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Research Article

An optimal lower bound for the size of periodic digraphs

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Abstract

A periodic digraph is the digraph associated with a periodic point of a continuous map from the unit interval to itself. This digraph encodes "covering" relation between minimal intervals in the corresponding orbit, which allows the application of purely combinatorial arguments in establishing results on the existence and co-existence of periods of periodic points (for example, in proving the famous Sharkovsky's theorem). In this article, an optimal lower bound for the size of periodic digraphs is provided and thus some previous results of Pavlenko are tightened.

Keywords: periodic digraph; Markov graph; Sharkovsky's theorem.

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1. Introduction

Combinatorial dynamics is a branch of dynamical systems that studies the existence of periodic points and the structure of their orbits for self-maps on various structures. A celebrated result that helped to shape this field is the Sharkovsky's theorem [8] which completely describes the coexistence of periods of periodic points of a continuous map from the unit interval to itself. It turned out that one can prove Sharkovsky's theorem using purely combinatorial arguments [2, 9]. These ideas are based on the following construction. Let $f : [0, 1] \to [0, 1]$ be a continuous map and $x \in [0, 1]$ be its n-periodic point. Clearly, the restriction of f to the corresponding orbit $\mathrm{orb}_f(x) = \{x, f(x), \dots, f^{n-1}(x)\}$ is a cyclic permutation of the latter. Consider the natural ordering $\mathrm{orb}_f(x) = \{x_1 < \cdots < x_n\}$. The corresponding periodic digraph Γ has the vertex set $V(\Gamma) = \{1, \dots, n-1\}$ and the arc set $A(\Gamma) = \{(i, j) : \min\{f(x_i), f(x_{i+1})\} \leq j < \max\{f(x_i), f(x_{i+1})\}\}$. The idea behind this construction is the following, each vertex $i \in V(\Gamma)$ corresponds to a minimal interval $[x_i, x_{i+1}]$, and there is an arc $i \to j$ in Γ provided the interval $[x_i, x_{i+1}]$ "covers" interval $[x_j, x_{j+1}]$ under f. This construction can be extended to a continuous vertex maps on topological trees, which enables obtain Sharkovsky-type results (not linear, but partial orders) in this more general setting [1].

The purely graph-theoretic properties of periodic digraphs were studied by Pavlenko in his three papers [5–7]. For example, the number of non-isomorphic periodic graphs with a given number of vertices was obtained in [5]. Characterizations of periodic digraphs and their induced subgraphs were proved in [6] and [7], respectively.

2. Preliminaries

A *graph* is a pair $G = (V, E)$, where $V = V(G)$ is the set of its *vertices* and $E = E(G)$ is the set of its *edges* which are unordered pairs of vertices. For convenience, instead of $\{u, v\}$ we will write uv for an edge in a graph. For a set of vertices $V' \subset V(G)$ in a graph G, by $E(V') = \{uv \in E(G) : u, v \in V'\}$ we denote the set of edges of G whose vertices are in V'.

A graph is *connected* provided there is a path between every pair of its vertices. The vertex set $V(G)$ of a connected graph G is naturally equipped with the standard metric d_G, where $d_G(u, v)$ equals the length of a shortest path $u - v$ path in G. For a pair of vertices $u, v \in V(G)$ in a connected graph G, the *metric interval* between u, v is the set $[u, v]_G = \{x \in V(G) : d_G(u, x) + d_G(x, v) = d_G(u, v)\}$the *metric interval* between u, v. Also, we put $A_G(u, v) = \{x \in V(G) : d_G(u, x) \leq d_G(v, x)\}$. The *Wiener index* of a connected graph is the value $W(G) = \sum_{\{u,v\} \subset V(G)} d_G(u, v)$.

A *tree* is a connected graph without cycles. Prominent example of trees are *path graphs* P_n, where $V(P_n) = \{1, \dots, n\}$ and $E(P_n) = \{ij : 1 \leq i = j - 1 \leq n - 1\}$. For example, $W(P_n) = \frac{n(n^2-1)}{6}$.

A *digraph* is a pair $D = (V, A)$, where $V = V(D)$ is the set of its *vertices* and $A = A(D) \subset V \times V$ is the set of its *arcs*. An arc of the form $u \to u$ in a digraph D is called a *loop* at vertex u.

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For a vertex $u \in V(D)$ in a digraph D put $N_D^+(u) = \{v \in V(D) : (u,v) \in A(D)\}$ and $N_D^-(u) = \{v \in V(D) : (v,u) \in A(D)\}$. The numbers $d_D^+(u) = |N_D^+(u)|$ and $d_D^-(u) = |N_D^-(u)|$ are called the *out-degree* and the *in-degree* of u, respectively.

Now let X be a tree and $\sigma : V(X) \to V(X)$ be a map from $V(X)$ to itself. The corresponding *Markov graph* is a digraph $\Gamma = \Gamma(X,\sigma)$ with the vertex set $V(\Gamma) = E(X)$ and the arc set $A(\Gamma) = \{(uv, xy) : xy \in E([\sigma(u), \sigma(v)]_X)\}$.

Example 2.1. *Let X be a tree with $V(X) = \{1,\dots,7\}$ and $E(X) = \{12, 23, 26, 34, 37, 45\}$. Consider the map*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 1 & 3 & 6 & 5 & 4 \end{pmatrix}.$$

The Markov graph $\Gamma(X,\sigma)$ is depicted in Figure 1.

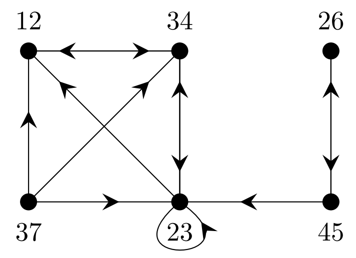

Figure 1: The Markov graph $\Gamma(X,\sigma)$ for the pair (X,σ) from Example 2.1.

Lemma 2.1. [3] *Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. Then for every pair of vertices $u, v \in V(X)$ and an edge $e \in E([\sigma(u), \sigma(v)]_X)$ there exists an edge $e' \in E([u,v]_X)$ with $e' \to e$ in $\Gamma(X,\sigma)$.*

Denote by $\mathcal{T}(X)$, $\mathcal{P}(X)$ and $\mathcal{C}(X)$ the classes of all maps, permutations and cyclic permutations of $V(X)$, respectively. The average number size of Markov graphs for maps in these classes can be explicitly calculated in terms of the Wiener index of X as follows.

Theorem 2.1. [4] *For any tree X with $n \geq 3$ vertices the next equalities hold:*

$$\frac{1}{n^n} \sum_{\sigma \in \mathcal{T}(X)} |A(\Gamma(X,\sigma))| = \frac{2(n-1)}{n^2} \cdot W(X),$$
$$\frac{1}{n!} \sum_{\sigma \in \mathcal{P}(X)} |A(\Gamma(X,\sigma))| = \frac{2}{n} \cdot W(X),$$
$$\frac{1}{(n-1)!} \sum_{\sigma \in \mathcal{C}(X)} |A(\Gamma(X,\sigma))| = \frac{2(n-3)}{(n-1)(n-2)} \cdot W(X) + \frac{n}{n-2}.$$

An *n-periodic digraph* is a Markov graph $\Gamma(X,\sigma)$ for a path $X = P_n$ and its cyclic permutation σ. The number of non-isomorphic n-periodic digraphs was obtained in [5].

Theorem 2.2. [5] *Let $d(n)$ denote the number of pairwise non-isomorphic n-periodic digraphs. Then*

$$d(n+1) = \begin{cases} \frac{1}{2}\left(n! - \varphi(n+1) + \dfrac{(n-1)!}{(k-1)!}\right) + 1 & \text{if } n = 2k-1, \\ \frac{1}{2}(n! - \varphi(n+1)) + 1 & \text{if } n = 2k, \end{cases}$$

where $\varphi(n)$ is Euler's totient function.

Also, periodic digraphs admit a nice graph-theoretic characterization. To present this result, for any digraph D, we define a self-map on the power set $\mathcal{A} : 2^{V(\Gamma)} \to 2^{V(\Gamma)}$ in such a way:

$$\mathcal{A}(V') := \bigcup_{v \in V'} (N_\Gamma^-(v) \backslash \bigcup_{w \in V' \backslash \{v\}} N_\Gamma^-(w))$$

for all subsets $V' \subset V(\Gamma)$. For example, for a two-element subset $V' = \{v_1, v_2\}$, we have $\mathcal{A}(V') = N_\Gamma^-(v_1) \triangle N_\Gamma^-(v_2)$. For a collection of sets $\mathcal{F}$, the corresponding *intersection graph* is an undirected graph with the vertex set $\mathcal{F}$ and two sets $A, B \in \mathcal{F}$ are being adjacent provided $A \cap B \neq \emptyset$.

Theorem 2.3. [6] *A digraph Γ with $n \geq 1$ vertices is a periodic digraph if and only if there exists a vertex $u \in V(\Gamma)$ with $d^-_\Gamma(u) = 2$ such that the singleton $\{u\}$ is an $(n+1)$-periodic point for $\mathcal{A}$, and the intersection graph for the collection of sets $\mathrm{orb}_\mathcal{A}(\{u\})$ is a path.*

The following bounds on the size of n-periodic digraphs can be easily obtained by examining in-degrees of their vertices.

Proposition 2.1. [6] *Let Γ be an n-periodic digraph with $n \geq 3$. Then $n \leq |A(\Gamma)| \leq \left\lfloor \frac{n^2}{2} \right\rfloor$.*

However, checking all the 5-periodic digraphs (a complete list of these digraphs can be found in [9]), we can conclude that the bounds from Proposition 2.1 are not optimal (every 5-periodic digraph have at least 6 arcs and at most 11 arcs). The aim of this paper is to present an optimal lower bound for the size of n-periodic digraphs.

3. Main result

The main result of this note is the following theorem.

Theorem 3.1. *For any n-periodic digraph Γ the next lower optimal bound holds*

$$|A(\Gamma)| \geq \left\lfloor \frac{3n-3}{2} \right\rfloor.$$

Proof. For $n = 1$ the bound is clear. Now let $n \geq 2$ and $X = P_n$ with $V(X) = \{1, \dots, n\}$ and $E(X) = \{ij : 1 \leq i = j-1 \leq n-1\}$. Consider a cyclic permutation σ of $V(X)$ and put $\Gamma = \Gamma(X, \sigma)$, $e_i = ij$ for all $1 \leq i = j-1 \leq n-1$.

By Lemma 2.1, $d^-_\Gamma(e_i) \geq 1$ for all $1 \leq i \leq n-1$. Put $V_1 = \{1 \leq i \leq n-1 : d^-_\Gamma(e_i) = 1\}$. Let us show that $|V_1| \leq \left\lfloor \frac{n}{2} \right\rfloor$. Indeed, assume $i \in V_1$. Then we have $N^-_\Gamma(e_i) = \{e_k\}$ for some $1 \leq k \leq n-1$. Clearly, $\sigma(k) \leq i$ and $\sigma(k+1) \geq i+1$, or $\sigma(k) \geq i+1$ and $\sigma(k+1) \leq i$. Moreover, $\sigma(A_X(k, k+1)) \subset A_X(i, i+1))$ and $\sigma(A_X(k+1, k)) \subset A_X(i+1, i)$, or $\sigma(A_X(k, k+1)) \subset A_X(i+1, i)$ and $\sigma(A_X(k+1, k)) \subset A_X(i, i+1)$ (again, see Lemma 2.1). Since σ is a permutation, then $k = |A_X(k, k+1)| = |A_X(i, i+1)| = i$ or $k = |A_X(k, k+1)| = |A_X(i, i+1)| = n-i$. In the first case, $\sigma(A_X(k, k+1)) \subset A_X(k, k+1)$, implying that $A_X(k, k+1)$ is a σ-invariant set, which contradicts the cyclicity of σ. Thus, we have the equality $k + i = n$. Now assume additionally that $i \neq \frac{n}{2}$. We show that in this case, $d^-_\Gamma(e_k) \geq 2$ and hence $k \notin V_1$. To the contrary, let $d^-_\Gamma(e_k) = 1$. By the same argument, $N^-_\Gamma(e_k) = \{e_i\}$. This means that $\sigma(i) \leq k$ and $\sigma(i+1) \geq k+1$, or $\sigma(i) \geq k+1$ and $\sigma(i+1) \leq k$. For convenience, let $k \leq i$. We consider the next four cases.

Case 1. $\sigma(k) \leq i$, $\sigma(k+1) \geq i+1$, $\sigma(i) \leq k$, $\sigma(i+1) \geq k+1$.
In this case, $e_i \in E([\sigma(k+1), \sigma(i)]_X)$. Then, by Lemma 2.1, there is an edge $e_m \in E([k+1, i]_X)$ with $e_m \to e_i$ in Γ. Since $k \leq i$, then $m \neq k$. This means that $d^-_\Gamma(k) \geq 2$ which is a contradiction.

Case 2. $\sigma(k) \leq i$, $\sigma(k+1) \geq i+1$, $\sigma(i) \geq k+1$, $\sigma(i+1) \leq k$.
Since $i \neq \frac{n}{2}$, $k \neq i$, and hence $\sigma(i) \leq i-1$ (otherwise, $\sigma(i+1) \leq k \leq i$ and $\sigma(i) \geq i+1$, which would imply the existence of a loop at e_i in Γ). Thus, $e_i \in E([\sigma(k+1), \sigma(i)]_X)$. Therefore, again $d^-_\Gamma(k) \geq 2$.

Case 3. $\sigma(k) \geq i+1$, $\sigma(k+1) \leq i$, $\sigma(i) \leq k$, $\sigma(i+1) \geq k+1$.
Here $\sigma(k+1) \geq k+2$ (otherwise, Γ would contain a loop at e_k). Similarly to previous cases, we have $e_k \in E([\sigma(k+1), \sigma(i)]_X)$, which again implies $d^-_\Gamma(k) \geq 2$.

Case 4. $\sigma(k) \geq i+1$, $\sigma(k+1) \leq i$, $\sigma(i) \geq k+1$, $\sigma(i+1) \leq k$.
In this case, $\sigma(A_X(k, k+1)) \subset A_X(i+1, i)$ and $\sigma(A_X(i, i+1)) \subset A_X(k, k+1)$. Therefore, the set $A_X(k, k+1) \cup A_X(i, i+1)$ is a proper σ-invariant set (which contradicts the cyclicity of σ).

Hence, in all cases we have $d^-_\Gamma(e_k) \geq 2$. In other words, $i \in V_1$ and $i \neq \frac{n}{2}$ implies $n - i \notin V_1$. This clearly means that $|V_1| \leq \left\lfloor \frac{n}{2} \right\rfloor$. Therefore,

$$|A(\Gamma)| = \sum_{i=1}^{n-1} d^-_\Gamma(e_i) \geq |V_1| + 2(n-1-|V_1|) = 2n - 2 - |V_1| \geq \left\lfloor \frac{3n-3}{2} \right\rfloor.$$

To prove the optimality of a given bound, let us construct a cyclic permutation which realizes this bound. At first, let n be an even number. If $n = 2$, then the unique cyclic permutation of $V(X)$ produces a periodic digraph with exactly

$\lfloor \frac{3n-3}{2} \rfloor = 1$ arc. Now assume $n \geq 4$. Consider the map

$$\sigma(i) = \begin{cases} n+1-i, \text{ if } 1 \leq i \leq \dfrac{n}{2}, \\ n-i, \text{ if } \dfrac{n}{2}+1 \leq i \leq n-1, \\ \dfrac{n}{2}, \text{ if } i = n \end{cases}$$

for $1 \leq i \leq n$. It is easily seen that σ is a permutation of $V(X)$. To prove that σ is a cyclic permutation, we show that for any $1 \leq k \leq n$ there is a number $m \in \mathbb{N}$ with $\sigma^m(1) = k$. Indeed, if $1 \leq k \leq \frac{n}{2}$, then $m = n-2k+2$. Similarly, if $\frac{n}{2}+1 \leq k \leq n-1$, then $m = 2k-n+1$. Finally, $\sigma(1) = n$. Putting $\Gamma = \Gamma(X, \sigma)$, we obtain

$$d^+_\Gamma(e_i) = \begin{cases} 1, \text{ if } 1 \leq i \leq \dfrac{n}{2}-1, \text{ or } \dfrac{n}{2}+1 \leq i \leq n-2, \\ 2, \text{ if } i = \dfrac{n}{2}, \\ \dfrac{n}{2}-1, \text{ if } i = n-1 \end{cases}$$

for all $1 \leq i \leq n-1$. This asserts the equality

$$|A(\Gamma)| = \sum_{i=1}^{n-1} d^+_\Gamma(e_i) = \frac{n}{2} - 1 + n - 2 - (\frac{n}{2}+1) + 1 + 2 + \frac{n}{2} - 1 = \frac{3n-4}{2}.$$

Now let n be an odd number. In this case, consider the map

$$\sigma(i) = \begin{cases} n+1-i, \text{ if } 1 \leq i \leq \dfrac{n-1}{2}, \\ n-i, \text{ if } \dfrac{n+1}{2} \leq i \leq n-1, \\ \dfrac{n+1}{2}, \text{ if } i = n \end{cases}$$

for $1 \leq i \leq n$. One can similarly show that σ is a cyclic permutation of $V(X)$. We have

$$d^+_\Gamma(e_i) = \begin{cases} 1, \text{ if } 1 \leq i \leq \dfrac{n-3}{2}, \text{ or } \dfrac{n+1}{2} \leq i \leq n-2, \\ 2, \text{ if } i = \dfrac{n}{2}, \\ \dfrac{n-1}{2}, \text{ if } i = n-1 \end{cases}$$

for all $1 \leq i \leq n-1$. This implies the equality

$$|A(\Gamma)| = \sum_{i=1}^{n-1} d^+_\Gamma(e_i) = \frac{n-3}{2} + n - 2 - \frac{n+1}{2} + 1 + 2 + \frac{n-1}{2} = \frac{3n-3}{2}.$$

□

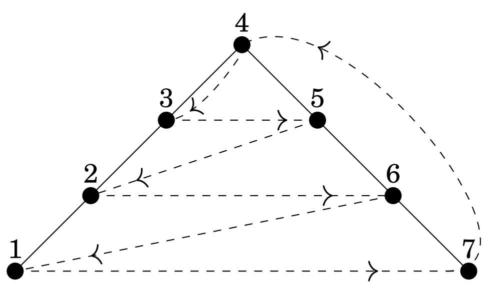

Figure 2: A cyclic permutation σ of P_7 which attains the lower bound from Theorem 3.1.

Example 3.1. *For a cyclic permutation $\sigma = (1743526)$ of a path P_7 (see Figure 2), the corresponding periodic digraph $\Gamma(P_7, \sigma)$ has $\lfloor \frac{3 \cdot 7 - 3}{2} \rfloor = 9$ arcs (see Figure 3).*

We also note that Theorem 2.1 implies that on average an n-periodic digraph has $\frac{2(n-3)}{(n-1)(n-2)} \cdot W(P_n) + \frac{n}{n-2} = \frac{n^2}{3}$ arcs.

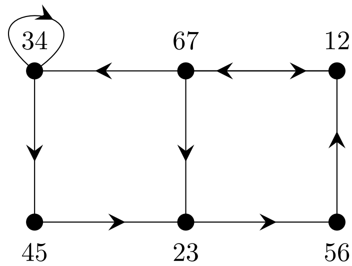

Figure 3: The Markov graph $\Gamma(X, \sigma)$ for the pair (X, σ) from Example 3.1.

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