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Research Article

Perimeter of a palindromic composition

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Abstract

We show that the number of palindromic compositions having perimeter $2n$ is asymptotic to $\frac{1+\eta}{\sqrt{22\pi n}}\sqrt{1+6\eta-3\eta^2}\eta^{-n-1}$, where $\eta=\frac{\sqrt[3]{17+3\sqrt{33}}}{3}-\frac{2}{3\sqrt[3]{17+3\sqrt{33}}}-\frac{1}{3}$.

Keywords: palindromic compositions; perimeter; generating functions.

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1. Introduction

Let n be any positive integer. A *composition* of n is a finite sequence $\sigma=\sigma_1\cdots\sigma_m$ of positive integers called parts such that $n=\sum_{i=1}^m \sigma_i$. Number m represents the number of parts of composition σ. A composition σ is palindromic if $\sigma_i=\sigma_{m+1-i}$ for all $i=1,2,\ldots,\lfloor m/2\rfloor$. Compositions and palindromic compositions have previously been extensively studied, see [8] and references contained therein. For any composition $\sigma=\sigma_1\cdots\sigma_m$, we associate a *bargraph* which is a column convex polyomino on a planar lattice grid $\mathbb{Z}^2$, where the lower edge lies on the x-axis and the i-th column contains exactly σ_i 1×1 square cells, for $i=1,2,\ldots,m$. For example, Figure 1 represents the bargraph of the palindromic composition 1432341. The perimeter is the number of edges on the boundary of the bargraph. For instance, the perimeter of the palindromic composition 1432341 is 26. The enumeration of bargraphs according to their area and perimeter has been well studied and is a basic problem in combinatorics (for instance, see [1–3, 5, 9, 10]). In particular, in [5] (see also [6]), it was shown that the generating function for the number of bargraphs (compositions) according to half the perimeter (i.e., semi-perimeter) is given by

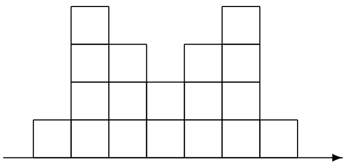

Figure 1: Bargraph of the palindromic composition 1432341.

$$\frac{1-2q-q^2-\sqrt{(1-2q-q^2)^2-4q^3}}{2q}.$$

The asymptotics of the coefficient of q^n in this generating function has been considered, and the dominant singularity ρ, which is the positive root of $1-4\rho+2\rho^2+\rho^4=0$, has been computed. By singularity analysis (for example, see [7]) the number of compositions with perimeter $2n$ is asymptotic to

$$\frac{1}{2}\sqrt{\frac{1-\rho-\rho^3}{\pi\rho n^3}}\rho^{-n},$$

where

$$\rho=\frac{\sqrt[3]{26+6\sqrt{33}}}{3}-\frac{8}{3\sqrt[3]{26+6\sqrt{33}}}-\frac{1}{3}.$$

In this note, we provide a similar result in the case of palindromic compositions. More precisely, we prove the theorem given on the next page.

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Theorem 1.1. *The generating function for the number of palindromic compositions according to the number of parts (counted by y) and the half of the perimeter (counted by q) is given by*

$$\begin{aligned}&-\frac{1+q-q(1-q)y}{2qy}+\frac{1+qy}{2qy(1-q-(1+q)q^2y^2)}\sqrt{(1-q^2)((1-q^2y^2)^2-q^2(1+q^2y^2)^2)}\\&=1+yq^2+y(y+1)q^3+y(y^2+y+1)q^4+y(y^3+2y^2+y+1)q^5+(y^5+2y^4+4y^3+y^2+y)q^6\\&\quad+y(y^5+3y^4+4y^3+5y^2+y+1)q^7+\cdots.\end{aligned}$$

Moreover, the number of palindromic compositions having perimeter $2n$ is asymptotic to

$$\frac{1+\eta}{\sqrt{22\pi n}}\sqrt{1+6\eta-3\eta^2}\eta^{-n-1},$$

where

$$\eta=\frac{\sqrt[3]{17+3\sqrt{33}}}{3}-\frac{2}{3\sqrt[3]{17+3\sqrt{33}}}-\frac{1}{3}.$$

To give the reader an insight into the underlying geometry of the above generating function, the following example is the full set of 4 palindromic compositions represented in the term $4y^4q^7$. These all have perimeter 14 (i.e., twice 7) and exactly 4 parts:

Example 1.1. $3333, 2332, 2112, 1331$ *are the four palindromic compositions counted by the term* $4y^4q^7$.

2. Proof of Theorem 1.1

Let $P(x,y,q)$ be the generating function for the number of palindromic compositions σ according to the sum of the parts x, the number of parts y, and the half the perimeter q of σ. In order to study this generating function, we define $P_1(x,y,q;a)$ (respectively, $P_2(x,y,q;a)$) to be the generating function of the number of palindromic compositions σ according to the same variable trackers x,y and q of each σ but such that the leftmost part of σ is a which appears exactly once (respectively, at least twice). Define

$$P_i(t;x,y,q)=\sum_{a\geq1}P_i(x,y,q;a)t^a,\quad i=1,2.$$

Since a palindromic composition σ with the leftmost part being a and appearing exactly once implies that $\sigma=a$, we have that

$$P_1(t;x,y,q)=\frac{q^2xyt}{1-qxt}.$$

Thus, from the definitions, we have

$$P(x,y,q)=1+P_1(1;x,y,q)+P_2(1;x,y,q)=1+\frac{q^2xy}{1-qx}+P_2(1;x,y,q). \tag{1}$$

For any palindromic composition σ such that the leftmost part of σ is a which appears at least twice, we have the following cases

- $\sigma=aa$;
- $\sigma=aba$ with $b\geq1$;
- $\sigma=ab\sigma'ba$ with $b\geq1$, where σ' is any palindromic composition.

We translate these to generating functions by considering either $1\leq b\leq a$ or $b\geq a+1$ and obtain

$$\begin{aligned}P_2(x,y,q;a)&=q^{a+2}x^{2a}y^2+\sum_{b=1}^{a}q^{2a-b+3}x^{2a+b}y^3+\sum_{b\geq a+1}q^{b+3}x^{2a+b}y^3\\&\quad+\sum_{b=1}^{a}q^{2a-2b+2}x^{2a}y^2P_2(x,y,q;b)+\sum_{b\geq a+1}q^2x^{2a}y^2P_2(x,y,q;b).\end{aligned}$$

After multiplying by t^a and summing over all $a\geq1$, while defining

$$P_2(t;x,y,q):=\sum_{a=1}^{\infty}P_2(x,y,q;a)t^a,$$

we get

$$P_2(t;x,y,q) = \frac{q^3x^2y^2t}{1-qx^2t} + \frac{q^2y^2}{1-q^2x^2t}P_2(x^2t;x,y,q)$$
$$+ \frac{q^2y^2}{1-x^2t}(x^2tP_2(1;x,y,q) - P_2(x^2t;x,y,q)) + \frac{(1-q^3x^3t)q^4x^3y^3t}{(1-qx)(1-qx^3t)(1-q^2x^2t)},$$

which leads to

$$P_2(t;x,y,q) = \frac{q^3x^2y^2t}{1-qx^2t} + \frac{(1-q^3x^3t)q^4x^3y^3t}{(1-qx)(1-qx^3t)(1-q^2x^2t)} + \frac{q^2x^2y^2t}{1-x^2t}P_2(1;x,y,q)$$
$$+ \frac{q^2(q^2-1)x^2y^2t}{(1-x^2t)(1-q^2x^2t)}P_2(x^2t;x,y,q). \tag{2}$$

Hence, by iterating (2) an infinite number of times under the assumption that $|x|<1$, we obtain

$$P_2(t;x,y,q) = \sum_{j\geq 1}\frac{q^{2j+1}(q^2-1)^{j-1}x^{j(j+1)}y^{2j}t^j}{\prod_{i=1}^{j-1}(1-x^{2i}t)(1-q^2x^{2i}t)}\left(\frac{1}{1-qx^{2j}t} + \frac{(1-q^3x^{2j+1}t)qxy}{(1-qx)(1-qx^{2j+1}t)(1-q^2x^{2j}t)}\right)$$
$$+ \sum_{j\geq 1}\frac{q^{2j}(q^2-1)^{j-1}x^{j(j+1)}y^{2j}t^j}{\prod_{i=1}^{j}(1-x^{2i}t)\prod_{i=1}^{j-1}(1-q^2x^{2i}t)}P_2(1;x,y,q).$$

By taking $t=1$, solving for $P_2(1;x,y,q)$ and using (1), we have the following result.

Theorem 2.1. *The generating function $P(x,y,q)$ is given by*

$$P(x,y,q) = 1 + \frac{q^2xy}{1-qx} + \frac{\sum_{j\geq 1}\frac{q^{2j+1}(q^2-1)^{j-1}x^{j(j+1)}y^{2j}}{\prod_{i=1}^{j-1}(1-x^{2i})(1-q^2x^{2i})}\left(\frac{1}{1-qx^{2j}} + \frac{(1-q^3x^{2j+1})qxy}{(1-qx)(1-qx^{2j+1})(1-q^2x^{2j})}\right)}{1-\sum_{j\geq 1}\frac{q^{2j}(q^2-1)^{j-1}x^{j(j+1)}y^{2j}}{\prod_{i=1}^{j}(1-x^{2i})\prod_{i=1}^{j-1}(1-q^2x^{2i})}}$$
$$= 1 + yq^2x + yq^3(y+1)x^2 + yq^4(y^2+1)x^3 + q^4y(qy^3+qy^2+q+y)x^4 + yq^6(y^2+1)^2x^5$$
$$+ yq^5(q^2y^5+q^2y^4+q^2y^3+q^2y^2+qy^3+q^2+y^2+y)x^6 + \cdots.$$

Example 2.1. *In Example 1.1, we listed all palindromic compositions with perimeter 14 (i.e., $q=7$). In $P(x,y,q)$ variable x marks the area. Of the four compositions in Example 1.1, only* 2112 *has x exponent 6 and thus the coefficient of $y^4x^6q^7$ in $P(x,y,q)$ is 1 as expected.*

Now, by considering (2) with $x=1$, we have

$$\left(1 - \frac{q^2(q^2-1)y^2t}{(1-t)(1-q^2t)}\right)P_2(t;1,y,q) = \frac{q^3y^2t}{1-qt} + \frac{(1-q^3t)q^4y^3t}{(1-q)(1-qt)(1-q^2t)} + \frac{q^2y^2t}{1-t}P_2(1;1,y,q). \tag{3}$$

We solve this functional equation through an application of the kernel method (see [4] for an introduction). To that end, define

$$t_0 = t_0(y,q) = \frac{1+q^2+(q^2-1)q^2y^2 - \sqrt{(1-q^2)((1-q^2y^2)^2 - q^2(1+q^2y^2)^2)}}{2q^2},$$

which is the root of

$$1 - \frac{q^2(q^2-1)y^2t_0}{(1-t_0)(1-q^2t_0)} = 0.$$

Then by substitution $t=t_0$ in (3), we obtain

$$P_2(1;1,y,q) = \frac{(t_0-1)q}{1-qt_0} + \frac{(1-q^3t_0)(t_0-1)q^2y}{(1-q)(1-qt_0)(1-q^2t_0)}.$$

Hence, by (1), we complete the proof of the first part of Theorem 1.1. By performing a singularity analysis (see [7, Section VI] for a comprehensive review), we complete the second part of Theorem 1.1.

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