_1}, x_{i_2}, \dots, x_{i_{|\Delta|}})$. Thus

$$R(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r) \leq 2^{|\Delta|} \prod_{i=1}^{r} 2^{|S'_i|-1} \leq 2^{|\Delta|} 2^{n-r} < 2^{n-r+\varepsilon r}.$$

☐

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