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Research Article

The edge partition dimension of graphs

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Abstract

The edge metric dimension was introduced in 2018 and since then, it has been extensively studied. In this paper, we present a different way to obtain resolving structures in graphs in order to gain more insight into the study of edge resolving sets and resolving partitions. We define the edge partition dimension of a connected graph and bound it for graphs of given order and for graphs with given maximum degree. We obtain exact values of the edge partition dimension for multipartite graphs. Some relations between the edge partition dimension and partition dimension/edge metric dimension are also presented. Moreover, several open problems for further research are stated.

Keywords: edge resolving partition; edge partition dimension; edge metric dimension; partition dimension.

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1. Introduction

Let G be a connected graph with *vertex set* $V(G)$ and *edge set* $E(G)$. The number of vertices of G is called the *order*. The *degree* of a vertex $v \in V(G)$ is the number of vertices adjacent to v. The *maximum degree* of G is the degree of a vertex which has the largest degree in G. We denote the *path, cycle* and *complete graph* with n vertices by P_n, C_n and K_n, respectively.

The *distance* $d_G(u, v)$ between two vertices u and v is the number of edges in a shortest path between u and v in G. A vertex $v \in V(G)$ is said to *distinguish* two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. A set $S \subset V(G)$ is a *resolving set* for G if any pair of vertices of G is distinguished by some element of S. A resolving set of minimum cardinality is called a *metric basis*, and its cardinality is the *metric dimension* of G, denoted by $\dim(G)$. Resolving sets were defined separately in [8] (where resolving sets were called locating sets), and in [3] (with the terminology of this article). The terminology of metric generators is also used in some works, and this was first introduced in [7]. The recent survey [9] contains a fairly complete compendium on the topic of metric dimension in graphs. The metric dimension was also considered in [6].

In concordance with the resolving sets for graphs, the concept of resolving partitions was presented in [2] and studied in several other further investigations. The survey [5] contains the most interesting contributions and open problems on this parameter. For any vertex $v \in V(G)$ and any set $W \subset V(G)$, the *distance* between v and W is defined as $d_G(v, W) = \min\{d_G(v, w) : w \in W\}$. A set $W \subset V(G)$ *distinguishes* two different vertices $u, v \in V(G)$ if $d_G(u, W) \neq d_G(v, W)$. An ordered vertex partition $\Pi = \{U_1, U_2, \ldots, U_k\}$ of a graph G is a *resolving partition* for G if every two different vertices of G are distinguished by some set of Π. The cardinality of a smallest resolving partition for G is the *partition dimension* of G, which is denoted by $\mathrm{pd}(G)$. Clearly, if $S = \{v_1, v_2, \ldots, v_k\}$ is a resolving set for a graph G, then the partition $\Pi = \{\{v_1\}, \{v_2\}, \ldots, \{v_k\}, V(G) \setminus S\}$ forms a resolving partition for G, which leads to the relationship $\mathrm{pd}(G) \leq \dim(G) + 1$, already known from [2].

On the other hand, in order to consider distinguishing the edges of a graph by using a set of landmarks standing on a set of vertices of the graph, the notion of edge metric dimension was introduced in [4]. The concept has become popular in the research area of metric dimension in graphs (see the survey [5] for more information on this fact). For a vertex $v \in V(G)$ and an edge $e = uw \in E(G)$, the *distance* between v and e is defined as $d_G(e, v) = \min\{d_G(u, v), d_G(w, v)\}$. A vertex $w \in V(G)$ *distinguishes* two edges $e_1, e_2 \in E(G)$ if $d_G(w, e_1) \neq d_G(w, e_2)$. A set S of vertices in a connected graph G is an *edge resolving set* for G if every two edges of G are distinguished by some vertex of S. The cardinality of a smallest edge resolving set for G is called the *edge metric dimension* and is denoted by $\mathrm{edim}(G)$. An *edge metric basis* for G is an edge resolving set for G of cardinality $\mathrm{edim}(G)$.

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In this paper, we present a different way to obtain resolving structures in graphs in order to gain more insight into the study of edge resolving sets and resolving partitions. Some of the principal antecedents of this new study of resolving set are the resolving partition defined in [2], the metric colorings presented in [1] and the strong resolving partitions described in [10]. To the best of our knowledge the next concept has not been presented elsewhere, although it seems very natural in concordance with several previous investigations on the topic.

For an edge $e \in E(G)$ and a set $W \subset V(G)$, the *distance* between e and W is defined as

$$d_G(e, W) = \min\{d_G(e, w) : w \in W\}.$$

A set W *distinguishes* two different edges $e, f \in E(G)$ if $d_G(e, W) \neq d_G(f, W)$. An ordered vertex partition $\Pi = \{U_1, \ldots, U_k\}$ of a graph G is an *edge resolving partition* for G if every two distinct edges of G are distinguished by some set of Π. An edge resolving partition of the smallest possible cardinality is called an *edge partition basis*, and its cardinality is the *edge partition dimension*, which is denoted by $\operatorname{epd}(G)$.

The following terminology is also useful for our exposition. For an edge $e \in E(G)$ and an ordered vertex partition $\Pi = \{U_1, U_2, \ldots, U_k\}$, the edge partition representation of e with respect to Π is the k-vector:

$$r(e|\Pi) = (d_G(e, U_1), d_G(e, U_2), \ldots, d_G(e, U_k)).$$

Clearly, a vertex partition Π of $V(G)$ is an edge resolving partition for G if and only if for every pair of distinct edges $e, f \in E(G)$ it follows that $r(e|\Pi) \neq r(f|\Pi)$. For a vertex $v \in V(G)$, the *open neighborhood* $N_G(v)$ of v is the set of vertices u of G such $uv \in E(G)$. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$.

2. Basic results and bounds

We first present a useful lemma that shall be applied in some situations. To see this, we say that two vertices u, v are *true twins* if $N_G[u] = N_G[v]$, and they are *false twins* if $N_G(u) = N_G(v)$. Clearly, the property of being (true or false) twins forms an equivalence relation in every graph G, where vertices that have no twins are singleton classes. From now on, we consider R_T as the twins equivalence relation in which each class is a set of vertices formed by either true twin vertices, or false twin vertices, or a singleton vertex.

Lemma 2.1. *Let $\Pi = \{U_1, U_2, \ldots, U_k\}$ be a partition of the vertices of a graph G of order at least 3. If there are two (true or false) twins u and v such that $u, v \in U_j$ for some $j \in \{1, 2, \ldots, k\}$, then Π is not an edge resolving partition for G.*

Proof. Let w be one of the common neighbors of u and v. It is clear that $uw, vw \in E(G)$, and it follows that

$$r(uw|\Pi) = r(vw|\Pi).$$

Thus, we readily see that Π is not an edge resolving partition for G. □

Corollary 2.1. *For any graph G of order at least 3, $\operatorname{epd}(G) \geq \max\{|S|, S$ is a class of the twins equivalence relation $R_T\}$.*

Let us present basic bounds for graphs with given order.

Theorem 2.1. *For any graph G of order $n \geq 3$, $2 \leq \operatorname{epd}(G) \leq n$. Moreover, $\operatorname{epd}(G) = 2$ if and only if G is P_n.*

Proof. The bounds are straightforward to observe. In order to prove the second assertion, we first readily observe that $\operatorname{epd}(P_n) = 2$, by just taking a partition of two sets, one of them formed by one single leaf of P_n and the other one containing the remaining vertices. On the other hand, assume G is a graph such that $\operatorname{epd}(G) = 2$ and let $\{U_1, U_2\}$ be an edge resolving partition. If there are two edges $e = uv$ and $f = xy$ such that, w.l.g., $u, x \in U_1$ and $v, y \in U_2$, then $d_G(e, U_1) = d_G(e, U_2) = d_G(f, U_1) = d_G(f, U_2) = 0$, which is not possible. Thus, since G is connected, there exists exactly one edge $h = u_1u_2$ such that $u_1 \in U_1$ and $u_2 \in U_2$. Suppose now that the maximum degree of G is at least 3 and let w be a closest vertex to u_1 of maximum degree and assume $w \in U_1$. Consider two vertices adjacent to w, say w_1, w_1', belonging to U_1 such that they are not lying in the shortest path connecting u_1 and w. So, the edges ww_1 and ww_1' satisfy that $d_G(ww_1, u_1) = d_G(ww_1', u_1)$, $d_G(ww_1, U_2) = d_G(ww_1, u_1) + 1$ and $d_G(ww_1', U_2) = d_G(ww_1', u_1) + 1$, which is a contradiction. So, the maximum degree of G is at most 2. If G is a cycle, then any vertex partition into two sets produces at least two edges having one endpoint in one set and the other endpoint in the other set, which is not possible. Therefore, G must be a path, and the proof is complete. □

Corollary 2.2. *For any graph G different from P_n, $3 \leq \operatorname{epd}(G) \leq n$.*

With respect to the upper bound of Theorem 2.1, it is easy to see that $\operatorname{epd}(K_n) = n$ by using Corollary 2.1, which shows the tightness of the bound. In contrast with its counterpart, the partition dimension (where the only graph of order n with $\operatorname{pd}(G) = n$ is $G = K_n$; see [2]), there are several other graphs G satisfying that $\operatorname{epd}(G) = n$, as we next show for the case of complete multipartite graphs $K_{n_1,n_2,\ldots,n_k}$ with at least three partite sets ($k \geq 3$).

Theorem 2.2. *If $n_1, n_2, \ldots, n_k$ are positive integers with $k \geq 3$ and $\sum_{i=1}^{k} n_i = n$, then $\operatorname{epd}(K_{n_1,n_2,\ldots,n_k}) = n$.*

Proof. By Theorem 2.1, $\operatorname{epd}(K_{n_1,n_2,\ldots,n_k}) \leq n$. From Lemma 2.1, it follows that any two vertices belonging to the same partite set of $K_{n_1,n_2,\ldots,n_k}$ belong to different sets in any edge resolving partition for $K_{n_1,n_2,\ldots,n_k}$.

On the other hand, consider two vertices u, v belonging to two different partite sets, say A, B, of $K_{n_1,n_2,\ldots,n_k}$ with $u \in A$ and $v \in B$, and such that u, v belong to a same set U_i of one edge resolving partition Π' for $K_{n_1,n_2,\ldots,n_k}$. Since $k \geq 3$, there exists another partite set, say C, of $K_{n_1,n_2,\ldots,n_k}$ such that the vertices u, v are adjacent to the vertices of C. Let $w \in C$. Consider now the edges uw and vw. Clearly,

$$d_{K_{n_1,n_2,\ldots,n_k}}(uw, U_i) = 0 = d_{K_{n_1,n_2,\ldots,n_k}}(vw, U_i) \text{ and } d_{K_{n_1,n_2,\ldots,n_k}}(uw, U_j) = 0 = d_{K_{n_1,n_2,\ldots,n_k}}(vw, U_j)$$

where U_j is a set of Π' such that $w \in \Pi'$. Moreover,

$$d_{K_{n_1,n_2,\ldots,n_k}}(uw, U_q) = 1 = d_{K_{n_1,n_2,\ldots,n_k}}(vw, U_q)$$

for any other $U_q \in \Pi'$ with $q \neq i, j$. Thus, the edges uw and vw are not resolved by Π', which is not possible. Consequently, every two vertices of $K_{n_1,n_2,\ldots,n_k}$ belong to different sets in any edge resolving partition of it, which completes the proof. □

The result above raises the question of characterizing the class of all graphs G for which $\operatorname{epd}(G) = n$, and we indeed wonder if there are graphs other than complete multipartite graphs $K_{n_1,n_2,\ldots,n_k}$ with at least three partite sets ($k \geq 3$) satisfying such equality. Thus, we state Problem 2.1.

Problem 2.1. *Characterize all the graphs G of order n with $\operatorname{epd}(G) = n$.*

In order to settle the study of complete multipartite graphs completely, let us consider the complete bipartite graphs K_{n_1,n_2} since they are not covered by Theorem 2.2.

Proposition 2.1. *If n_1, n_2 are positive integers, then $\operatorname{epd}(K_{n_1,n_2}) = n_1 + n_2 - 1$.*

Proof. If $n_1 = n_2 = 1$, clearly $\operatorname{epd}(K_{1,1}) = 1$. So, assume that $(n_1, n_2) \neq (1, 1)$.

Let V_1 and V_2 be the partite sets of K_{n_1,n_2} such that $|V_1| = n_1$ and $|V_2| = n_2$. First observe that, by Lemma 2.1, any two vertices of V_1, as well as any two vertices of V_2, belong to different sets in any edge resolving partition for K_{n_1,n_2}. Now, let $\Pi = \{U_1, U_2, \ldots, U_k\}$ be an edge resolving partition for K_{n_1,n_2}. Suppose there are two pairs of vertices $u, u' \in V_1$ and $v, v' \in V_2$ such that $u, v \in U_i$ and $u', v' \in U_j$ for some distinct $U_i, U_j \in \Pi$. Hence, the edges $e = uv'$ and $f = u'v$ satisfy that $d_{K_{n_1,n_2}}(e, U_\ell) = 1 = d_{K_{n_1,n_2}}(f, U_\ell)$ for every $\ell \neq i, j$, and $d_{K_{n_1,n_2}}(e, U_j) = d_{K_{n_1,n_2}}(e, U_i) = 0 = d_{K_{n_1,n_2}}(f, U_i) = d_{K_{n_1,n_2}}(f, U_j)$, and this is a contradiction. Consequently, there could be at most one pair of vertices u, v, with $u \in V_1$ and $v \in V_2$, belonging to a same set of the partition Π. This means that Π must have at least n_1+n_2-1 sets, which leads to $\operatorname{epd}(K_{n_1,n_2}) \geq n_1+n_2-1$. To obtain the equality, we only need to construct an edge resolving partition for K_{n_1,n_2} of cardinality $n_1 + n_2 - 1$ which can be easily done. □

Our next result provides a lower bound on the edge partition dimension for graphs with given maximum degree.

Theorem 2.3. *For any graph G with maximum degree $\Delta \geq 2$, $\operatorname{epd}(G) \geq \lceil \log_2 \Delta \rceil + 1$.*

Proof. Let $\Pi = \{U_1, U_2, \ldots, U_k\}$ be an edge partition basis for G. Let v be a vertex of degree Δ in G. We denote the vertices adjacent to v by $v_1, v_2, \ldots, v_\Delta$. We can assume that $v \in U_1$. So, the first entry of $r(vv_i|\Pi)$ is 0 for every $i = 1, 2, \ldots, \Delta$.

Now, for every i, l, j such that $1 \leq i < l \leq \Delta$ and $2 \leq j \leq k$, the edges vv_i and vv_l are incident, therefore the j-th entries of $r(vv_i|\Pi)$ and $r(vv_l|\Pi)$ differ by at most 1. Thus, there are at most 2^{k-1} different possibilities for the representations of the edges $vv_1, vv_2, \ldots, vv_\Delta$ with respect to Π. Thus, $\Delta \leq 2^{k-1}$ and therefore $\operatorname{epd}(G) \geq \lceil \log_2 \Delta \rceil + 1$. □

Let us present the class of trees T_k for $k \geq 3$ which attain the lower bound presented in Theorem 2.3. Let v be a vertex of maximum degree in T_k. One vertex in $N_{T_k}(v)$ is a pendant vertex, $k-1$ vertices in $N_{T_k}(v)$ are each adjacent to one pendant vertex, $\binom{k-1}{2}$ vertices in $N_{T_k}(v)$ are each adjacent to two pendant vertices. In general, for $j \in \{0, 1, \ldots, k-1\}$, $\binom{k-1}{j}$ vertices in $N_{T_k}(v)$ are each adjacent to j pendant vertices. See Figure 1 for a representative example, in which the tree T_3 is presented.

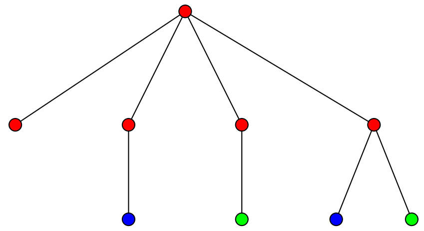

Figure 1: The tree T_k for $k = 3$ with maximum degree $\Delta = 2^{k-1} = 4$, order $1 + \sum_{j=0}^{k-1}(j+1)\binom{k-1}{j} = 9$ and $\mathrm{epd}(T_3) = 3$ (vertices equally colored represent the set of an edge partition basis).

The maximum degree of T_k is $\Delta = \sum_{j=0}^{k-1}\binom{k-1}{j} = 2^{k-1}$ and the order of T_k is

$$1 + \sum_{j=0}^{k-1}\binom{k-1}{j} + \sum_{j=0}^{k-1} j\binom{k-1}{j} = 1 + \sum_{j=0}^{k-1}(j+1)\binom{k-1}{j}.$$

Let us show that the edge partition dimension of T_k is k.

Theorem 2.4. *For $k \geq 3$, $\mathrm{epd}(T_k) = k$.*

Proof. By Theorem 2.3, since T_k has maximum degree $\Delta = 2^{k-1}$, it follows $\mathrm{epd}(T_k)) \geq \lceil \log_2 2^{k-1} \rceil + 1 = k$. We next present an edge resolving partition $\Pi = \{U_1, U_2, \ldots, U_k\}$ for T_k to show that $\mathrm{epd}(T_k) = k$.

Let us denote the vertices adjacent to the vertex v of maximum degree $\Delta = 2^{k-1}$ by $v_1, v_2, \ldots, v_\Delta$. Let

$$U_1 = \{v, v_1, v_2, \ldots, v_\Delta\}.$$

For $j \in \{0, 1, \ldots, k-1\}$, there are $\binom{k-1}{j}$ vertices in $N_{T_k}(v)$, each adjacent to j pendant vertices. Those j vertices (adjacent to the same vertex in $N_{T_k}(v)$) belong to j different sets of Π. Moreover, the pendant vertices adjacent to vertices in $N_{T_k}(v)$ of same degree belong to $\binom{k-1}{j}$ different combinations of the sets $U_2, U_3 \ldots, U_k$.

We now show that $\Pi = \{U_1, U_2, \ldots, U_k\}$ is an edge resolving partition for T_k. The first entry of $r(e|\Pi)$ for any edge $e \in E(T_k)$ is 0, because every edge is incident with a vertex in U_1. The distance between v and any other vertex is at most 2. Therefore, all the other entries of $r(vv_i|\Pi)$ (for any $i \in \{1, 2, \ldots, \Delta\}$) are 1 or 2. Any vertex v_i is adjacent to pendant vertices belonging to a unique combination of sets from Π. Thus, any edge vv_i has a unique representation in terms of Π.

Note that the edges vv_i for $i \in \{1, 2, \ldots, \Delta\}$ are the only edges of T_k having representations with only one entry equal to 0. The edges which are not incident with v have representations with two entries equal to 0. Such two edges cannot have the same representations if they are incident with pendant vertices from different sets in Π.

Hence, we need to consider those edges not incident with v which are incident with pendant vertices from the same set U_p, where $2 \leq p \leq k$. For any such edge $v_i w$, where w is a pendant vertex, it follows that v_i is adjacent to pendant vertices belonging to a unique combination of sets from Π, therefore the edge $v_i w$ has a unique representation. □

From Theorem 2.3, we obtain the following corollary.

Corollary 2.3. *If G is a graph with edge partition dimension k and maximum degree Δ, then $\Delta \leq 2^{k-1}$.*

3. Edge partition dimension versus edge metric dimension

It is natural to think that the edge metric dimension and the edge partition dimension are closely related. For instance, if $S = \{v_1, v_2, \ldots, v_r\}$ is an edge metric basis of G, then it is straightforward to observe that the partition $\Pi = \{\{v_1\}, \{v_2\}, \ldots, \{v_r\}, V(G) \setminus S\}$ represents an edge resolving partition for G. Thus, the following primary relationship is deduced.

Theorem 3.1. *For any graph G,* $\mathrm{epd}(G) \leq \mathrm{edim}(G) + 1$.

The bound above is tight. A trivial example can be seen by just considering a path P_n for which $\mathrm{epd}(P_n) = 2 = \mathrm{edim}(P_n) + 1$ (see [4]). Also, the complete graph K_n has the edge partition dimension equal to $\mathrm{edim}(K_n) + 1$. Let us present some other examples attaining such equality. For instance, if G is a graph such that $\mathrm{edim}(G) = 2$, then by Theorem 3.1 and Corollary 2.2, we obtain Proposition 3.1.

Proposition 3.1. *If G is a graph with* $\mathrm{edim}(G) = 2$*, then* $\mathrm{epd}(G) = 3$.

There are several classes of graphs G such that $\mathrm{edim}(G) = 2$, thus by Proposition 3.1, $\mathrm{epd}(G) = 3$. We mention some of them in Corollary 3.1. Note that for $n \geq 3$, the corona graph $P_n \odot K_1$ is obtained from P_n by joining each vertex of P_n to a new vertex. So, $P_n \odot K_1$ has $2n$ vertices. For $n, s \geq 2$, a grid graph $P_n \square P_s$ is the Cartesian product of P_n and P_s.

Corollary 3.1. *For* $n \geq 3$,

- $\mathrm{epd}(C_n) = 3$,
- $\mathrm{epd}(P_n \odot K_1) = 3$,
- $\mathrm{epd}(P_n \square P_s) = 3$ *for* $s \geq 2$.

After having Proposition 3.1 for $\mathrm{edim}(G) = 2$, we consider graphs G for which $\mathrm{edim}(G) > 2$. We are interested in finding graphs G such that $\mathrm{edim}(G) > 2$ and $\mathrm{epd}(G) = 3$, or more generally, in finding graphs G for which $\mathrm{epd}(G) = k$ and $\mathrm{edim}(G) > k - 1$ where $k \geq 3$. We use the graph T_k (previously defined) to study this relation.

In the proof of Theorem 3.2, we use pendant paths. A pendant path is a path in G such that all its internal vertices have degree 2 in G and its two terminal vertices have degree 1 and at least 3 in G. If v is that vertex of degree at least 3, then we call that path a pendant path of v. We denote the number of pendant paths of v by l_v.

Theorem 3.2. *For* $k \geq 3$, $\mathrm{edim}(T_k) = 2^{k-2}(k-3) + k$.

Proof. From [4], $\mathrm{edim}(T) = \sum_{v \in V(T), l_v > 1}(l_v - 1)$ for any tree T that is not a path. In order to show our result, we need to show that for T_k, we have

$$\sum_{v \in V(T_k), l_v > 1}(l_v - 1) = 2^{k-2}(k-3) + k.$$

From the definition of T_k, we know that the vertex v which has maximum degree in T_k is adjacent to

$$\binom{k-1}{0} + \binom{k-1}{1} + \cdots + \binom{k-1}{k-1}$$

vertices. There are $\binom{k-1}{1} + 2\binom{k-1}{2} + \cdots + (k-1)\binom{k-1}{k-1}$ at distance 2 from v in T_k. It follows that

$$\mathrm{edim}(T_k) = \sum_{v \in V(T_k), l_v > 1}(l_v - 1) = k - 1 + \binom{k-1}{2} + 2\binom{k-1}{3} + \cdots + (k-2)\binom{k-1}{k-1} = 2^{k-2}(k-3) + k.$$

□

By Theorems 2.4 and 3.2, we have $\mathrm{epd}(T_k) = k$ and $\mathrm{edim}(T_k) = 2^{k-2}(k-3) + k$, respectively. Thus, $\mathrm{edim}(T_k) - \mathrm{epd}(T_k) = 2^{k-2}(k-3) \geq 0$ for $k \geq 3$. This means that there is a graph G with $\mathrm{epd}(G) = 3$ and $\mathrm{edim}(G) > 2$. For $k = 3$, we have $\mathrm{epd}(T_3) = 3 = \mathrm{edim}(T_3)$. Notice that, the construction of T_k allows us to claim that if $k \geq 4$, then there is a graph G such that $\mathrm{epd}(G) = k$ and $\mathrm{edim}(G) > k$. Now we raise the following question.

Problem 3.1. *Is it true that if* $\mathrm{epd}(G) = 3$ *for a given graph* G, *then* $2 \leq \mathrm{edim}(G) \leq 3$?

4. Edge partition dimension versus partition dimension

It would be natural to think that the edge partition dimension and the partition dimension of graphs are somehow related. However, to deduce such a relationship does not appear to be a simple task. For instance, it is known that for any tree T, we have $\dim(T) = \mathrm{edim}(T)$, but for the partition case, an analogous result does not follow. We show that $\mathrm{epd}(T)$ is not necessarily equal to $\mathrm{pd}(T)$. Let us consider double stars S_{n_1,n_2} with $n_1 + n_2$ pendant vertices.

Theorem 4.1. *If* $n_1 \geq n_2 \geq 1$ *and* $n_1 \geq 3$, *then* $\mathrm{pd}(S_{n_1,n_2}) = n_1$ *and* $\mathrm{epd}(S_{n_1,n_2}) = n_1 + 1$.

Proof. Assume the pendant vertices adjacent to the two non-pendant vertices u and v are $u_1, u_2, \ldots, u_{n_1}$ and $v_1, v_2, \ldots, v_{n_2}$, respectively.

We show that $\mathrm{pd}(S_{n_1,n_2}) = n_1$. First, any two pendant vertices adjacent to u must be in different sets of some vertex resolving partition Π, for otherwise they cannot be resolved by Π. So, $\mathrm{pd}(S_{n_1,n_2}) \geq n_1$.

Let $\Pi = \{U_1, U_2, \dots, U_{n_1}\}$, where $u_i \in U_i$ for $i = 1, 2, \dots, n_1$, $v_i \in U_i$ for $i = 1, 2, \dots, n_2$, $u \in U_1$ and $v \in U_3$. There are three vertices u, u_1, v_1 in U_1. We have

$$r(u|\{U_2, U_3\}) = (1, 1),$$
$$r(u_1|\{U_2, U_3\}) = (2, 2),$$
$$r(v_1|\{U_2, U_3\}) = (x, 1),$$

where $x = 2$ if $n_2 \geq 2$, and $x = 3$ if $n_2 = 1$. For the vertices in U_3, we have

$$r(u_3|\{U_1, U_2\}) = (2, 2),$$
$$r(v_3|\{U_1, U_2\}) = (2, 1) \text{ for } n_2 \geq 3,$$
$$r(v|\{U_1, U_2\}) = (1, y),$$

where $y = 1$ if $n_2 \geq 2$, and $y = 2$ if $n_2 = 1$. For $i = 2$ and $i = 4, 5, \dots, n_2$, $U_i = \{u_i, v_i\}$ and we obtain

$$d_{S_{n_1,n_2}}(u_i, U_1) = 1 \text{ and } d_{S_{n_1,n_2}}(v_i, U_1) = 2.$$

Thus, u_i and v_i are resolved by Π. There is only one vertex in U_i for $i = n_2 + 1, n_2 + 2, \dots, n_1$, and so Π is a vertex resolving partition for S_{n_1,n_2}. Therefore, $\mathrm{pd}(S_{n_1,n_2}) = n_1$.

We now prove that $\mathrm{epd}(S_{n_1,n_2}) = n_1 + 1$. First, we show that $\mathrm{epd}(S_{n_1,n_2}) \geq n_1 + 1$. Suppose to the contrary that $\mathrm{epd}(S_{n_1,n_2}) = k \leq n_1$. Let $\Pi' = \{U_1, U_2, \dots, U_k\}$ be an edge resolving partition for S_{n_1,n_2}. By Lemma 2.1, the vertices $u_1, u_2, \dots, u_{n_1}$ are in different sets of Π', say $u_i \in U_i$ where $i = 1, 2, \dots, n_1$ (so $k \geq n_1$). Without loss of generality, we assume that $u \in U_1$. Then for $v \in U_i$ where $1 \leq i \leq n_1$, we obtain $r(uu_i|\Pi') = r(uv|\Pi')$ which is a contradiction. Thus, $\mathrm{epd}(S_{n_1,n_2}) \geq n_1 + 1$.

Let $\Pi = \{U_1, U_2, \dots, U_{n_1+1}\}$, where $u_i \in U_i$ for $i = 1, 2, \dots, n_1$, $v_i \in U_{i+1}$ for $i = 1, 2, \dots, n_2$, $u \in U_1$ and $v \in U_{n_1+1}$. Hence, the first and (n_1+1)-th entry of $r(uv|\Pi)$ is 0, only the first entry of $r(uu_1|\Pi)$ is 0, while only the (n_1+1)-th entry of $r(vv_{n_2}|\Pi)$ is 0. For $i = 2, 3, \dots, n_1$, the first and i-th entry of $r(uu_i|\Pi)$ is 0. Also, for $i = 1, 2, \dots, n_2 - 1$, the $(i + 1)$-th and $(n_1 + 1)$-th entry of $r(vv_i|\Pi)$ is 0. Thus, Π is an edge resolving partition for S_{n_1,n_2}, and so $\mathrm{epd}(S_{n_1,n_2}) = n_1 + 1$, as required. □

Double stars are the only trees of diameter 3. We suggest studying relations between $\mathrm{epd}(T)$ and $\mathrm{pd}(T)$ for trees T with greater diameters.

Problem 4.1. *Study relations between* $\mathrm{epd}(T)$ *and* $\mathrm{pd}(T)$ *for trees* T *with diameter at least* 4.

For double stars S_{n_1,n_2}, we have $\mathrm{epd}(S_{n_1,n_2}) > \mathrm{pd}(S_{n_1,n_2})$. It would be interesting to know if there is any tree T with $\mathrm{epd}(T) < \mathrm{pd}(T)$.

Problem 4.2. *Is there any tree* T *such that* $\mathrm{epd}(T) < \mathrm{pd}(T)$ *or is it true that* $\mathrm{epd}(T) \geq \mathrm{pd}(T)$ *for every tree* T*?*

A similar question can be asked about general graphs.

Problem 4.3. *Is there any graph* G *with* $\mathrm{epd}(G) < \mathrm{pd}(G)$*?*

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