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Research Article

Parameterized binding numbers and degree sequence theorems

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Abstract

We define an extension of the standard binding number of a graph which introduces parameters into the computation. We call the result a *parameterized binding number*. This extension is motivated by a number of theorems that use bounds on the order of neighbor sets of vertices to determine the existence of cycles or factors within the graph. We demonstrate how this extended binding number can be integrated into such theorems. Additionally, we present theorems that provide sufficient conditions on the degree sequence of a graph which guarantees a prescribed lower bound on parameterized binding numbers. These degree sequence theorems are shown to be best possible in a certain sense. Finally, we show how these degree conditions can be combined with known theorems to produce sufficient conditions which guarantee certain cycles or factors within the graph.

Keywords: binding number; degree sequence; best monotone theorem; cycle; factor.

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1. Introduction

We will consider finite, simple graphs G with vertex set $V(G)$. Our terminology and notation will be standard, except where indicated. In particular, given two disjoint graphs G and H, the complete join will be denoted by $G + H$. Given a graph H and a positive integer m, we use mH to denote m disjoint copies of H. If $S \subset V(G)$, we use $G - S$ to refer to the induced subgraph $\langle V(G) - S\rangle$. For any $v \in V(G)$, the degree of v is given by $\deg_G(v)$, or simply $\deg(v)$ if it is clear to which graph G we are referring. For $S \subseteq V(G)$, the *neighbor set of* S, denoted by $N(S)$, is the set of all vertices of G adjacent to at least one vertex in S.

In [5], Woodall introduced the *binding number* of a graph G, which can be defined as

$$\mathrm{bind}(G) = \min\left\{ \frac{|N(S)|}{|S|} \;\middle|\; \emptyset \neq S \subseteq V(G),\ N(S) \neq V(G) \right\}.$$

We say that G is b-*binding* if $\mathrm{bind}(G) \geq b$, for some $b \geq 0$. Among other results, Woodall proved the significant fact that $\frac{3}{2}$-binding implies the graph is Hamiltonian [5].

The degree sequence of a graph G, denoted by $\pi(G)$, is a list of degrees of all vertices of G in nondecreasing order. We often use an exponential shorthand notation for degree sequences such that the power indicates the multiplicity of the degree, e.g., $(2, 2, 2, 2, 5, 5) = 2^4 5^2$. Given a finite sequence of nonnegative integers π, we say that π is *graphical* if there exists a graph G such that $\pi(G) = \pi$, and we call G a *realization* of π. Note that it is possible for a graphical sequence to have multiple distinct realizations. Given a graph property P, a graphical sequence is *forcibly* P if every realization has the property P. By P-*theorem* we mean a theorem that applies conditions to a graphical sequence to determine if it is forcibly P. A classic such result for hamiltonicity is the following.

Theorem 1.1 ([3]). *Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If $d_i \leq i \Rightarrow d_{n-i} \geq n - i$ for all $1 \leq i < \frac{n}{2}$, then π is forcibly Hamiltonian.*

Given two graphical sequences of the same length $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$ and $\pi' = (d'_1 \leq d'_2 \leq \cdots \leq d'_n)$, we say π' *majorizes* π, denoted by $\pi' \geq \pi$, if $d'_i \geq d_i$ for all $1 \leq i \leq n$. Observe that if a graphical sequence satisfies the conditions of Theorem 1.1, then any majorizing sequence will also satisfy the conditions. We say a P-theorem is *monotone increasing* (or simply *montone* for the purposes of this paper) if when it declares π forcibly P, it also declares any majorizing sequence $\pi' \geq \pi$ forcibly P. Thus, Theorem 1.1 is a monotone Hamiltonian-theorem. In addition to being monotone, Theorem 1.1 has the property that if π fails to satisfy any of the conditions, then there exists $\pi' \geq \pi$ that is not forcibly P. To see this, assume

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$\pi = (d_1 \le d_2 \le \cdots \le d_n)$ fails Theorem 1.1 for some i. Then $\pi \le i^i(n-i-1)^{n-2i}(n-1)^i$, which has the nonHamiltonian realization $K_i + (K_i \cup K_{n-2i})$. We call a monotone P-theorem *weakly-optimal* if whenever a graphical sequence π fails the conditions of the theorem, there is a graphical sequence $\pi' \ge \pi$ that is not forcibly P. Thus, Theorem 1.1 is weakly-optimal. The significance of weak optimality is due to the following theorem.

Theorem 1.2 ([1]). *Let T, T_0 be monotone P-theorems, with T_0 weakly-optimal. If T declares a degree sequence π to be forcibly P, then so does T_0.*

Given a graph property P, Theorem 1.2 implies that if T_0 is a weakly-optimal monotone P-theorem and T is any other monotone P-theorem, then the set of sequences declared forcibly P by T is a subset of those declared forcibly P by T_0. It is also known that a weakly-optimal monotone P-theorem is unique [1]. Thus, we call the weakly-optimal monotone P-theorem the *best monotone P-theorem*. So, Theorem 1.1 is the best monotone Hamiltonian-theorem. Best monotone theorems for additional graph properties can be found in [1].

2. A parameterized binding number

Let $\alpha(\ge 1), \beta, \gamma \in \mathbb{Z}$ be given and let G be a graph with $n \doteq |V(G)|$. Then we define a *parameterized binding number of* G, denoted $\mathrm{bind}\{\alpha,\beta,\gamma\}(G)$, as

$$\mathrm{bind}\{\alpha,\beta,\gamma\}(G) = \min\left\{ \frac{\alpha|N(S)| + \beta n + \gamma}{|S|} \;\middle|\; \emptyset \ne S \subseteq V(G),\ N(S) \ne V(G) \right\},$$

where $\mathrm{bind}\{1,0,0\}(G)$ is the standard binding number $\mathrm{bind}(G)$.

A graph G is $b-(\alpha,\beta,\gamma)$*binding* if $\mathrm{bind}\{\alpha,\beta,\gamma\}(G) \ge b$, and we call $\emptyset \ne S \subseteq V(G)$ an (α,β,γ)*binding set* of G if

$$\mathrm{bind}\{\alpha,\beta,\gamma\}(G) = \frac{\alpha|N(S)| + \beta n + \gamma}{|S|}.$$

Let $S_1 = \{v\}$ for $v \in K_n$ and let $S_2 = V(nK_1)$. Then for any graph G on n vertices, we have

$$\mathrm{bind}\{\alpha,\beta,\gamma\}(G) \le \mathrm{bind}\{\alpha,\beta,\gamma\}(K_n) = \frac{\alpha|N(S_1)| + \beta n + \gamma}{|S_1|} = \frac{\alpha(n-1) + \beta n + \gamma}{1} = (\alpha+\beta)n - \alpha + \gamma$$

and

$$\mathrm{bind}\{\alpha,\beta,\gamma\}(G) \ge \mathrm{bind}\{\alpha,\beta,\gamma\}(nK_1) = \frac{\alpha|N(S_2)| + \beta n + \gamma}{|S_2|} = \frac{\beta n + \gamma}{n}.$$

Thus, it only makes sense to consider b such that $(\beta n+\gamma)/n \le b \le (\alpha+\beta)n - \alpha + \gamma$, or $n \ge (b+\alpha-\gamma)/(\alpha+\beta)$ and $bn \ge \beta n + \gamma$.

Using the Fundamental Lemma of [6] we obtain the following.

Theorem 2.1. *Let G be a graph and let b, α, β, and γ be rational numbers such that b, $\alpha \ge 0$. Then*

$$b|N(X)| \ge \alpha|X| + (b-\alpha-\beta)n - \gamma$$

for every non-empty subset X of $V(G)$, if and only if

$$\alpha|N(X)| \ge b|X| - \beta n - \gamma$$

for every subset X of $V(G)$ such that $N(X) \ne V(G)$.

This implies a corollary relating lower bounds on parameterized binding numbers to inequalities on neighbor sets.

Corollary 2.1. *If $\mathrm{bind}\{\alpha,\beta,\gamma\}(G) \ge b$, then $b|N(X)| \ge \alpha|X| + (b-\alpha-\beta)n - \gamma$ for every non-empty subset X of $V(G)$.*

The definition of parameterized binding numbers is motivated by the previous corollary and the following results.

Theorem 2.2 ([6]). *Let G be a 2-connected graph such that $\delta(G) \ge \frac{1}{3}(n+2)$ and $3|N(X)| \ge |X| + n - 1$ for every non-empty independent subset X of $V(G)$. Then G is Hamiltonian.*

Theorem 2.3 ([6]). *Let G be a graph with an even number of vertices and such that $4|N(X)| \ge 2|X| + n + 1$ for every non-empty independent subset X of $V(G)$. Then G has a 1-factor.*

Theorem 2.4 ([6]). *Let $k \geq 2$ be an integer. Let G be a graph with n vertices and suppose that, if k is odd, then n is even and G is connected. Suppose that*

$$(2k-1)|N(X)| \geq |X| + (k-1)n - 1$$

for every non-empty independent subset X of $V(G)$, and G has minimum degree

$$\delta \geq \frac{(k-1)(n+1)}{2k-1}.$$

Suppose further that, if $n < 4k-6$, then

$$\delta > n + 2k - 2\sqrt{kn+2}.$$

Then G has a k-factor.

Theorem 2.5 ([7]). *Let G be a graph such that $3|N(X)| \geq |X| + n + 1$ for every non-empty independent subset X of $V(G)$. Then G contains a triangle.*

By using Corollary 2.1, we can rephrase the above theorems in terms of parameterized binding numbers. Of course, since $\text{bind}\{\alpha, \beta, \gamma\}$ is not limited to independent subsets of $V(G)$, the conditions of these analogous theorems are slightly more restrictive.

Theorem 2.6. *Let G be a 2-connected graph with $n \geq 3$, $\delta(G) \geq \frac{1}{3}(n+2)$, and $\text{bind}\{1,1,1\}(G) \geq 3$. Then G is Hamiltonian.*

The lower bound of 3 on $\text{bind}\{1,1,1\}(G)$ is best possible by considering the nonHamiltonian graphs $G_a = K_{a-1,a}$ for $a \geq 4$. These are 2-connected graphs with $\delta(G_a) = a - 1 = (n-1)/2 \geq (n+2)/3$ and, by taking $S = V(aK_1)$,

$$\text{bind}\{1,1,1\}(G_a) = \frac{a - 1 + 2a - 1 + 1}{a} = 3 - \frac{1}{a} < 3.$$

The lower bound of $\frac{1}{3}(n+2)$ on the minimum degree is to avoid the graphs $G_k = (k+1)K_2 + kK_1$, for $k \geq 2$, and $G_m = (K_3 \cup mK_2) + mK_1$, for $m \geq 2$. These are 2-connected graphs with

$$\text{bind}\{1,1,1\}(G_k) = \frac{(2k+1) + (3k+2) + 1}{k+1} = 5 - \frac{1}{k+1} > 3,$$

$$\text{bind}\{1,1,1\}(G_m) = \frac{(3m+2) + (3m+3) + 1}{2k+1} = 3 + \frac{3}{2k+1} > 3,$$

$\delta(G_k) = k + 1 = \frac{1}{3}(n+1)$, and $\delta(G_m) = k + 1 = \frac{n}{3}$.

The property of 2-connected is needed to avoid the graphs $G_p = K_1 + (K_p \cup K_{n-p-1})$, for $\frac{1}{3}(n+2) \leq p \leq \frac{1}{2}(n-1)$. These graphs are not 2-connected with

$$\text{bind}\{1,1,1\}(G_p) = \frac{n - 1 + n + 1}{n - p} = \frac{2n}{n-p} \geq \frac{2n}{n - \frac{1}{3}(n+2)} = \frac{3n}{n-1} > 3,$$

and $\delta(G_p) = p \geq \frac{1}{3}(n+2)$.

Theorem 2.7. *Let G be a graph with n even and $\text{bind}\{2,1,-1\}(G) \geq 4$. Then G has a 1-factor.*

The lower bound of 4 is best possible by considering the graphs $G_b = K_{b-2,b}$ for $b \geq 3$. These graphs do not have a 1-factor and, by taking $S = V(bK_1)$,

$$\text{bind}\{2,1,-1\}(G_a) = \frac{2(b-2) + 2b - 2 - 1}{b} = 4 - \frac{7}{b} < 4.$$

Theorem 2.8. *Let $k \geq 2$ be an integer. Let G be a graph with n vertices and suppose that, if k is odd, then n is even and G is connected. Suppose that*

$$\text{bind}\{1, k-1, 1\}(G) \geq 2k - 1,$$

and G has minimum degree

$$\delta \geq \frac{(k-1)(n+1)}{2k-1}.$$

Suppose further that, if $n < 4k-6$, then

$$\delta > n + 2k - 2\sqrt{kn+2}.$$

Then G has a k-factor.

A discussion of how the minimum degree bounds are best possible is provided in [6]. The bound $\mathrm{bind}\{1,k-1,1\}(G) \geq 2k-1$ is best possible by considering the graphs $G = K_{n-r-2(sk-1)} + (rK_1 \cup (sk-1)K_2)$, with $n = 2r + 2(2sk - s - 2)$ and $r, s \geq 1$. These graphs do not have a k-factor and, by letting $S = V(rK_1 + (sk-1)K_2)$, we have

$$\begin{aligned}\mathrm{bind}\{1,k-1,1\}(G) &= \frac{|N(S)| + (k-1)n + 1}{|S|} = \frac{n - r + (k-1)n + 1}{r + 2sk - 2} = \frac{kn - r + 1}{r + 2sk - 2} \\ &= \frac{2rk + 2k(2sk - s - 2) - r + 1}{r + 2sk - 2} = \frac{2k(r + 2sk - 2) - (r + 2sk - 2) - 1}{r + 2sk - 2} < 2k - 1.\end{aligned}$$

Theorem 2.9. *Let G be a graph with $n \geq 3$ and $\mathrm{bind}\{1,1,-1\} \geq 3$. Then G contains a triangle.*

The lower bound of 3 is best possible by considering the graphs $G_a = K_{a-1,a}$ for $a \geq 3$. These are triangle-free and, by taking $S = V(aK_1)$,

$$\mathrm{bind}\{1,1,-1\}(G_a) = \frac{a - 1 + 2a - 1 - 1}{a} = 3 - \frac{3}{a} < 3.$$

In the next section, we prove best monotone theorems for $b-(\alpha,\beta,\gamma)$binding. As a result, the degrees of G may be used as a way of verifying the parameterized binding number conditions of Theorems 2.2, 2.3, 2.4, or 2.5. Of course, we can also apply this approach to similar theorems that can be phrased in terms of appropriate parameterized binding numbers.

3. Degree sequence theorems

The following three theorems appear in [2]. The first is a best possible minimum degree condition for b-binding, the second and third are best monotone theorems for b-binding when $0 < b \leq 1$ or $b \geq 1$, respectively.

Theorem 3.1. *Let $b > 0$. If a graph G satisfies $\delta(G) \geq \frac{bn}{b+1}$, then $\mathrm{bind}(G) \geq b$.*

Theorem 3.2. *Let $0 < b \leq 1$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq \lceil b+1 \rceil = 2$. If*

(i) $d_i \leq \lceil bi \rceil - 1 \;\Rightarrow\; d_{n-\lceil bi \rceil+1} \geq n - i$, *for* $1 \leq i \leq \left\lfloor \frac{n}{b+1} \right\rfloor$, *and*

(ii) $d_{\lfloor \frac{n}{b+1} \rfloor + 1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor$,

then π is forcibly b-binding.

Theorem 3.3. *Let $b \geq 1$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq \lceil b+1 \rceil$. If*

(i) $d_i \leq n - \left\lceil \frac{n-i}{b} \right\rceil - 1 \;\Rightarrow\; d_{\lfloor \frac{n-i}{b} \rfloor + 1} \geq n - i$, *for* $1 \leq i \leq \left\lfloor \frac{n}{b+1} \right\rfloor$, *and*

(ii) $d_{\lfloor \frac{n}{b+1} \rfloor + 1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor$,

then π is forcibly b-binding.

Here, we prove the analogous theorems for the parameterized binding numbers. Each one reduces to the corresponding theorem for the standard binding number when $\alpha = 1$ and $\beta = \gamma = 0$.

Theorem 3.4. *Let $\alpha(\geq 1), \beta, \gamma \in \mathbb{Z}$ and $b > 0$. If a graph G on $n \geq 1$ vertices satisfies $\delta(G) \geq \frac{(b-\beta)n-\gamma}{b+\alpha}$ and $bn > \beta n + \gamma$, then $\mathrm{bind}\{\alpha,\beta,\gamma\}(G) \geq b$.*

To see that Theorem 3.4 is best possible, consider $G = K_{\lceil \frac{(b-\beta)n-\gamma}{b+\alpha} \rceil - 1} + \overline{K_{\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \rfloor + 1}}$. Then, $\delta = \left\lceil \frac{(b-\beta)n-\gamma}{b+\alpha} \right\rceil - 1$, and taking $S = V\left(\overline{K_{\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \rfloor + 1}}\right)$, we have

$$\mathrm{bind}\{\alpha,\beta,\gamma\}(G) \leq \frac{\alpha\left(\left\lceil \frac{(b-\beta)n-\gamma}{b+\alpha} \right\rceil - 1\right) + \beta n + \gamma}{\left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor + 1} < \frac{\alpha\left(\frac{(b-\beta)n-\gamma}{b+\alpha}\right) + \beta n + \gamma}{\frac{(\alpha+\beta)n+\gamma}{b+\alpha}} = b.$$

Proof of Theorem 3.4. Let $S \subseteq V(G)$ be an (α,β,γ)binding set of G. If $\mathrm{bind}\{\alpha,\beta,\gamma\}(G) < b$, then

$$|S| > \frac{\alpha|N(S)| + \beta n + \gamma}{b} \geq \frac{\alpha\delta(G) + \beta n + \gamma}{b}.$$

Let $Y = V(G) - N(S) \neq \emptyset$. Then $|S| \cap |N(Y)| = \emptyset$, and $|S| + |N(Y)| \leq n$. Thus $|S| \leq n - |N(Y)| \leq n - \delta(G)$, since $|N(Y)| \geq \delta(G)$. But $\frac{\alpha\delta(G)+\beta n+\gamma}{b} < |S| \leq n - \delta(G)$, or $\delta(G) < \frac{(b-\beta)n-\gamma}{b+\alpha}$. This completes the proof by contraposition. □

The next theorem we present is the best monotone theorem for b-(α,β,γ)binding when $0 \le b \le \alpha$.

Theorem 3.5. *Let $\alpha(\ge 1),\beta,\gamma \in \mathbb{Z}$. Let $0 < b \le \alpha$ and $\pi = (d_1 \le \cdots \le d_n)$ be a graphical sequence, with $n \ge \left\lceil \frac{b+\alpha-\gamma}{\alpha+\beta} \right\rceil$ and $bn > \beta n + \gamma$. If*

(i) $d_i \le \left\lceil \frac{bi-\beta n-\gamma}{\alpha} \right\rceil - 1 \Rightarrow d_{n-\lceil \frac{bi-\beta n-\gamma}{\alpha} \rceil+1} \ge n-i$, for $\left\lfloor \frac{\beta n+\gamma}{b} \right\rfloor + 1 \le i \le \left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor$, and

(ii) $d_{\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \rfloor+1} \ge n - \left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor$,

then π is forcibly b-(α,β,γ)binding.

Before proving Theorem 3.5, we show that it would be best monotone b-(α,β,γ)binding. Clearly it is monotone, and so it suffices to show that it is weakly-optimal.

If condition (i) fails for some i, then consider $G' = K_{\lceil \frac{bi-\beta n-\gamma}{\alpha} \rceil -1} + \left(K_{n-i-\lceil \frac{bi-\beta n-\gamma}{\alpha} \rceil+1} \cup \overline{K_i}\right)$, whose degrees majorize π. Taking $S = V(\overline{K_i})$, we find that

$$\operatorname{bind}\{\alpha,\beta,\gamma\}(G') \le \frac{\alpha|N(S)|+\beta n+\gamma}{|S|} = \frac{\alpha\left(\left\lceil \frac{bi-\beta n-\gamma}{\alpha} \right\rceil - 1\right)+\beta n+\gamma}{i} < \frac{\alpha\left(\frac{bi-\beta n-\gamma}{\alpha}\right)+\beta n+\gamma}{i} = b.$$

If condition (ii) fails, consider $G' = K_{n-\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \rfloor-1} + \overline{K_{\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \rfloor+1}}$, whose degrees majorize π. Taking $S = V\left(\overline{K_{\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \rfloor+1}}\right)$ we find that

$$\operatorname{bind}\{\alpha,\beta,\gamma\}(G') \le \frac{\alpha|N(S)|+\beta n+\gamma}{|S|} = \frac{\alpha\left(n-\left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor - 1\right)+\beta n+\gamma}{\left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor + 1} < \frac{\alpha\left(n-\frac{(\alpha+\beta)n+\gamma}{b+\alpha}\right)+\beta n+\gamma}{\frac{(\alpha+\beta)n+\gamma}{b+\alpha}} = b.$$

Proof of Theorem 3.5. Suppose π satisfies (i) and (ii), but has a realization G with $\operatorname{bind}\{\alpha,\beta,\gamma\}(G) < b$. Let $S \subseteq V(G)$ be an (α,β,γ)binding set of G. Partition $V(G)$ into $A \doteq S - N(S)$, $B \doteq N(S) - S$, $C \doteq S \cap N(S)$, and $D \doteq V(G) - (S \cup N(S))$, so that $S = A \cup C$ and $N(S) = B \cup C$. Clearly, A is an independent set. Since

$$\operatorname{bind}\{\alpha,\beta,\gamma\}(G) = \frac{\alpha(|B|+|C|)+\beta n+\gamma}{|A|+|C|} < b \le \alpha,$$

we have $|A| > |B| + (\beta n+\gamma)/\alpha \ge (\beta n+\gamma)/\alpha > 0$. So, $A \ne \emptyset$. Also, $N(A) \subseteq B$, and so $N(A) \ne V(G)$. If $|C| > 0$, then

$$\operatorname{bind}\{\alpha,\beta,\gamma\}(G) \le \frac{\alpha|N(A)|+\beta n+\gamma}{|A|} \le \frac{\alpha|B|+\beta n+\gamma}{|A|} < \frac{\alpha(|B|+|C|)+\beta n+\gamma}{|A|+|C|} = \operatorname{bind}\{\alpha,\beta,\gamma\}(G),$$

a contradiction. Hence, $C = \emptyset$, $\operatorname{bind}\{\alpha,\beta,\gamma\}(G) = \frac{\alpha|B|+\beta n+\gamma}{|A|} < b$, and $|A| \ge \lfloor (\beta n+\gamma)/b \rfloor + 1$.

We consider two cases.

Case 1. $|A| \ge \left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor + 1$.

In this case, $d_{\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \rfloor+1} \le d_{|A|} \le |B| = n-|A|-|D| \le n - \left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor - 1$, contradicting condition (ii).

Case 2. $\left\lfloor \frac{\beta n+\gamma}{b} \right\rfloor + 1 \le |A| \le \left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor$.

Since $\frac{\alpha|B|+\beta n+\gamma}{|A|} = \operatorname{bind}\{\alpha,\beta,\gamma\}(G) < b$, we have

$$n-|D| = |A|+|B| < |A| + \frac{b|A|-\beta n-\gamma}{\alpha} \le \frac{(b+\alpha)|A|-\beta n-\gamma}{\alpha} \le \frac{(\alpha+\beta)n+\gamma-\beta n-\gamma}{\alpha} = n,$$

so that $|D| \ne \emptyset$. So each vertex in A has degree at most $n-|A|-|D| \le n-|A|-1$, and each vertex in D has degree at most $|B|+|D|-1 = n-|A|-1$. Thus, $d_{|A|+|D|} \le n-|A|-1$. Set $i \doteq |A|$, so

$$\left\lfloor \frac{\beta n+\gamma}{b} \right\rfloor + 1 \le i \le \left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor.$$

Since $|B| < (b|A|-\beta n-\gamma)/\alpha$, we have $|B| \le \left\lceil \frac{b|A|-\beta n-\gamma}{\alpha} \right\rceil - 1 = \left\lceil \frac{bi-\beta n-\gamma}{\alpha} \right\rceil - 1$. But then

$$d_i = d_{|A|} \le |B| \le \left\lceil \frac{bi-\beta n-\gamma}{\alpha} \right\rceil - 1,$$

while

$$d_{n-\lceil \frac{bi-\beta n-\gamma}{\alpha} \rceil+1} = d_{n-\lceil \frac{b|A|-\beta n-\gamma}{\alpha} \rceil+1} \le d_{n-|B|} = d_{|A|+|D|} \le n-|A|-1 = n-i-1,$$

contradicting condition (i). □

We now give the best monotone $b-(\alpha,\beta,\gamma)$binding theorem for $b \geq \alpha$.

Theorem 3.6. *Let $\alpha(\geq 1), \beta, \gamma \in \mathbb{Z}$. Let $b \geq \alpha$ and $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq \left\lceil \frac{b+\alpha-\gamma}{\alpha+\beta} \right\rceil$ and $bn > \beta n + \gamma$. If*

(i) $d_i \leq n - \left\lfloor \frac{(\alpha+\beta)n-\alpha i+\gamma}{b} \right\rfloor - 1 \Rightarrow d_{\lfloor \frac{(\alpha+\beta)n-\alpha i+\gamma}{b} \rfloor+1} \geq n-i$, *for* $\max\left\{1, \left\lfloor \frac{\beta n+\gamma-(b-\alpha)n}{\alpha} \right\rfloor + 1\right\} \leq i \leq \left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor$, *and*

(ii) $d_{\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \rfloor+1} \geq n - \left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor$,

then π is forcibly b-(α,β,γ)binding.

Before proving Theorem 3.6, we show that it would be best monotone b-(α,β,γ)binding. Clearly it is monotone, and so it suffices to show that it is weakly-optimal.

If condition (i) fails for some i, then consider $G' = K_{n-\lfloor \frac{(\alpha+\beta)n-\alpha i+\gamma}{b} \rfloor-1} + \left(K_{\lfloor \frac{(\alpha+\beta)n-\alpha i+\gamma}{b} \rfloor-i+1} \cup \overline{K_i}\right)$, whose degrees majorize π. Taking $S = V\left(K_{\lfloor \frac{(\alpha+\beta)n-\alpha i+\gamma}{b} \rfloor-i+1} \cup \overline{K_i}\right)$, we find that

$$\operatorname{bind}\{\alpha,\beta,\gamma\}(G') \leq \frac{\alpha|N(S)|+\beta n+\gamma}{|S|} = \frac{\alpha(n-i)+\beta n+\gamma}{\left\lfloor \frac{(\alpha+\beta)n-\alpha i+\gamma}{b} \right\rfloor + 1} < \frac{(\alpha+\beta)n-\alpha i+\gamma}{\left(\dfrac{(\alpha+\beta)n-\alpha i+\gamma}{b}\right)} = b.$$

Here, condition (ii) is the same as in Theorem 3.5, so the proof that it is weakly-optimal is identical.

To prove Theorem 3.6, we will require the following lemma.

Lemma 3.1. *If π satisfies conditions (i) and (ii) of Theorem 3.6 for some $b \geq \alpha \geq 1$, then π is forcibly α-(α,β,γ)binding.*

Proof of Lemma 3.1. Assume π satisfies conditions (i) and (ii) of Theorem 3.6. To show π is forcibly α-(α,β,γ)binding, it suffices, by Theorem 3.5, to show

(1) $d_i \leq i - \left\lfloor \frac{\beta n+\gamma}{\alpha} \right\rfloor - 1 \Rightarrow d_{n-i-\lfloor \frac{\beta n+\gamma}{\alpha} \rfloor+1} \geq n-i$, for $\left\lfloor \frac{\beta n+\gamma}{\alpha} \right\rfloor + 1 \leq i \leq \left\lfloor \frac{(\alpha+\beta)n+\gamma}{2\alpha} \right\rfloor$, and

(2) $d_{\lfloor \frac{(\alpha+\beta)n+\gamma}{2\alpha} \rfloor+1} \geq n - \left\lfloor \frac{(\alpha+\beta)n+\gamma}{2\alpha} \right\rfloor$.

If $\left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor + 1 \leq i \leq \left\lfloor \frac{(\alpha+\beta)n+\gamma}{2\alpha} \right\rfloor$, then condition (ii) in Theorem 3.6 gives

$$d_i \geq d_{\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \rfloor+1} \geq n - \left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor \geq n-(i-1).$$

Note that $i \leq \frac{(\alpha+\beta)n+\gamma}{2\alpha}$ implies $n-i+1 > i - \left\lfloor \frac{\beta n+\gamma}{\alpha} \right\rfloor$. Thus $d_i \geq n-i+1 > i - \left\lfloor \frac{\beta n+\gamma}{\alpha} \right\rfloor$, so that (1) is vacuously satisfied.

Assume instead that $\left\lfloor \frac{\beta n+\gamma}{\alpha} \right\rfloor + 1 \leq i \leq \left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor$. Since $b \geq \alpha$, we also know $i \geq \left\lfloor \frac{\beta n+\gamma}{b} \right\rfloor + 1$, i.e., the lower bound on i in (i) of Theorem 3.6. Now,

$$i - \left\lfloor \frac{\beta n+\gamma}{\alpha} \right\rfloor = i - \left\lfloor \frac{(\alpha+\beta)n-\alpha i+\gamma-\alpha n+\alpha i}{\alpha} \right\rfloor = n - \left\lfloor \frac{(\alpha+\beta)n-\alpha i+\gamma}{\alpha} \right\rfloor \leq n - \left\lfloor \frac{(\alpha+\beta)n-\alpha i+\gamma}{b} \right\rfloor.$$

If

$$d_i \leq i - \left\lfloor \frac{\beta n+\gamma}{\alpha} \right\rfloor \leq n - \left\lfloor \frac{(\alpha+\beta)n-\alpha i+\gamma}{b} \right\rfloor,$$

then (i) of Theorem 3.6 implies

$$d_{n-i-\lfloor \frac{\beta n+\gamma}{\alpha} \rfloor+1} \geq d_{\lfloor \frac{(\alpha+\beta)n-\alpha i+\gamma}{b} \rfloor+1} \geq n-i.$$

So (1) holds for this range of i values as well.

For (2), note that by $b \geq \alpha$ and condition (ii) in Theorem 3.6 we have

$$d_{\lfloor \frac{(\alpha+\beta)n+\gamma}{2\alpha} \rfloor+1} \geq d_{\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \rfloor+1} \geq n - \left\lfloor \frac{(\alpha+\beta)n+\gamma}{b+\alpha} \right\rfloor \geq n - \left\lfloor \frac{(\alpha+\beta)n+\gamma}{2\alpha} \right\rfloor,$$

which is (2). Thus, π is forcibly α-(α,β,γ)binding. □

Proof of Theorem 3.6. Suppose π satisfies (i) and (ii), but has a realization G with $\text{bind}\{\alpha,\beta,\gamma\}(G) < b$. Let S be a largest (α,β,γ)-binding set of G, so that $S \neq \emptyset$, $N(S) \neq V(G)$, and $\text{bind}\{\alpha,\beta,\gamma\}(G) = \frac{\alpha|N(S)|+\beta n+\gamma}{|S|} < b$. Partition $V(G)$ into $A \doteq S - N(S)$, $B \doteq N(S) - S$, $C \doteq S \cap N(S)$, and $D \doteq V(G) - (S \cup N(S))$, so that $S = A \cup C$ and $N(S) = B \cup C$. Clearly, A is an independent set.

Claim 1. $|C| \geq |D|$.

Proof of Claim 1: Suppose $|D| > |C|$. Define $S' \doteq A \cup D$, so $N(S') \subseteq B \cup D$. Since $N(S) = B \cup C \neq V(G)$, we have $S' \neq \emptyset$. Also, $S = A \cup C \neq \emptyset$, so that $N(S') \neq V(G)$. Since $|D| > |C|$ and π is forcibly α-(α,β,γ)binding by Lemma 3.1, we have

$$\frac{\alpha(|B|+|D|)+\beta n+\gamma}{|A|+|D|} \leq \frac{\alpha(|B|+|C|)+\beta n+\gamma}{|A|+|C|}.$$

Thus,

$$\begin{aligned}\text{bind}\{\alpha,\beta,\gamma\}(G) &\leq \frac{\alpha|N(S')|+\beta n+\gamma}{|S'|} \leq \frac{\alpha(|B|+|D|)+\beta n+\gamma}{|A|+|D|}\\ &\leq \frac{\alpha(|B|+|C|)+\beta n+\gamma}{|A|+|C|} = \frac{\alpha|N(S)|+\beta n+\gamma}{|S|} = \text{bind}\{\alpha,\beta,\gamma\}(G),\end{aligned}$$

and S' is an (α,β,γ)binding set of G. However, $|D| > |C|$ implies $|S'| > |S|$, contradicting our choice of S. ■

Note that each vertex in A has degree at most $|B| = n - (|A|+|C|+|D|)$, and each vertex in D has degree at most $|B|+|D|-1 = n-(|A|+|C|)$. Therefore, since $|A|+|D| \geq 1$ (otherwise, $N(S) = V(G)$),

$$d_{|A|+|D|} \leq n-(|A|+|C|) \leq n-(|A|+|D|). \tag{1}$$

Also, when $C \neq \emptyset$, each vertex in C has degree at most $|B|+|C|-1 = n-(|A|+|D|+1) \geq |B|$, and so

$$d_{|A|+|C|} < n-(|A|+|D|), \text{ if } |C| \geq 1. \tag{2}$$

Since $|A|+|C| \leq n$ and $|B|+|C| = n-(|A|+|D|)$, we have

$$\frac{\alpha(n-(|A|+|D|))+\beta n+\gamma}{n} \leq \frac{\alpha(|B|+|C|)+\beta n+\gamma}{|A|+|C|} < b.$$

Thus $|A|+|D| > \frac{\beta n+\gamma-(b-\alpha)n}{\alpha}$, or $|A|+|D| \geq \max\left\{1, \left\lfloor\frac{\beta n+\gamma-(b-\alpha)n}{\alpha}\right\rfloor+1\right\}$.

Case 1. $|A|+|D| \geq \left\lfloor\frac{(\alpha+\beta)n+\gamma}{b+\alpha}\right\rfloor+1$.

By (1),

$$d_{\left\lfloor\frac{(\alpha+\beta)n+\gamma}{b+\alpha}\right\rfloor+1} \leq d_{|A|+|D|} \leq n-(|A|+|D|) < n-\left\lfloor\frac{(\alpha+\beta)n+\gamma}{b+\alpha}\right\rfloor,$$

contradicting condition (ii).

Case 2. $\max\left\{1, \left\lfloor\frac{\beta n+\gamma-(b-\alpha)n}{\alpha}\right\rfloor+1\right\} \leq |A|+|D| \leq \left\lfloor\frac{(\alpha+\beta)n+\gamma}{b+\alpha}\right\rfloor$.

Note that $|C| \geq 1$, else $|D| = |C| = 0$ by Claim 1, and $b > \frac{\alpha|N(S)|+\beta n+\gamma}{|S|} = \frac{\alpha(n-|A|)+\beta n+\gamma}{|A|}$, or $|A| > \frac{(\alpha+\beta)n+\gamma}{b+\alpha}$, contradicting the current case. Since $b > \frac{\alpha|N(S)|+\beta n+\gamma}{|S|}$, we have

$$|A|+|C| = |S| > \frac{\alpha|N(S)|+\beta n+\gamma}{b} = \frac{\alpha(n-(|A|+|D|))+\beta n+\gamma}{b},$$

or

$$|A|+|C| \geq \left\lfloor\frac{\alpha(n-(|A|+|D|))+\beta n+\gamma}{b}\right\rfloor+1. \tag{3}$$

Set $i \doteq |A|+|D|$, so $\max\left\{1, \left\lfloor\frac{\beta n+\gamma-(b-\alpha)n}{\alpha}\right\rfloor+1\right\} \leq i \leq \left\lfloor\frac{(\alpha+\beta)n+\gamma}{b+\alpha}\right\rfloor$. By (1) and (3), we have

$$d_i = d_{|A|+|D|} \leq n-(|A|+|C|) \leq n-\left\lfloor\frac{\alpha(n-(|A|+|D|))+\beta n+\gamma}{b}\right\rfloor-1 = n-\left\lfloor\frac{(\alpha+\beta)n-\alpha i+\gamma}{b}\right\rfloor-1.$$

While by (2) and (3),

$$d_{\left\lfloor\frac{(\alpha+\beta)n-\alpha i+\gamma}{b}\right\rfloor+1} = d_{\left\lfloor\frac{\alpha(n-(|A|+|D|))+\beta n+\gamma}{b}\right\rfloor+1} \leq d_{|A|+|C|} < n-(|A|+|D|) = n-i.$$

The above contradicts condition (i). □

As examples of additional degree sequence theorems we can produce, consider the following corollaries, which combine Theorem 3.6 with Theorems 2.8 and 2.9, respectively.

Corollary 3.1. *Let $k \geq 2$ be an integer. Let G be a graph with degree sequence $\pi(G) = (d_1 \leq \cdots \leq d_n)$ and suppose that, if k is odd, then n is even and G is connected. If*

(i) $d_1 \geq \frac{(k-1)(n+1)}{2k-1}$, *and if* $n < 4k-6$, *then* $d_1 > n + 2k - 2\sqrt{kn+2}$,

(ii) $d_i \leq n - \left\lfloor \frac{kn-i+1}{2k-1} \right\rfloor - 1 \Rightarrow d_{\left\lfloor \frac{kn-i+1}{2k-1} \right\rfloor + 1} \geq n - i$, *for* $1 \leq i \leq \left\lfloor \frac{kn+1}{2k-1} \right\rfloor$, *and*

(iii) $d_{\lfloor \frac{kn+1}{2k} \rfloor + 1} \geq n - \lfloor \frac{kn+1}{2k} \rfloor$,

then G contains a k-factor.

Corollary 3.2. *Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence with $n \geq 3$. If*

(i) $d_i \leq n - \lfloor \frac{2n-i-1}{3} \rfloor - 1 \Rightarrow d_{\lfloor \frac{2n-i-1}{3} \rfloor + 1} \geq n - i$, *for* $1 \leq i \leq \lfloor \frac{2n-1}{4} \rfloor$, *and*

(ii) $d_{\lfloor \frac{2n-1}{4} \rfloor + 1} \geq n - \lfloor \frac{2n-1}{4} \rfloor$,

then π forcibly contains a triangle.

It is known that $\frac{3}{2}$-binding implies the graph contains a triangle, and that this is best possible [4]. Thus, we can also generate a monotone theorem for containing a triangle by setting $b = \frac{3}{2}$ in Theorem 3.3. This yields

Corollary 3.3. *Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence with $n \geq 3$. If*

(i) $d_i \leq n - \lfloor \frac{2n-2i}{3} \rfloor - 1 \Rightarrow d_{\lfloor \frac{2n-2i}{3} \rfloor + 1} \geq n - i$, *for* $1 \leq i \leq \lfloor \frac{2n}{5} \rfloor$, *and*

(ii) $d_{\lfloor \frac{2}{5} \rfloor + 1} \geq n - \lfloor \frac{2n}{5} \rfloor$,

then π forcibly contains a triangle.

It can be shown that any graphical sequence that satisfies the conditions of Corollary 3.3 will also satisfy the conditions of Corollary 3.2. So, Corollary 3.2 would be considered a stronger result than Corollary 3.3. In general, the results for appropriate parameterized binding numbers will improve analogous results for the standard binding number.

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