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Research Article

Some new results on the edge-strength and strength of graphs

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Abstract

An edge numbering f of a graph G of size m is a labeling that assigns distinct elements of the set $\{1, 2, \ldots, m\}$ to the edges of G. The edge-strength estr(G) of G is defined by $\mathrm{estr}(G) = \min\{\mathrm{estr}_f(G) \mid f \text{ is an edge numbering of } G\}$, where $\mathrm{estr}_f(G) = \max\{f(e_1) + f(e_2) \mid e_1, e_2 \text{ are adjacent edges of } G\}$. In this paper, formulas for $\mathrm{estr}(G)$ are presented when G is either the forest whose components are stars of order at least three or the complete bipartite graph whose partite sets consist of at least two vertices. The edge-strength of a graph G is the strength of the line graph of G, and thus this work extends the known results about the edge-strength and strength of graphs.

Keywords: edge-strength; strength; line graph; graph labeling; combinatorial optimization.

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1. Introduction

Only graphs without loops or multiple edges are considered in this paper. Undefined graph theoretical notation and terminology can be found in [2] or [19].

If S is a nonempty subset of the vertex set $V(G)$ of a graph G, then the subgraph $\langle S \rangle$ of G *induced by* S is the graph having vertex set S and whose edge set consists of those edges of G incident with two elements of S. A subgraph H of G is called *induced* if $H = \langle S \rangle$ for some subset S of $V(G)$.

The *degree of* a *vertex* v in a graph G is the number of edges of G incident with v, which is denoted by $\deg v$. The *minimum degree* of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$. A graph G is *regular* of degree r if $\deg v = r$ for each $v \in V(G)$. Such graphs are called r*-regular*. An r-regular subgraph H of a graph G is an r*-factor* of G if $V(H) = V(G)$.

The graph with n vertices and no edges is referred to as the *empty graph*. For integers $k \geq 2$, a graph G is a k*-partite graph* if $V(G)$ can be partitioned into k nonempty subsets $V_1, V_2, \ldots, V_k$ (called *partite sets*) such that no edge of G joins vertices in the same set. For $k = 2$, such graphs are called *bipartite graphs*. If G is a k-partite graph with partite sets $V_1, V_2, \ldots, V_k$ such that every vertex of V_i is joined to every vertex of V_j, where $1 \leq i < j \leq k$, then G is called a *complete* k*-partite graph*. If $|V_i| = n_i$, then this graph is denoted by $K_{n_1, n_2, \ldots, n_k}$ and called a *complete multipartite graph*. If $n_i = t$ for all i, then the complete k-partite graph is also denoted by $K_{k(t)}$. A *complete bipartite graph* with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. The graph $K_{1,n}$ is called a *star*.

We use the notation $[a, b]$ for the interval of integers x such that $a \leq x \leq b$. A *numbering* f of a graph G of order n is a labeling that assigns distinct elements of the set $[1, n]$ to the vertices of G. The *strength* $\mathrm{str}(G)$ of G is defined by

$$\mathrm{str}(G) = \min\{\mathrm{str}_f(G) \mid f \text{ is a numbering of } G\},$$

where $\mathrm{str}_f(G) = \max\{f(u) + f(v) \mid uv \in E(G)\}$. This type of numberings was introduced in [8] as a generalization of the problem of finding whether a graph is super edge-magic or not (see [3] for the definition of a super edge-magic graph, and also consult either [1] or [4] for alternative and often more useful definitions of the same concept). If G is an empty graph, then $\mathrm{str}(G)$ is undefined (or we could define $\mathrm{str}(G) = +\infty$). For further detail about the strength of graphs, the authors suggest that the reader consult the results in [6, 9–13].

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Another problem concerns labeling the edges of a graph in terms of its size rather than its order. An *edge numbering* f of a graph G of size m is a labeling that assigns distinct elements of the set $[1, m]$ to the edges of G. The *edge-strength* estr(G) of G is defined by

$$\mathrm{estr}\,(G) = \min\{\mathrm{estr}_f\,(G) \mid f \text{ is an edge numbering of } G\},$$

where $\mathrm{estr}_f\,(G) = \max\{f\,(e_1) + f\,(e_2) \mid e_1, e_2 \text{ are adjacent edges of } G\}$. The determination of $\mathrm{estr}\,(G)$ can be transformed into a problem dealing with strengths, namely, from the definitions it is immediate that

$$\mathrm{estr}\,(G) = \mathrm{str}\,(L\,(G)),$$

where $L\,(G)$ is the line graph of G. The *line graph* $L\,(G)$ of a graph G is the graph whose vertices can be put in one-to-one correspondence with the edges of G in such a way that two vertices of $L\,(G)$ are adjacent if and only if the corresponding edges of G are adjacent. The line graph $L\,(G)$ is empty if and only if every component of G is either K_1 or K_2. In such a case, $\mathrm{estr}\,(G)$ is undefined (or we could define $\mathrm{estr}\,(G) = +\infty$). This type of numberings was recently studied in [14, 15].

In the following definitions, we assume that G_1 and G_2 are two graphs with disjoint vertex sets. The *union* $G = G_1 \cup G_2$ has $V\,(G) = V\,(G_1) \cup V\,(G_2)$ and $E\,(G) = E\,(G_1) \cup E\,(G_2)$. The *cartesian product* $G = G_1 \times G_2$ has $V\,(G) = V\,(G_1) \times V\,(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E\,(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E\,(G_1)$.

In this paper, we provide formulas for $\mathrm{estr}\,(G)$ and $\mathrm{str}\,(H)$ when $G = K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_k}$ and $H = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}$ $(2 \leq n_1 \leq n_2 \leq \cdots \leq n_k)$. We also present formulas for $\mathrm{estr}\,(G)$ and $\mathrm{str}\,(H)$ when $G = K_{m,n}$ and $H = K_m \times K_n$ $(2 \leq n < m)$. These extend what was known about the edge-strength and strength of graphs.

There are other related parameters that have been studied in the area of graph labelings. Excellent sources for more information on this topic are found in the extensive survey by Gallian [5], which also includes information on other kinds of graph labeling problems as well as their applications.

2. Results on forests and complete bipartite graphs

In this section, we study the edge-strength of certain forests and complete bipartite graphs as well as the strength of the corresponding graphs.

In the process of settling the problem (proposed in [8]) of finding sufficient conditions for a graph G of order n with $\delta\,(G)$ to ensure that $\mathrm{str}\,(G) = n + \delta\,(G)$, the following class of graphs was defined in [11]. For integers $n \geq 2$, let F_n be the graph with $V\,(F_n) = \{v_i \mid i \in [1, n]\}$ and $E\,(F_n) = \{v_iv_j \mid i \in [1, \lfloor n/2 \rfloor] \text{ and } j \in [1 + i, n + 1 - i]\}$. Let $S = \{v_i \mid i \in [1, \lfloor n/2 \rfloor + 1]\}$. Then the subgraph H of F_n induced by S has the vertex set $V\,(H) = \{v_i \mid i \in [1, \lfloor n/2 \rfloor + 1]\}$ and the edge set

$$E\,(H) = \{v_iv_j \mid i \in [1, \lfloor n/2 \rfloor] \text{ and } i < j \leq \lfloor n/2 \rfloor + 1\}.$$

Thus, $|V\,(H)| = \lfloor n/2 \rfloor + 1$ and

$$|E\,(H)| = \lfloor n/2 \rfloor + (\lfloor n/2 \rfloor - 1) + \cdots + 1 = \binom{\lfloor n/2 \rfloor + 1}{2}.$$

Consequently, $H = K_{\lfloor n/2 \rfloor + 1}$ and $K_{\lfloor n/2 \rfloor + 1} \subseteq F_n$. This gives us the following lemma.

Lemma 2.1. *For every integer $n \geq 2$, F_n contains $K_{\lfloor n/2 \rfloor + 1}$ as a subgraph.*

The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V\,(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. The following result established in [11] provides a necessary and sufficient condition for a graph G of order n to hold the inequality $\mathrm{str}\,(G) \leq 2n - k$, where $k \in [2, n - 1]$.

Theorem 2.1. *Let G be a graph of order n. Then $\mathrm{str}\,(G) \leq 2n - k$ if and only if $\overline{G}$ contains F_k as a subgraph, where $k \in [2, n - 1]$.*

The following theorem that is a dual of Theorem 2.1 provides a necessary and sufficient condition for a graph G of order n to hold the inequality $\mathrm{str}\,(G) \geq 2n - k + 1$, where $k \in [2, n - 1]$.

Theorem 2.2. *Let G be a graph of order n. Then $\mathrm{str}\,(G) \geq 2n - k + 1$ if and only if $\overline{G}$ does not contain F_k as a subgraph, where $k \in [2, n - 1]$.*

The preceding result proves to be useful in our study of edge-strength of graphs.

The next two results were established in [14].

Theorem 2.3. *For every integer $n \geq 2$,*

$$\operatorname{estr}(K_{1,n}) = 2n - 1.$$

Theorem 2.4. *For every two integers $m \geq 2$ and $n \geq 2$,*

$$\operatorname{estr}(K_{1,m} \cup K_{1,n}) = 2(m+n) - 3.$$

The above results are now generalized to forests whose components are stars of order at least three. Our proof uses the concepts of the clique number and the independence number. The *clique number* $\omega(G)$ of a graph G is maximum order among complete subgraphs of G. A set of vertices in a graph G is *independent* if no two of them are adjacent. The *independence number* $\beta(G)$ of a graph G is the maximum cardinality among the independence sets of vertices of G. Clearly, $\omega(G) = \beta\left(\overline{G}\right)$ for every graph G.

Theorem 2.5. *For every integers $n_1, n_2, \ldots, n_k$ with $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$,*

$$\operatorname{estr}(K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_k}) = 2(n_1 + n_2 + \cdots n_k) - 2k + 1.$$

Proof. Let $G = K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_k}$ and $H = L(G)$. Then $H = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}$ so that $\overline{H} = K_{n_1,n_2,\ldots,n_k}$. From the definitions, it is clear that a clique in a graph is an independent set in its complement. It follows that

$$\omega\left(\overline{H}\right) = \beta\left(\overline{\overline{H}}\right) = \beta(H) = k.$$

Thus, $\overline{H}$ does not contain K_{k+1}. However, by Lemma 2.1, F_{2k} contains K_{k+1} as a subgraph. Consequently, $\overline{H}$ does not contain F_{2k} as a subgraph. It is now immediate from Theorem 2.2 that

$$\operatorname{estr}(G) = \operatorname{str}(H) \geq 2(n_1 + n_2 + \cdots n_k) - 2k + 1.$$

To show that $\operatorname{estr}(G) = 2(n_1 + n_2 + \cdots n_k) - 2k + 1$, it suffices to verify the existence of a numbering f of H for which $\operatorname{str}_f(H) = 2(n_1 + n_2 + \cdots n_k) - 2k + 1$. For each $i \in [1,k]$, let $V(K_{n_i}) = \left\{x_i^s \mid s \in [1, n_i]\right\}$ and $E(K_{n_i}) = \left\{x_i^s x_i^t \mid 1 \leq s < t \leq n_i\right\}$. Then H can be defined as the graph with $V(H) = \bigcup_{i=1}^{k} V(K_{n_i})$ and $E(H) = \bigcup_{i=1}^{k} E(K_{n_i})$. Further, let $\sigma_i = n_1 + n_2 + \cdots n_i$, where $i \in [1,k]$. Then the labeling $f : V(H) \to [1, \sigma_k]$ such that

$$f(w) = \begin{cases} \sigma_k + 1 - i & \text{if } w = x_i^1 \text{ and } i \in [1,k] \\ \sigma_i + 2 - s - i & \text{if } w = x_i^s,\ i \in [1,k] \text{ and } s \in [2, n_i] \end{cases}$$

has the property that

$$\begin{aligned} \operatorname{str}_f(H) &= \max\{f(u) + f(v) \mid uv \in E(H)\} \\ &= f\left(x_k^1\right) + f\left(x_k^2\right) = (\sigma_k + 1 - k) + (\sigma_k - k) \\ &= 2\sigma_k - 2k + 1 = 2(n_1 + n_2 + \cdots n_k) - 2k + 1. \end{aligned}$$

□

The proof of the preceding theorem also provides the following corollary. This significantly extends the result found in [8] that $\operatorname{str}(K_n) = 2n - 1$ for every integer $n \geq 2$.

Corollary 2.1. *For every integers $n_1, n_2, \ldots, n_k$ with $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$,*

$$\operatorname{str}(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}) = 2(n_1 + n_2 + \cdots n_k) - 2k + 1.$$

Unless $m = n = 4$, the line graph of $K_{m,n}$ was independently characterized by Moon [16] and Hoffman [7]. Furthermore, the following result was discovered by Palmer [17].

Theorem 2.6. *For every two positive integers m and n,*

$$L(K_{m,n}) = K_m \times K_n.$$

The graph considered in Theorem 2.6 is sometimes referred to as a *rook's graph* (see [18] for the definition of a rook's graph and its properties).

Theorem 2.3 is now extended to complete bipartite graphs whose partite sets consist of at least two vertices and three vertices.

Theorem 2.7. *For every two integers m and n with $2 \le n < m$,*

$$\operatorname{estr}(K_{m,n}) = 2mn - 2n + 1.$$

Proof. Let $G = K_{m,n}$ and $H = L(G)$. Then Theorem 2.6 yields $H = K_m \times K_n$ so that $\overline{H} = K_{n(m)} - F$, where F is an $(n-1)$-factor of $K_{n(m)}$. It follows that $\omega\left(\overline{H}\right) = n$, that is, $\overline{H}$ does not contain K_{n+1} as a subgraph. However, Lemma 2.1 guarantees that F_{2n} contains K_{n+1} as a subgraph. Thus, $\overline{H}$ does not contain F_{2n} as a subgraph and so the inequality

$$\operatorname{estr}(G) = \operatorname{str}(H) \ge 2mn - 2n + 1$$

follows from Theorem 2.2.

To establish the reverse inequality, define the graph $\overline{H}$ with

$$V\left(\overline{H}\right) = \left\{ x_i^j \mid i \in [1, m] \text{ and } j \in [1, n] \right\} \text{ and } E(\overline{H}) = E(K_{n(m)}) - E(F),$$

where

$$E\left(K_{n(m)}\right) = \left\{ x_i^j x_{i'}^{j'} \mid i, i' \in [1, m] \text{ and } 1 \le j \le j' \le n \right\} \text{ and } E(F) = \left\{ x_i^j x_i^{j'} \mid i \in [1, m] \text{ and } 1 \le j \le j' \le n \right\}.$$

Further, let $S = \{ y_i \mid i \in [1, 2n-1] \}$, where

$$y_i = x_i^i \,(i \in [1, n]) \text{ and } y_i = x_{2n+2-i}^{2n+1-i} \,(i \in [n+1, 2n-1]).$$

Then the subgraph H' of $\overline{H}$ induced by S has the vertex set $V(H') = \{ y_i \mid i \in [1, 2n-1] \}$ and the edge set

$$E(H') = \{ y_i y_j \mid i \in [1, \lfloor (2n-1)/2 \rfloor] \text{ and } j \in [1+i, 2n-i] \}.$$

Thus, $H' = F_{2n-1}$ and $F_{2n-1} \subseteq \overline{H}$. It is now immediate from Theorem 2.1 that

$$\operatorname{estr}(G) = \operatorname{str}(H) \le 2mn - (2n-1) = 2mn - 2n + 1,$$

which completes the proof. □

This result also has a rather immediate corollary. This generalizes the result found in [14] that $\operatorname{str}(K_m \times K_2) = 4m - 3$ for every integer $m \ge 3$.

Corollary 2.2. *For every two integers m and n with $2 \le n < m$,*

$$\operatorname{str}(K_m \times K_n) = 2mn - 2n + 1.$$

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